Journal of Mathematical Research & Exposition
Vol.24, No.4, 649–652, November, 2004

A Direct Proof and Extensions of An Inequality *

YANG Shi-jun[1,2]
(1. Dept. of Math., Hangzhou Normal College, Hangzhou 310012, China;
2. Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, China)

Abstract: In this note, we discuss a well-known inequality which attracts considerable interest recently. We first obtain a full version of the inequality, then we give a concise and elementary proof which reveals its very essential. This naturally leads us to some more general extensions of it. Meanwhile, we point out some mistakes in the existing literature concerning the inequality.

Key words: Inequality; Hadamard inequality; Chebyshev inequality; extensions.

Classification: AMS(2000) 26D20/CLC number: O178

Document code: A **Article ID**: 1000-341X(2004)04-0649-04

1. Introduction

By considering monotonicity of the function $f(x)=\frac{(1+u)^x-(1-u)^x}{2xu}, 0<u<1$, Chen[1] gave a simple proof to the following inequalities (see Corollary 1.2 in [1]).

Lemma 1[1] *For* $0<a<b$, *we have*

$$\frac{b^{x+y}-a^{x+y}}{b^x-a^x}\geq\frac{x+y}{x}\Big(\frac{a+b}{2}\Big)^y,\quad x\geq 1, y>0, x+y\geq 2; \tag{1}$$

$$\frac{b^{x+y}-a^{x+y}}{b^x-a^x}<\frac{x+y}{x}\Big(\frac{a+b}{2}\Big)^y,\quad 0<x<1, y>0, x+y\leq 2. \tag{2}$$

Inequalities (1) and (2) are minor extensions of those in [4]–[6]. Similar results are also considered in [2,3] and references therein.

The main purpose of this note is to prove the following extensions.

Theorem 2 *If* $0<a<b$, *then the inequality*

$$\frac{b^s-a^s}{b^r-a^r}\geq\frac{s}{r}\Big(\frac{a+b}{2}\Big)^{s-r} \tag{3}$$

*Received date: 2002-10-14
Foundation item: Supported by the Special Funds for Major State Basic Research Projects (19990328) and NNSFC (10275054)
Biography: YANG Shi-jun (1962-), male, Ph.D., Associate Professor.

holds for

(a) $r \geq 1, s \geq 2, s > r$,

or

(b) $r \leq 2, s \leq 1, s < r$, *and* $s/r > 0$

with equality holding if and only if $r = 1, s = 2$. *The inequality (3) reverses for*

(c) $r > s \geq 2$,

or

(d) $r \leq 1, s \leq 2, s > r$, *and* $s/r > 0$.

Remark

- Setting $r = x, s = x + y$ in conditions (a) and (d) respectively, we easily obtain the inequalities (1) and (2).
- Note that the condition $s/r > 0$ in (a) and (d) can not be omitted.
  Counterexample. Let $a > 0, b > 0, r = 1, s = -1$,
  $$\frac{b^{-1} - a^{-1}}{b - a} = -\frac{1}{ab} \leq -\Big(\frac{a+b}{2}\Big)^{-2}.$$
  This contradicts the inequality (3). If $r = -1, s = 1$, the inequality (3) fails to reverse.
- Similar results to our theorem were independently obtained by Y.H.Kim (see Lemma in [2]). But there are mistakes in [2] since the condition $s/r > 0$ is neglected. This makes Theorems 1–4 in [2] fail to be valid. Furthermore, our proof is much simpler and easier to be generalized (see below) than that of [2].

2. A simple proof

We shall need the following famous inequalities in our proof.

Lemma 3 *Suppose that f is convex on the interval $[a, b]$, then we have*

$$\int_a^b f(x)\mathrm{d}x \geq f\Big(\frac{a+b}{2}\Big)(b-a). \tag{4}$$

Lemma 4 *Let $f(x), g(x)$ be integrable functions on the interval $[a, b]$, both increasing or both decreasing. Furthermore, let $p(x)$ be positive integrable function. Then*

$$\int_a^b p(x)f(x)g(x)\mathrm{d}x \int_a^b p(x)\mathrm{d}x \geq \int_a^b p(x)f(x)\mathrm{d}x \int_a^b p(x)g(x)\mathrm{d}x. \tag{5}$$

If one of the functions f and g is nonincreasing and the other is nondecreasing, then the inequality in (5) is reversed.

Inequalities (4) and (5) are called Hadamard and Chebyshev integral inequality, respectively. We are now ready to prove our theorem.

Proof Note first that $f(x) = x^{\mu}$ is convex on $[a, b]$ when $\mu \geq 1$ or $\mu \leq 0$ and is concave when $0 < \mu < 1$.

Case (a). If $r < 2$. Both $-rx^{r-1}$ and sx^{s-1} are convex on the interval $[a,b]$. It follows from Inequality (4) that

$$b^s - a^s = \int_a^b sx^{s-1}\mathrm{d}x \geq s(b-a)\Big(\frac{a+b}{2}\Big)^{s-1} \geq 0, \tag{6}$$

$$0 \leq b^r - a^r = \int_a^b rx^{r-1}\mathrm{d}x \leq r(b-a)\Big(\frac{a+b}{2}\Big)^{r-1}. \tag{7}$$

Now the inequality (3) follows easily from Inequalities (6) and (7).

If $r \geq 2$. We note that both x^{s-r} and x^{r-2} are increasing in $[a,b]$ and x^{s-r+1} is convex on $[a,b]$. So we have by Inequalities (4) and (5) that

$$\begin{aligned}\frac{1}{2}(b^2-a^2)(b^s-a^s) &= \int_a^b x\mathrm{d}x \int_a^b sx^{s-1}\mathrm{d}x = \int_a^b x\mathrm{d}x \int_a^b x\, x^{r-2}\, sx^{s-r}\mathrm{d}x \\ &\geq \int_a^b x^{r-1}\mathrm{d}x \int_a^b sx^{s-r+1}\mathrm{d}x \geq \frac{s}{r}(b^r-a^r)\left(\frac{a+b}{2}\right)^{s-r+1}(b-a).\end{aligned}$$

Now dividing both sides by $\frac{1}{2}(b^2-a^2)(b^r-a^r)$, we have inequality (3). The condition for equality can be easily deduced.

Case (b). If $1 < r \leq 2$, then $0 < r-1 \leq 1$. In this case, x^{s-1} is convex whereas x^{r-1} is concave on the interval $(0,\infty)$.

$$\frac{b^s-a^s}{b^r-a^r} = \frac{\int_a^b sx^{s-1}\mathrm{d}x}{\int_a^b rx^{r-1}\mathrm{d}x} \geq \frac{s}{r}\Big(\frac{a+b}{2}\Big)^{s-r}.$$

If $r \leq 1$, then both x^{s-r} and x^{r-1} are decreasing in $[a,b]$ and x^{s-r} is convex on $[a,b]$. Note that $(b^r-a^r)/r > 0$. It follows from Inequalities (4) and (5) that

$$\begin{aligned}(b-a)\frac{b^s-a^s}{s} &= \int_a^b \mathrm{d}x \int_a^b x^{s-1}\mathrm{d}x = \int_a^b \mathrm{d}x \int_a^b x^{s-r}\, x^{r-1}\mathrm{d}x \\ &\geq \int_a^b x^{s-r}\mathrm{d}x \int_a^b x^{r-1}\mathrm{d}x \geq (b-a)\Big(\frac{a+b}{2}\Big)^{s-r}\frac{b^r-a^r}{r},\end{aligned}$$

which yields the desired conclusion.

Cases (c) and (d) follow from Cases (a) and (b) by changing s and r respectively. □

3. Generalizations

In this section, we assume $g(x) \neq 0$ unless otherwise stated or implied. Similar arguments to those of the previous section leads to the following theorem.

Theorem 5 *Let $f,g : [a,b] \to \mathbf{R}$ be defined. The inequality*

$$\frac{\int_a^b f(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x} \geq \frac{f(\frac{a+b}{2})}{g(\frac{a+b}{2})} \tag{8}$$

holds if one of the following conditions is satisfied:

(i). $f(x)$ is convex and $g(x)$ is concave on $[a,b]$ and $\int_a^b g(x)\mathrm{d}x > 0$;

(ii). Both $g(x)$ and $f(x)/g(x)$ are nonincreasing or nondecreasing in $[a,b]$ and $f(x)/g(x)$ is convex on $[a,b]$;

(iii). $g(x)$ is nonnegative integrable and symmetric with respect to $x=(a+b)/2$ and $f(x)/g(x)$ is convex on $[a,b]$.

Remark 6 The condition (iii), which is due to Fejér, can be found in [7].

Corollary 7 *Let $f, g : [a,b] \to \mathbf{R}$ be both three times differentiable satisfying $f'''(x) > 0, g'''(x) < 0, \forall x \in [a,b], g(b)-g(a) > 0$, then we have*

$$\frac{f(b)-f(a)}{g(b)-g(a)} \geq \frac{f'(\frac{a+b}{2})}{g'(\frac{a+b}{2})}. \tag{9}$$

The inequality (9) reverses if $f'''(x) < 0, g'''(x) > 0, g(\frac{a+b}{2}) > 0, \forall x \in [a,b]$.

Acknowledgement The author would like to thank Professor Wang Xing-hua whose comments help to improve the original manuscript greatly.

References:

[1] CHEN Dao-qi. *Extensions of some inequalities* [J]. Appl. Math. J. Chinese Univ. Ser. B., 2001, **16**(3): 241–243.

[2] KIM Young-Ho. *A simple proof and extensions of an inequality* [J]. J. Math. Anal. Appl., 2000, **245**: 294–296.

[3] QI Feng, LUO Q M. *A simple proof of monotonicity for extended mean values* [J]. J. Math. Anal. Appl., 1998, **224**: 356–359.

[4] QI Feng, XU Sen-lin. *Refinements and extensions of an inequality II* [J]. J. Math. Anal. Appl., 1997, **211**: 616–620.

[5] QI Feng, XU Sen-lin. *The function $(b^x - a^x)/x$: inequalities and properties* [J]. Proc. Amer. Math. Soc., 1998, **126**(11): 3355–3359.

[6] LIU Zheng. *Remarks on refinements and extensions of an inequality* [J]. J. Math. Anal. Appl., 1999, **234**: 529–533.

[7] MITRINOVIĆ D S. *Analytical Inequalities* [M]. Springer-Verlag, 1970.

一个不等式的简单证明与推广

杨 士 俊 [1,2]

(1. 杭州师范学院数学系， 浙江 杭州 310012; 2. 浙江大学浙江近代物理中心， 浙江 杭州 310027)

摘 要: 本文讨论了一个引人注目而被广泛讨论的、与平均值有关的不等式．我们首先获得了它的完整形式，给出了一个非常简洁而能揭示问题本质的证明．由此我们自然地引出了它的最为一般的形式．同时，我们还指出文献中存在的一些谬误．

关键词: 不等式； Hadamard 不等式； Chebyshev 不等式；推广．