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G^1 Continuous Condition of B-Spline Surfaces with Double Knots

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Abstract: The conditions for G^1 continuity between two adjacent bicubic B-spline surfaces with double interior knots along their common boundary curve are obtained in this paper, which are directly represented by the control points of the two B-spline surfaces. As stated by Shi Xi-quan and Zhao Yan, a local scheme of constructing G^1 continuous B-spline surface models with single interior knots does not exist; we may achieve a local scheme of (true) G^1 continuity over an arbitrary B-spline surface network using these conditions.

Key words: B-spline surface patches; double knots; geometric continuity.
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1. Introduction

Geometric continuity between adjacent parametric surface patches has been playing a very important role in CAD/CAM, geometric modeling and reverse engineering, etc.. This is not only because geometric continuity provides free shape parameters which can be used to construct and modify very complicated geometric objects, but also because this type of continuity is in practice the essential continuity between surface patches, i.e., it avoids dependencies on the parametrizations of the involved patches. It is interesting to note that despite the fact that B-spline surfaces are popular representation of choice in CAD, geometric modeling, animation and reverse engineering industries, etc., to authors' knowledge, very little research has addressed the issue of G^1 continuity of B-spline surfaces [Shi and Wang, 1999]. In recent years, a fair amount of research on G^1 continuity of a number of other surface formulations has been conducted. Examples include rectangular Bézier patches, triangular Bézier patches, combinations of rectangular and triangular Bézier patches, Catmull-Rom patches, Gregory patches and Hermite patches (DeRose, 1990; DeRose and Barsky, 1988; Farin, 1982; Hahn, 1989; Kahmann, 1983; Liu and Hoschek, 1989; Peters, 1990; Piper, 1987; Shirman and Séquin, 1990; Shirman and Séquin, 1991; Watkins, 1988; Ye et al., 1996), etc.. Unfortunately, none of the above literatures studies the issue on the conditions for G^1 continuity of B-spline surfaces.

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As stated in [10], a local scheme of constructing G^1 smooth B-spline surface models with single interior knots does not exist. In this paper, we obtain the conditions for G^1 continuity between two adjacent bicubic B-spline surfaces with double interior knots along their common boundary curve, these conditions will result in a local scheme of achieving true G^1 continuity over an arbitrary B-spline surface network. These conditions are directly represented by the control points of the two B-spline surfaces.

2. Two adjacent B-spline surfaces

For two adjacent bicubic B-spline surfaces

$$\begin{aligned} S_1(u,v) &= \sum_{i=0}^{m_1}\sum_{j=0}^{n_1} \mathbf{P}_{i,j} N_{i,3}(u) N_{j,3}(v), \\ S_2(u,v) &= \sum_{i=0}^{m_2}\sum_{j=0}^{n_2} \mathbf{Q}_{i,j} N_{i,3}(u) N_{j,3}(v) \end{aligned} \tag{1}$$

defined over $I^2 = [0,1] \times [0,1]$ with the common boundary curve $C(v) = S_1(0,v) = S_2(0,v)$, and the following clamped knot vectors respectively

$$\begin{cases} U_1 &= \{0,0,0,0,u_{1,4},u_{1,5},\ldots,u_{1,m_1},1,1,1,1\}, \\ V_1 &= \{0,0,0,0,v_{1,4},v_{1,5},\ldots,v_{1,n_1},1,1,1,1\}, \end{cases}$$

and

$$\begin{cases} U_2 &= \{0,0,0,0,u_{2,4},u_{2,5},\ldots,u_{2,m_2},1,1,1,1\}, \\ V_2 &= \{0,0,0,0,v_{2,4},v_{2,5},\ldots,v_{2,n_2},1,1,1,1\}, \end{cases}$$

by knot refinement, we can make $S_1(u,v)$ and $S_2(u,v)$ have the same knot vectors in u-direction and v-direction, that is $U_1 = U_2$ and $V_1 = V_2$. Without loss of generality, we assume both of $S_1(u,v)$ and $S_2(u,v)$ have the same knot vectors in u-direction and v-direction below

$$U = V = \{\hat{t_0},\ldots,\hat{t}_{m+4}\} = \{0,0,0,0,\hat{t}_4,\ldots,\hat{t}_m,1,1,1,1\}, \quad n \geq 7,$$

where $\hat{t}_0 = \hat{t}_1 = \hat{t}_2 = \hat{t}_3 = 0$, $\hat{t}_{m+1} = \hat{t}_{m+2} = \hat{t}_{m+3} = \hat{t}_{m+4} = 1$ and $U(V)$ has at least two distinct pairs of interior double knots, that is $\hat{t}_i = \hat{t}_{i+1}$ and $\hat{t}_j = \hat{t}_{j+1}$ for $4 \leq i < j \leq m$. The numbers of the control points in u-direction and v-direction are $m+1$.

Before we deduce the conditions for G^1 continuity between $S_1(u,v)$ and $S_2(u,v)$ along their common boundary curve $C(v) = S_1(0,v) = S_2(0,v)$, we define three curves

$$\begin{cases} C_1(v) &= \left.\dfrac{\partial S_1(u,v)}{\partial u}\right|_{u=0} &= \dfrac{3}{\hat{t}_4}\displaystyle\sum_{j=0}^{m}(\mathbf{P}_{1,j} - \mathbf{P}_{0,j})N_{j,3}(v), \\ C_2(v) &= \left.\dfrac{\partial S_2(u,v)}{\partial u}\right|_{u=0} &= \dfrac{3}{\hat{t}_4}\displaystyle\sum_{j=0}^{m}(\mathbf{Q}_{1,j} - \mathbf{Q}_{0,j})N_{j,3}(v), \qquad v \in [0,1], \\ C_0(v) &= \left.\dfrac{\partial S_1(u,v)}{\partial v}\right|_{u=0} &= 3\displaystyle\sum_{j=0}^{m-1}\dfrac{\mathbf{P}_{0,j+1} - \mathbf{P}_{0,j}}{\hat{t}_{j+4} - \hat{t}_{j+1}}N_{j,2}(v), \end{cases} \tag{2}$$

that is,

$$
\begin{aligned}
C_1(v) &= \frac{3}{\hat{t}_4}\sum_{j=0}^{m}\mathbf{P}_j N_{j,3}(v),\\
C_2(v) &= \frac{3}{\hat{t}_4}\sum_{j=0}^{m}\mathbf{Q}_j N_{j,3}(v),\\
C_0(v) &= 3\sum_{j=0}^{m-1}\frac{\mathbf{T}_j}{\hat{t}_{j+4}-\hat{t}_{j+1}}N_{j,2}(v),
\end{aligned}
\tag{3}
$$

where $\mathbf{P}_j = \mathbf{P}_{1,j} - \mathbf{P}_{0,j}$, $\mathbf{Q}_j = \mathbf{Q}_{1,j} - \mathbf{Q}_{0,j}$ and $\mathbf{T}_j = \mathbf{P}_{0,j+1} - \mathbf{P}_{0,j}$ are respectively the *B-spline control points* of $C_1(v)$, $C_2(v)$ and $C_0(v)$. Obviously, $C_1(v)$ and $C_2(v)$ are two cubic B-spline curves with the knot vector V, and $C_0(v)$ is a quadratic B-spline curve with the knot vector $V_c = \{0,0,0,\hat{t}_4,\ldots,\hat{t}_m,1,1,1\}$.

Since the conditions for G^1 continuity of two adjacent B-spline surfaces depend on the structures of U and V, we only consider the following two typical cases of U and V here.

Case 1. All interior knots are double. This case means the knot vectors U and V are of the form

$$U = V = \{0,0,0,0,t_1,t_1,\ldots,t_n,t_n,1,1,1,1\}, \tag{4}$$

where $t_1 < \ldots < t_n$ $(n \geq 2)$, and $m = 2n+3$.

Case 2. Two pairs of interior knots are double. In this case, the knot vectors U and V have the following form

$$U = V = \{0,0,0,0,t_1,t_1,t_2,\ldots,t_{n-1},t_n,t_n,1,1,1,1\}, \tag{5}$$

where $t_1 < t_2 < \ldots < t_{n-1} < t_n$ $(n \geq 3)$, and $m = n+5$.

The derivation of the G^1 conditions for the rest cases of knot vectors is similar to the above two cases, it will not be discussed in this paper.

3. Decomposition of B-spline curves

In this section, for the two cases of U and V, we assume $h = t_{j+1} - t_j = 1/(n+1)$ for $j = 0,\ldots,n$ $(t_0 = 0, t_{n+1} = 1)$, and decompose the B-spline curves $C_1(v)$, $C_2(v)$ and $C_0(v)$ defined in (2) into their constituent Bézier curves.

3.1. All interior knots are double

Inserting $t_1,\ldots,t_n$ into V for the cubic B-spline curve $C_1(v)$ in (3) (refer to Page 143–144 in [8]), we have

$$C_1(v) = \frac{3}{h}\sum_{j=0}^{3(n+1)}\widehat{\mathbf{P}}_j N_{j,3}(v) \tag{6}$$

with the knot vector

$$\widehat{V} = \{0,0,0,0,t_1,t_1,t_1,\ldots,t_n,t_n,t_n,1,1,1,1\}, \tag{7}$$

and the control points $\widehat{\mathbf{P}}_j$ as follows

$$\begin{aligned}
&\widehat{\mathbf{P}}_0 = \mathbf{P}_0, \\
&\widehat{\mathbf{P}}_{3j+1} = \mathbf{P}_{2j+1}, \qquad && j = 0, \ldots, n; \\
&\widehat{\mathbf{P}}_{3j+2} = \mathbf{P}_{2j+2}, && j = 0, \ldots, n; \\
&\widehat{\mathbf{P}}_{3j} = \tfrac{1}{2}(\mathbf{P}_{2j} + \mathbf{P}_{2j+1}), && j = 1, \ldots, n; \\
&\widehat{\mathbf{P}}_{3(n+1)} = \mathbf{P}_{2n+3}.
\end{aligned} \tag{8}$$

Analogously, insert $t_1, \ldots, t_n$ into V for $C_2(v)$ in (3) to get

$$C_2(v) = \frac{3}{h} \sum_{j=0}^{3(n+1)} \widehat{\mathbf{Q}}_j N_{j,3}(v), \tag{9}$$

on the knot vector $\widehat{V}$, where $\widehat{\mathbf{Q}}_j$ are defined as same as (8).

Similarly, directly from (3) we obtain

$$C_0(v) = \frac{3}{h} \sum_{j=0}^{2(n+1)} \widehat{\mathbf{T}}_j N_{j,2}(v) \tag{10}$$

with the knot vector

$$\hat{V}_c = \{0, 0, 0, t_1, t_1, \ldots, t_n, t_n, 1, 1, 1\}, \tag{11}$$

and the control points $\widehat{\mathbf{T}}_j$ below

$$\begin{aligned}
&\widehat{\mathbf{T}}_0 = \mathbf{T}_0, \\
&\widehat{\mathbf{T}}_{2j+1} = \mathbf{T}_{2j+1}, \qquad && j = 0, \ldots, n; \\
&\widehat{\mathbf{T}}_{2j} = \tfrac{1}{2}\mathbf{T}_{2j}, && j = 1, \ldots, n; \\
&\widehat{\mathbf{T}}_{2(n+1)} = \mathbf{T}_{2(n+1)}.
\end{aligned} \tag{12}$$

3.2. Two pairs of interior knots are double

In this case, inserting $t_1, t_2, t_2, \ldots, t_{n-1}, t_{n-1}, t_n$ into V for the cubic B-spline curve $C_1(v)$ in (3), we have

$$C_1(v) = \frac{3}{h} \sum_{j=0}^{3(n+1)} \widehat{\mathbf{P}}_j N_{j,3}(v) \tag{13}$$

defined on the knot vector $\widehat{V}$ of (7) where the control points $\widehat{\mathbf{P}}_j$ will be defined late.

Analogously, insert $t_1, t_2, t_2, \ldots, t_{n-1}, t_{n-1}, t_n$ into V for $C_2(v)$ in (3) to get

$$C_2(v) = \frac{3}{h} \sum_{j=0}^{3(n+1)} \widehat{\mathbf{Q}}_j N_{j,3}(v), \tag{14}$$

with the knot vector $\widehat{V}$, where $\widehat{\mathbf{Q}}_j$ are similarly defined as $\widehat{\mathbf{P}}_j$.

Similarly, we insert $t_2, \ldots, t_{n-1}$ into V_c for $C_0(v)$ in (3) to get

$$C_0(v) = \frac{3}{h} \sum_{j=0}^{2(n+1)} \widehat{\mathbf{T}}_j N_{j,2}(v) \tag{15}$$

where the knot vector $\hat{V}_c$ of (11), and the control points $\widehat{\mathbf{T}}_j$ will be computed late.

$\widehat{\mathbf{P}}_j$ and $\widehat{\mathbf{T}}_j$ are determined by the following equations:

Case $n=3$,

$$\begin{array}{lll}\widehat{\mathbf{P}}_i=\mathbf{P}_i \quad i=0,1,2, & \widehat{\mathbf{P}}_3=\frac{1}{2}(\mathbf{P}_3+\mathbf{P}_2), & \widehat{\mathbf{P}}_4=\mathbf{P}_3,\\ \widehat{\mathbf{P}}_5=\frac{1}{2}(\mathbf{P}_4+\mathbf{P}_3), & \widehat{\mathbf{P}}_6=\frac{1}{4}(\mathbf{P}_5+2\mathbf{P}_4+\mathbf{P}_3), & \widehat{\mathbf{P}}_7=\frac{1}{2}(\mathbf{P}_5+\mathbf{P}_4),\\ \widehat{\mathbf{P}}_8=\mathbf{P}_5, \quad \widehat{\mathbf{P}}_9=\frac{1}{2}(\mathbf{P}_6+\mathbf{P}_5), & \widehat{\mathbf{P}}_i=\mathbf{P}_i & i=10,11,12.\end{array} \tag{16}$$

$$\begin{array}{l}\widehat{\mathbf{T}}_i=\mathbf{T}_i, \quad i=0,1, \quad \widehat{\mathbf{T}}_i=\mathbf{T}_i, \quad i=2,3, \quad \widehat{\mathbf{T}}_4=\frac{1}{4}(\mathbf{T}_4+\mathbf{T}_3),\\ \widehat{\mathbf{T}}_i=\frac{1}{2}\mathbf{T}_{i-1}, \quad i=5,6, \quad \widehat{\mathbf{T}}_i=\mathbf{T}_{i-1}, \quad i=7,8.\end{array} \tag{17}$$

Case $n=4$,

$$\begin{array}{l}\widehat{\mathbf{P}}_i=\mathbf{P}_i \quad i=0,1,2, \quad \widehat{\mathbf{P}}_3=\frac{1}{2}(\mathbf{P}_3+\mathbf{P}_2), \quad \widehat{\mathbf{P}}_4=\mathbf{P}_3,\\ \widehat{\mathbf{P}}_5=\frac{1}{2}(\mathbf{P}_4+\mathbf{P}_3), \quad \widehat{\mathbf{P}}_6=\frac{1}{12}(2\mathbf{P}_5+7\mathbf{P}_4+3\mathbf{P}_3), \quad \widehat{\mathbf{P}}_7=\frac{1}{3}(\mathbf{P}_5+2\mathbf{P}_4),\\ \widehat{\mathbf{P}}_8=\frac{1}{3}(2\mathbf{P}_5+\mathbf{P}_4), \quad \widehat{\mathbf{P}}_9=\frac{1}{12}(3\mathbf{P}_6+7\mathbf{P}_5+2\mathbf{P}_4), \quad \widehat{\mathbf{P}}_{10}=\frac{1}{2}(\mathbf{P}_6+\mathbf{P}_5),\\ \widehat{\mathbf{P}}_{11}=\mathbf{P}_6, \quad \widehat{\mathbf{P}}_{12}=\frac{1}{2}(\mathbf{P}_7+\mathbf{P}_6), \quad \widehat{\mathbf{P}}_i=\mathbf{P}_{i-6} \quad i=13,14,15.\end{array} \tag{18}$$

$$\begin{array}{l}\widehat{\mathbf{T}}_i=\mathbf{T}_i, \quad i=0,1, \quad \widehat{\mathbf{T}}_i=\mathbf{T}_i, \quad i=2,3, \quad \widehat{\mathbf{T}}_4=\frac{1}{6}\mathbf{T}_4+\frac{1}{4}\mathbf{T}_3, \quad \widehat{\mathbf{T}}_5=\frac{1}{3}\mathbf{T}_4,\\ \widehat{\mathbf{T}}_6=\frac{1}{4}\mathbf{T}_5+\frac{1}{6}\mathbf{T}_4, \quad \widehat{\mathbf{T}}_i=\frac{1}{2}\mathbf{T}_{i-2}, \quad i=7,8, \quad \widehat{\mathbf{T}}_i=\mathbf{T}_{i-2}, \quad i=9,10.\end{array} \tag{19}$$

Case $n\geq 5$,

$$\begin{array}{l}\widehat{\mathbf{P}}_i=\mathbf{P}_i \ \ i=0,1,2, \ \ \widehat{\mathbf{P}}_3=\frac{1}{2}(\mathbf{P}_3+\mathbf{P}_2), \ \ \widehat{\mathbf{P}}_4=\mathbf{P}_3, \ \ \widehat{\mathbf{P}}_5=\frac{1}{2}(\mathbf{P}_4+\mathbf{P}_3),\\ \widehat{\mathbf{P}}_6=\frac{1}{12}(2\mathbf{P}_5+7\mathbf{P}_4+3\mathbf{P}_3), \ \ \widehat{\mathbf{P}}_{3j+1}=\frac{1}{3}(\mathbf{P}_{j+3}+2\mathbf{P}_{j+2}), \ \ j=2,\ldots,n-2,\\ \widehat{\mathbf{P}}_{3j+2}=\frac{1}{3}(2\mathbf{P}_{j+3}+\mathbf{P}_{j+2}), \ \ j=2,\ldots,n-2,\\ \widehat{\mathbf{P}}_{3j}=\frac{1}{6}(\mathbf{P}_{j+3}+4\mathbf{P}_{j+2}+\mathbf{P}_{j+1}), \ \ j=3,\ldots,n-2,\\ \widehat{\mathbf{P}}_{3(n-1)}=\frac{1}{12}(3\mathbf{P}_{n+2}+7\mathbf{P}_{n+1}+2\mathbf{P}_n), \ \ \widehat{\mathbf{P}}_{3n-2}=\frac{1}{2}(\mathbf{P}_{n+2}+\mathbf{P}_{n+1}),\\ \widehat{\mathbf{P}}_{3n-1}=\mathbf{P}_{n+2}, \ \ \widehat{\mathbf{P}}_{3n}=\frac{1}{2}(\mathbf{P}_{n+3}+\mathbf{P}_{n+2}), \ \ \widehat{\mathbf{P}}_{3n+i}=\mathbf{P}_{n+2+i}, \ \ i=1,2,3.\end{array} \tag{20}$$

$$\begin{array}{l}\widehat{\mathbf{T}}_i=\mathbf{T}_i, \quad i=0,1, \ \ \widehat{\mathbf{T}}_i=\frac{1}{2}\mathbf{T}_i, \ \ i=2,3, \ \ \widehat{\mathbf{T}}_4=\frac{1}{6}\mathbf{T}_4+\frac{1}{4}\mathbf{T}_3,\\ \widehat{\mathbf{T}}_{2j+1}=\frac{1}{3}\mathbf{T}_{j+2}, \ \ j=2,\ldots,n-2 \ \ \widehat{\mathbf{T}}_{2j}=\frac{1}{6}(\mathbf{T}_{i+2}+\mathbf{T}_{i+1}), \ \ j=3,\ldots,n-2\\ \widehat{\mathbf{T}}_{2(n-1)}=\frac{1}{4}\mathbf{T}_{n+1}+\frac{1}{6}\mathbf{T}_n, \ \ \widehat{\mathbf{T}}_{2n-2+i}=\frac{1}{2}\mathbf{T}_{n+i}, \ \ i=1,2, \ \ \widehat{\mathbf{T}}_{2n+i}=\mathbf{T}_{n+2+i}, \ \ i=1,2.\end{array} \tag{21}$$

4. Conditions for G^1 continuity

Let the extracted Bézier curves of $C_1(v)$, $C_2(v)$ and $C_0(v)$ defined in (6), (9) and (10) or (13), (14) and (15) on the interval $[t_j,t_{j+1}]$ be respectively denoted by

$$\begin{aligned}C_{1,j}(t)&:=\sum_{i=0}^{3}\widehat{\mathbf{P}}_{3j+i}B_{i,3}(\hat{t}),\\ C_{2,j}(t)&:=\sum_{i=0}^{3}\widehat{\mathbf{Q}}_{3j+i}B_{i,3}(\hat{t}),\\ C_{0,j}(t)&:=\sum_{i=0}^{2}\widehat{\mathbf{T}}_{2j+i}B_{i,2}(\hat{t}),\end{aligned} \tag{22}$$

where $\hat{t} = (t - t_j)/(t_{j+1} - t_j)$, $t \in [t_j, t_{j+1}]$ $(j = 0, \ldots, n)$ and $t_0 = 0, t_{n+1} = 1$. Therefore, $S_1(u,v)$ and $S_2(u,v)$ are G^1 smooth joint along their common boundary curve $C(v)$ if and only if there exist analytic functions $h_j(\hat{t}), f_j(\hat{t})$ and $g_j(\hat{t})$ for $j = 0, \ldots, n$ such that

$$h_j(\hat{t}) \sum_{i=0}^{3} \widehat{\mathbf{Q}}_{3j+i} B_{i,3}(\hat{t}) \;=\; f_j(\hat{t}) \sum_{i=0}^{3} \widehat{\mathbf{P}}_{3j+i} B_{i,3}(\hat{t}) \;+\; g_j(\hat{t}) \sum_{i=0}^{2} \widehat{\mathbf{T}}_{2j+i} B_{i,2}(\hat{t}) \tag{23}$$

where $h_j(\hat{t}) f_j(\hat{t}) < 0$.

In almost all existing literatures related to constructing G^1 smooth surface models, it is usual to take

$$\left\{ \begin{array}{rcl} h_j(\hat{t}) & = & 1, \\ f_j(\hat{t}) & = & -1, \\ g_j(\hat{t}) & = & b_j(1-\hat{t}) + c_j \hat{t}, \end{array} \right. \qquad j = 0, \ldots, n \tag{24}$$

to obtain a sufficient condition for G^1 continuity. As usual, we apply (24) to (23) to yield the conditions for G^1 continuity between $S_1(u,v)$ and $S_2(u,v)$ as follows

$$\left\{ \begin{array}{l} \widehat{\mathbf{Q}}_{3j} = -\widehat{\mathbf{P}}_{3j} + b_j \widehat{\mathbf{T}}_{2j}, \\ 3\widehat{\mathbf{Q}}_{3j+1} = -3\widehat{\mathbf{P}}_{3j+1} + 2b_j \widehat{\mathbf{T}}_{2j+1} + c_j \widehat{\mathbf{T}}_{2j}, \\ 3\widehat{\mathbf{Q}}_{3j+2} = -3\widehat{\mathbf{P}}_{3j+2} + b_j \widehat{\mathbf{T}}_{2j+2} + 2c_j \widehat{\mathbf{T}}_{2j+1}, \\ \widehat{\mathbf{Q}}_{3j+3} = -\widehat{\mathbf{P}}_{3j+3} + c_j \widehat{\mathbf{T}}_{2j+2}, \end{array} \right. \qquad j = 0, \ldots, n. \tag{25}$$

From the first and the last equations in (25), we get

$$b_{j+1} = c_j, \quad j = 0, \ldots, n-1. \tag{26}$$

Rewrite (25) as the following form

$$\left\{ \begin{array}{l} \widehat{\mathbf{Q}}_{3j} = -\widehat{\mathbf{P}}_{3j} + b_j \widehat{\mathbf{T}}_{2j}, \\ 3\widehat{\mathbf{Q}}_{3j+1} = -3\widehat{\mathbf{P}}_{3j+1} + 2b_j \widehat{\mathbf{T}}_{2j+1} + c_j \widehat{\mathbf{T}}_{2j}, \\ 3\widehat{\mathbf{Q}}_{3j+2} = -3\widehat{\mathbf{P}}_{3j+2} + b_j \widehat{\mathbf{T}}_{2j+2} + 2c_j \widehat{\mathbf{T}}_{2j+1}, \end{array} \right. \qquad j = 0, \ldots, n-1; \tag{27}$$

and

$$\left\{ \begin{array}{l} \widehat{\mathbf{Q}}_{3n} = -\widehat{\mathbf{P}}_{3n} + b_n \widehat{\mathbf{T}}_{2n}, \\ 3\widehat{\mathbf{Q}}_{3n+1} = -3\widehat{\mathbf{P}}_{3n+1} + 2b_n \widehat{\mathbf{T}}_{2n+1} + c_n \widehat{\mathbf{T}}_{2n}, \\ 3\widehat{\mathbf{Q}}_{3n+2} = -3\widehat{\mathbf{P}}_{3n+2} + b_n \widehat{\mathbf{T}}_{2n+2} + 2c_n \widehat{\mathbf{T}}_{2n+1}, \\ \widehat{\mathbf{Q}}_{3(n+1)} = -\widehat{\mathbf{P}}_{3(n+1)} + c_n \widehat{\mathbf{T}}_{2(n+1)}. \end{array} \right. \tag{28}$$

For brevity in the following discussion, we now write $\bar{\mathbf{Q}}_j$ for $\mathbf{Q}_j + \mathbf{P}_j$, that is

$$\bar{\mathbf{Q}}_j = \mathbf{Q}_j + \mathbf{P}_j, \quad j = 0, \ldots, 2n+3. \tag{29}$$

4.1. G^1 conditions for the case of all interior double knots

From (29), substituting (8), $\widehat{\mathbf{Q}}_j$ in (9) and (12) into (27) and (28) for $\widehat{\mathbf{P}}_j$, $\widehat{\mathbf{Q}}_j$ and $\widehat{\mathbf{T}}_j$ yields

$$\left\{ \begin{array}{l} \bar{\mathbf{Q}}_0 = b_0 \mathbf{T}_0, \\ 3\bar{\mathbf{Q}}_1 = 2b_0 \mathbf{T}_1 + c_0 \mathbf{T}_0, \\ 3\bar{\mathbf{Q}}_2 = \frac{1}{2} b_0 \mathbf{T}_2 + 2c_0 \mathbf{T}_1, \end{array} \right. \tag{30}$$

$$\begin{cases} \bar{\mathbf{Q}}_{2j} + \bar{\mathbf{Q}}_{2j+1} = c_{j-1}\mathbf{T}_{2j}, \\ 3\bar{\mathbf{Q}}_{2j+1} = 2c_{j-1}\mathbf{T}_{2j+1} + \frac{1}{2}c_j\mathbf{T}_{2j}, \\ 3\bar{\mathbf{Q}}_{2j+2} = \frac{1}{2}c_{j-1}\mathbf{T}_{2j+2} + 2c_j\mathbf{T}_{2j+1}, \end{cases} \quad j = 1,\ldots,n-1, \tag{31}$$

and

$$\begin{cases} \bar{\mathbf{Q}}_{2n} + \bar{\mathbf{Q}}_{2n+1} = c_{n-1}\mathbf{T}_{2n}, \\ 3\bar{\mathbf{Q}}_{2n+1} = 2c_{n-1}\mathbf{T}_{2n+1} + \frac{1}{2}c_n\mathbf{T}_{2n}, \\ 3\bar{\mathbf{Q}}_{2n+2} = c_{n-1}\mathbf{T}_{2n+2} + 2c_n\mathbf{T}_{2n+1}, \\ \bar{\mathbf{Q}}_{2n+3} = c_n\mathbf{T}_{2n+2}. \end{cases} \tag{32}$$

This follows that (30)–(32) are equivalent to

$$\begin{cases} \bar{\mathbf{Q}}_0 = b_0\mathbf{T}_0, \\ 3\bar{\mathbf{Q}}_1 = 2b_0\mathbf{T}_1 + c_0\mathbf{T}_0, \\ 3\bar{\mathbf{Q}}_2 = \frac{1}{2}b_0\mathbf{T}_2 + 2c_0\mathbf{T}_1 \\ 3\bar{\mathbf{Q}}_{2j+1} = 2c_{j-1}\mathbf{T}_{2j+1} + \frac{1}{2}c_j\mathbf{T}_{2j}, \quad j = 1,\ldots,n-1, \\ 3\bar{\mathbf{Q}}_{2j+2} = \frac{1}{2}c_{j-1}\mathbf{T}_{2j+2} + 2c_j\mathbf{T}_{2j+1}, \\ 3\bar{\mathbf{Q}}_{2n+1} = 2c_{n-1}\mathbf{T}_{2n+1} + \frac{1}{2}c_n\mathbf{T}_{2n}, \\ 3\bar{\mathbf{Q}}_{2n+2} = c_{n-1}\mathbf{T}_{2n+2} + 2c_n\mathbf{T}_{2n+1}, \\ \bar{\mathbf{Q}}_{2n+3} = c_n\mathbf{T}_{2n+2}. \end{cases} \tag{33}$$

Subsequently,

$$2c_j(\mathbf{T}_{2j+1} + \mathbf{T}_{2j+3}) = (3c_j - \frac{c_{j-1} + c_{j+1}}{2})\mathbf{T}_{2j+2}, \quad j = 0,\ldots,n-1, \tag{34}$$

where $c_{-1} = b_0$.

In (34), let

$$3c_j - \frac{1}{2}(c_{j-1} + c_{j+1}) = 2k_jc_j \tag{35}$$

yields

$$\mathbf{T}_{2j+1} + \mathbf{T}_{2j+3} = k_j\mathbf{T}_{2j+2}, \quad j = 0,\ldots,n-1. \tag{36}$$

Let $\mathbf{P}_{0,0}, \mathbf{P}_{0,1}, \ldots, \mathbf{P}_{0,2n+2}, \mathbf{P}_{0,2n+3}$ denote the control points. From (36) and $\mathbf{T}_j = \mathbf{P}_{0,j+1} + \mathbf{P}_{0,j}$, we have

$$(1 + k_j)\mathbf{P}_{0,2j+2} + \mathbf{P}_{0,2j+4} = \mathbf{P}_{0,2j+1} + (1 + k_j)\mathbf{P}_{0,2j+3}, \quad j = 0,\ldots,n-1. \tag{37}$$

When $n = 1$,from (38) we obtain

$$\mathbf{P}_{0,4} = \mathbf{P}_{0,1} - (1 + k_0)\mathbf{P}_{0,2} + (1 + k_0)\mathbf{P}_{0,3}. \tag{38}$$

When $n = 2$, from (38) we obtain

$$\begin{cases} \mathbf{P}_{0,3} = \dfrac{-(1+k_0)\mathbf{P}_{0,1} + (1+k_0)(1+k_1)\mathbf{P}_{0,2} + (1+k_1)\mathbf{P}_{0,5} - \mathbf{P}_{0,6}}{k_1 + k_0(1+k_1)}, \\ \mathbf{P}_{0,4} = \dfrac{-\mathbf{P}_{0,1} + (1+k_0)\mathbf{P}_{0,2} + (1+k_0)(1+k_1)\mathbf{P}_{0,5} - (1+k_0)\mathbf{P}_{0,6}}{k_1 + k_0(1+k_1)}. \end{cases} \tag{39}$$

When $n=3$, from (38) we obtain

$$\left\{\begin{array}{l}\mathbf{P}_{0,4}=\mathbf{P}_{0,1}-(1+k_0)\mathbf{P}_{0,2}+(1+k_0)\mathbf{P}_{0,3}.\\ \mathbf{P}_{0,5}=\dfrac{-(1+k_1)\mathbf{P}_{0,3}+(1+k_1)(1+k_2)\mathbf{P}_{0,4}+(1+k_2)\mathbf{P}_{0,7}-\mathbf{P}_{0,8}}{k_2+k_1(1+k_2)},\\ \mathbf{P}_{0,6}=\dfrac{-\mathbf{P}_{0,3}+(1+k_1)\mathbf{P}_{0,4}+(1+k_1)(1+k_2)\mathbf{P}_{0,7}-(1+k_1)\mathbf{P}_{0,8}}{k_2+k_1(1+k_2)}.\end{array}\right. \tag{40}$$

When $n>3$, we can rewrite $n=2r$ or $n=2r+1$, from (38) we obtain

$$\left\{\begin{array}{ll}\mathbf{P}_{0,2j+2}=\mathbf{P}_{0,2j-1}-(1+k_{j-1})\mathbf{P}_{0,2j}+(1+k_{j-1})\mathbf{P}_{0,2j+1}, & j=1,\dots,r-1.\\ \mathbf{P}_{0,2r+1}=\frac{-(1+k_{r-1})\mathbf{P}_{0,2r-1}+(1+k_{r-1})(1+k_r)\mathbf{P}_{0,2r}+(1+k_r)\mathbf{P}_{0,2r+3}-\mathbf{P}_{0,2r+4}}{k_r+k_{r-1}(1+k_r)}, & \\ \mathbf{P}_{0,2r+2}=\frac{-\mathbf{P}_{0,2r-1}+(1+k_{r-1})\mathbf{P}_{0,2r}+(1+k_{r-1})(1+k_r)\mathbf{P}_{0,2r+3}-(1+k_{r-1})\mathbf{P}_{0,2r+4}}{k_r+k_{r-1}(1+k_r)}, & \\ \mathbf{P}_{0,2j-1}=(1+k_{j-1})\mathbf{P}_{0,2j}-(1+k_{j-1})\mathbf{P}_{0,2j+1}+\mathbf{P}_{0,2j+2}. & j=r+2,\dots,n.\end{array}\right. \tag{41}$$

Theorem 1 *For the G^1 condition (24), the boundary control points of two G^1 adjacent bicubic patches have to satisfy the equations (33) and (34)in case $c_i\neq 0, i=0,\dots,n$.*

4.2. G^1 conditions for the case of two pair interior double knots

If $n=3$, substitution (16),$\widehat{\mathbf{Q}}_j$ in (9) and (17) into (25) and (26) for $\widehat{\mathbf{P}}_j$, $\widehat{\mathbf{Q}}_j$ and $\widehat{\mathbf{T}}_j$ yields

$$\left\{\begin{array}{l}\bar{\mathbf{Q}}_0=b_0\mathbf{T}_0,\\ 3\bar{\mathbf{Q}}_1=2b_0\mathbf{T}_1+c_0\mathbf{T}_0,\\ 3\bar{\mathbf{Q}}_2=\dfrac{1}{2}b_0\mathbf{T}_2+2c_0\mathbf{T}_1,\end{array}\right. \tag{42}$$

$$\left\{\begin{array}{l}\bar{\mathbf{Q}}_3+\bar{\mathbf{Q}}_2=c_0\mathbf{T}_2,\\ 3\bar{\mathbf{Q}}_3=c_0\mathbf{T}_3+\dfrac{1}{2}c_1\mathbf{T}_2,\\ 3(\bar{\mathbf{Q}}_4+\bar{\mathbf{Q}}_3)=\dfrac{1}{2}c_0(\mathbf{T}_4+\mathbf{T}_3)+2c_1\mathbf{T}_3,\end{array}\right. \tag{43}$$

$$\left\{\begin{array}{l}(\bar{\mathbf{Q}}_5+2\bar{\mathbf{Q}}_4+\bar{\mathbf{Q}}_3)=c_1(\mathbf{T}_4+\mathbf{T}_3),\\ 3(\bar{\mathbf{Q}}_5+\bar{\mathbf{Q}}_4)=2c_1\mathbf{T}_4+\dfrac{1}{2}c_2(\mathbf{T}_4+\mathbf{T}_3),\\ 3\bar{\mathbf{Q}}_5=\dfrac{1}{2}c_1\mathbf{T}_5+c_2\mathbf{T}_4,\end{array}\right. \tag{44}$$

$$\left\{\begin{array}{l}\bar{\mathbf{Q}}_6+\bar{\mathbf{Q}}_5=\dfrac{1}{2}c_2\mathbf{T}_5,\\ 3\bar{\mathbf{Q}}_6=2c_2\mathbf{T}_6+\dfrac{1}{2}c_3\mathbf{T}_5,\\ 3\bar{\mathbf{Q}}_7=c_2\mathbf{T}_7+2c_3\mathbf{T}_6,\\ \bar{\mathbf{Q}}_8=c_3\mathbf{T}_7.\end{array}\right. \tag{45}$$

From (42,3),(43,1)and (43,2) we have

$$c_0+c_2=2c_1. \tag{46}$$

Therefore, (49)–(52) equal

$$\left\{\begin{array}{l}
\bar{\mathbf{Q}}_0 = b_0\mathbf{T}_0,\\
3\bar{\mathbf{Q}}_1 = 2b_0\mathbf{T}_1 + c_0\mathbf{T}_0,\\
3\bar{\mathbf{Q}}_2 = \frac{1}{2}b_0\mathbf{T}_2 + 2c_0\mathbf{T}_1,\\
3\bar{\mathbf{Q}}_3 = c_0\mathbf{T}_3 + \frac{1}{2}c_1\mathbf{T}_2,\\
3\bar{\mathbf{Q}}_4 = \frac{1}{2}c_0\mathbf{T}_4 - (2c_1 - \frac{1}{2}c_0)\mathbf{T}_3 - \frac{1}{2}c_1\mathbf{T}_2,\\
3\bar{\mathbf{Q}}_5 = \frac{1}{2}c_1\mathbf{T}_5 + c_2\mathbf{T}_4,\\
3\bar{\mathbf{Q}}_6 = 2c_2\mathbf{T}_6 + \frac{1}{2}C_3\mathbf{T}_5,\\
3\bar{\mathbf{Q}}_7 = C_2\mathbf{T}_7 + 2c_3\mathbf{T}_6,\\
\bar{\mathbf{Q}}_8 = c_3\mathbf{T}_7.
\end{array}\right. \tag{47}$$

and

$$\left\{\begin{array}{l}
c_0\mathbf{T}_3 - (3c_0 - \frac{1}{2}(c_1 + b_0))\mathbf{T}_2 + 2c_0 = 0,\\
\frac{1}{2}c_1\mathbf{T}_5 - c_1\mathbf{T}_4 + c_1\mathbf{T}_3 - \frac{1}{2}c_1\mathbf{T}_2 = 0,\\
2c_2\mathbf{T}_6 - (3c_2 - \frac{1}{2}(c_1 + c_3))\mathbf{T}_5 + c_2\mathbf{T}_4 = 0.
\end{array}\right. \tag{48}$$

In (48), let

$$\left\{\begin{array}{l}
3c_0 - \frac{1}{2}(b_0 + c_1) = k_0c_0,\\
3c_2 - \frac{1}{2}(c_1 + c_3) = k_2c_2,
\end{array}\right. \tag{49}$$

where k_0, k_2 are constants. From (48) and $\mathbf{T}_j = \mathbf{P}_{0,j+1} + \mathbf{P}_{0,j}$, we have

$$\left\{\begin{array}{l}
\mathbf{P}_{0,4} - (1 + k_0)\mathbf{P}_{0,3} + (2 + k_0)\mathbf{P}_{0,2} - 2\mathbf{P}_{0,1} = 0,\\
\mathbf{P}_{0,6} - 3\mathbf{P}_{0,5} + 4\mathbf{P}_{0,4} - 3\mathbf{P}_{0,3} + \mathbf{P}_{0,2} = 0,\\
2\mathbf{P}_{0,7} - (2 + k_2)\mathbf{P}_{0,6} + (1 + k_2)\mathbf{P}_{0,5} - \mathbf{P}_{0,4} = 0.
\end{array}\right. \tag{50}$$

We obtain from (50)

$$\left\{\begin{array}{l}
\mathbf{P}_{0,3} = \frac{-2(1+4k_2)\mathbf{P}_{0,1}+(1+k_0+7k_2+4k_0k_2)\mathbf{P}_{0,2}+(5+2k_2)\mathbf{P}_{0,6}-6\mathbf{P}_7}{-2+k_2+k_0(1+4k_2)},\\
\mathbf{P}_{0,4} = \frac{-6(1+k_2)\mathbf{P}_{0,1}-(5+k2_0)(1+k_2)\mathbf{P}_{0,2}+(1+k_0)(5+2k_2)\mathbf{P}_{0,6}-6(1+k_0)\mathbf{P}_{0,7}}{-2+k_2+k_0(1+4k_2)},\\
\mathbf{P}_{0,5} = \frac{-6\mathbf{P}_{0,1}-(5+2k_0)\mathbf{P}_{0,2}+(1+7k_0+k_2+4k_0k_2)\mathbf{P}_{0,6}-(2+8k_0)\mathbf{P}_{0,7}}{-2+k_2+k_0(1+4k_2)}.
\end{array}\right. \tag{51}$$

If $n = 4$, substitution (18), $\widehat{\mathbf{Q}}_j$ in (9) and (19) into (25) and (26) for $\widehat{\mathbf{P}}_j$, $\widehat{\mathbf{Q}}_j$ and $\widehat{\mathbf{T}}_j$ yields

$$\left\{\begin{array}{l}
\bar{\mathbf{Q}}_0 = b_0\mathbf{T}_0,\\
3\bar{\mathbf{Q}}_1 = 2b_0\mathbf{T}_1 + c_0\mathbf{T}_0,\\
3\bar{\mathbf{Q}}_2 = \frac{1}{2}b_0\mathbf{T}_2 + 2c_0\mathbf{T}_1,
\end{array}\right. \tag{52}$$

$$\left\{\begin{array}{l}
\bar{\mathbf{Q}}_3 + \bar{\mathbf{Q}}_2 = c_0\mathbf{T}_2,\\
3\bar{\mathbf{Q}}_3 = c_0\mathbf{T}_3 + \frac{1}{2}c_1\mathbf{T}_2,\\
3(\bar{\mathbf{Q}}_4 + \bar{\mathbf{Q}}_3) = 2c_0(\frac{1}{6}\mathbf{T}_4 + \frac{1}{4}\mathbf{T}_3) + 2c_1\mathbf{T}_3,
\end{array}\right. \tag{53}$$

$$\left\{\begin{array}{l}\frac{1}{12}(2\bar{\mathbf{Q}}_5+7\bar{\mathbf{Q}}_4+3\bar{\mathbf{Q}}_3)=c_1(\frac{1}{6}\mathbf{T}_4+\frac{1}{4}\mathbf{T}_3),\\ \bar{\mathbf{Q}}_5+2\bar{\mathbf{Q}}_4=\frac{2}{3}c_1\mathbf{T}_4+c_2(\frac{1}{6}\mathbf{T}_4+\frac{1}{4}\mathbf{T}_3),\\ 2\bar{\mathbf{Q}}_5+\bar{\mathbf{Q}}_4=c_1(\frac{1}{4}\mathbf{T}_5+\frac{1}{6}\mathbf{T}_4)+\frac{2}{3}c_2\mathbf{T}_4,\end{array}\right. \tag{54}$$

$$\left\{\begin{array}{l}\frac{1}{12}(3\bar{\mathbf{Q}}_6+7\bar{\mathbf{Q}}_5+2\bar{\mathbf{Q}}_4)=c_2(\frac{1}{4}\mathbf{T}_5+\frac{1}{6}\mathbf{T}_4),\\ 3(\bar{\mathbf{Q}}_6+\bar{\mathbf{Q}}_5)=2c_2\mathbf{T}_5+2c_3(\frac{1}{4}\mathbf{T}_5+\frac{1}{6}\mathbf{T}_4),\\ 3\bar{\mathbf{Q}}_6=\frac{1}{2}c_2\mathbf{T}_6+c_3\mathbf{T}_5,\end{array}\right. \tag{55}$$

$$\left\{\begin{array}{l}\bar{\mathbf{Q}}_7+\bar{\mathbf{Q}}_6=\frac{1}{2}c_2\mathbf{T}_6,\\ 3\bar{\mathbf{Q}}_7=2c_3\mathbf{T}_7+\frac{1}{2}c_4\mathbf{T}_6,\\ 3\bar{\mathbf{Q}}_8=c_3\mathbf{T}_8++2c_4\mathbf{T}_7,\\ \bar{\mathbf{Q}}_9=c_4\mathbf{T}_8.\end{array}\right. \tag{56}$$

From (53,3),(54),(55,1) and (55,2) we have

$$\left\{\begin{array}{l}c_0+c_2=2c_1,\\ c_1+c_3=2c_2.\end{array}\right. \tag{57}$$

Therefore, (53)–(56) equal

$$\left\{\begin{array}{l}\bar{\mathbf{Q}}_0=b_0\mathbf{T}_0,\\ 3\bar{\mathbf{Q}}_1=2b_0\mathbf{T}_1+c_0\mathbf{T}_0,\\ 3\bar{\mathbf{Q}}_2=\frac{1}{2}b_0\mathbf{T}_2+2c_0\mathbf{T}_1,\\ 3\bar{\mathbf{Q}}_3=c_0\mathbf{T}_3+\frac{1}{2}c_1\mathbf{T}_2,\\ 3\bar{\mathbf{Q}}_4=\frac{1}{3}c_0\mathbf{T}_4+(2c_1-\frac{1}{2}c_0)\mathbf{T}_3-\frac{1}{2}c_1\mathbf{T}_2,\\ 3\bar{\mathbf{Q}}_5=\frac{1}{2}c_1\mathbf{T}_5+(\frac{7}{6}c_2-\frac{1}{3}c_1)\mathbf{T}_4-\frac{1}{4}c_2\mathbf{T}_3,\\ 3\bar{\mathbf{Q}}_6=\frac{1}{2}c_2\mathbf{T}_6+C_3\mathbf{T}_5,\\ 3\bar{\mathbf{Q}}_7=2C_3\mathbf{T}_7+\frac{1}{2}c_4\mathbf{T}_6,\\ 3\bar{\mathbf{Q}}_8=c_3\mathbf{T}_8+2c_4\mathbf{T}_7,\\ \bar{\mathbf{Q}}_9=c_4\mathbf{T}_8,\end{array}\right. \tag{58}$$

and

$$\left\{\begin{array}{l}c_0\mathbf{T}_3-(3c_0-\frac{1}{2}(c_1+b_0))\mathbf{T}_2+2c_0=0,\\ \frac{1}{4}c_1\mathbf{T}_5-\frac{1}{2}c_1\mathbf{T}_4+c_1\mathbf{T}_3-\frac{1}{2}c_1\mathbf{T}_2=0,\\ \frac{1}{2}c_2\mathbf{T}_6-c_2\mathbf{T}_5+\frac{1}{2}c_2\mathbf{T}_4-\frac{1}{4}c_2\mathbf{T}_3=0,\\ 2c_3\mathbf{T}_7-(3c_3-\frac{1}{2}(c_2+c_4))\mathbf{T}_6+c_3\mathbf{T}_5=0.\end{array}\right. \tag{59}$$

In (59), let

$$\left\{\begin{array}{l}3c_0-\frac{1}{2}(b_0+c_1)=k_0c_0,\\ 3c_3-\frac{1}{2}(c_2+c_4)=k_3c_3,\end{array}\right. \tag{60}$$

where k_0, k_3 are constants. From (60) and $\mathbf{T}_j = \mathbf{P}_{0,j+1} + \mathbf{P}_{0,j}$, we have

$$\left\{\begin{array}{l} \mathbf{P}_{0,4} - (1+k_0)\mathbf{P}_{0,3} + (2+k_0)\mathbf{P}_{0,2} - 2\mathbf{P}_{0,1} = 0, \\ \mathbf{P}_{0,6} - 3\mathbf{P}_{0,5} + 6\mathbf{P}_{0,4} - 4\mathbf{P}_{0,3} + 2\mathbf{P}_{0,2} = 0, \\ 2\mathbf{P}_{0,7} - 6\mathbf{P}_{0,6} + 6\mathbf{P}_{0,5} - 3\mathbf{P}_{0,4} + \mathbf{P}_{0,3} = 0, \\ 2\mathbf{P}_{0,8} - (2+k_3)\mathbf{P}_{0,7} + (1+k_3)\mathbf{P}_{0,6} - \mathbf{P}_{0,5} = 0. \end{array}\right. \tag{61}$$

We obtain from (61)

$$\left\{\begin{array}{l} \mathbf{P}_{0,3} = \frac{(12-54k_3)\mathbf{P}_{0,1}+3(-4+14k_3+k_0(-2+9k_3))\mathbf{P}_{0,2}+(20+6k_3)\mathbf{P}_{0,7}-24\mathbf{P}_{0,8}}{-4+6k_3+k_0(-6+27k_3)}, \\ \mathbf{P}_{0,4} = \frac{(4-42k_3)\mathbf{P}_{0,1}+(-4+30k_3+k_0(-2+9k_3))\mathbf{P}_{0,2}+2(1+k_0)(10+3k_3)\mathbf{P}_{0,7}-24(1+k_0)\mathbf{P}_{0,8}}{-4+6k_3+k_0(-6+27k_3)}, \\ \mathbf{P}_{0,5} = \frac{-12(1+k_3)\mathbf{P}_{0,1}-8(1+k_3)\mathbf{P}_{0,2}+(16+54k_0+6k_3+21k_0k_3)\mathbf{P}_{0,7}-(20+66k_0)\mathbf{P}_{0,8}}{-4+6k_3+k_0(-6+27k_3)}, \\ \mathbf{P}_{0,6} = \frac{-12\mathbf{P}_{0,1}+8\mathbf{P}_{0,2}+(8+42k_0+6k_3+27k_0k_3)\mathbf{P}_{0,7}-(12+54k_0)\mathbf{P}_{0,8}}{-4+6k_3+k_0(-6+27k_3)}. \end{array}\right. \tag{62}$$

If $n = 5$, substitution (20), $\widehat{\mathbf{Q}}_j$ in (9) and (21) into (25) and (26) for $\widehat{\mathbf{P}}_j$, $\widehat{\mathbf{Q}}_j$ and $\widehat{\mathbf{T}}_j$ yields

$$\left\{\begin{array}{l} \bar{\mathbf{Q}}_0 = b_0\mathbf{T}_0, \\ 3\bar{\mathbf{Q}}_1 = 2b_0\mathbf{T}_1 + c_0\mathbf{T}_0, \\ 3\bar{\mathbf{Q}}_2 = \frac{1}{2}b_0\mathbf{T}_2 + 2c_0\mathbf{T}_1, \end{array}\right. \tag{63}$$

$$\left\{\begin{array}{l} \bar{\mathbf{Q}}_3 + \bar{\mathbf{Q}}_2 = c_0\mathbf{T}_2, \\ 3\bar{\mathbf{Q}}_3 = c_0\mathbf{T}_3 + \frac{1}{2}c_1\mathbf{T}_2, \\ 3(\bar{\mathbf{Q}}_4 + \bar{\mathbf{Q}}_3) = 2c_0(\frac{1}{6}\mathbf{T}_4 + \frac{1}{4}\mathbf{T}_3) + 2c_1\mathbf{T}_3, \end{array}\right. \tag{64}$$

$$\left\{\begin{array}{l} \frac{1}{12}(2\bar{\mathbf{Q}}_5 + 7\bar{\mathbf{Q}}_4 + 3\bar{\mathbf{Q}}_3) = c_1(\frac{1}{6}\mathbf{T}_4 + \frac{1}{4}\mathbf{T}_3), \\ \bar{\mathbf{Q}}_5 + 2\bar{\mathbf{Q}}_4 = \frac{2}{3}c_1\mathbf{T}_4 + c_2(\frac{1}{6}\mathbf{T}_4 + \frac{1}{4}\mathbf{T}_3), \\ 2\bar{\mathbf{Q}}_5 + \bar{\mathbf{Q}}_4 = \frac{1}{6}c_1(\mathbf{T}_5 + \mathbf{T}_4) + \frac{2}{3}c_2\mathbf{T}_4, \end{array}\right. \tag{65}$$

$$\left\{\begin{array}{l} (\bar{\mathbf{Q}}_6 + 4\bar{\mathbf{Q}}_5 + \bar{\mathbf{Q}}_4) = c_2(\mathbf{T}_5 + \mathbf{T}_4), \\ \bar{\mathbf{Q}}_6 + 2\bar{\mathbf{Q}}_5 = \frac{2}{3}c_2\mathbf{T}_5 + \frac{1}{6}c_3(\mathbf{T}_5 + \mathbf{T}_4), \\ 2\bar{\mathbf{Q}}_6 + \bar{\mathbf{Q}}_5 = c_2(\frac{1}{4}\mathbf{T}_6 + \frac{1}{6}\mathbf{T}_5) + \frac{2}{3}c_3\mathbf{T}_5, \end{array}\right. \tag{66}$$

$$\left\{\begin{array}{l} \frac{1}{12}(3\bar{\mathbf{Q}}_7 + 7\bar{\mathbf{Q}}_6 + 2\bar{\mathbf{Q}}_5) = c_3(\frac{1}{4}\mathbf{T}_6 + \frac{1}{6}\mathbf{T}_5), \\ \frac{3}{2}(\bar{\mathbf{Q}}_7 + \bar{\mathbf{Q}}_6) = c_3\mathbf{T}_6 + c_4(\frac{1}{4}\mathbf{T}_6 + \frac{1}{6}\mathbf{T}_4), \\ 3\bar{\mathbf{Q}}_7 = \frac{1}{2}c_3\mathbf{T}_7 + c_4\mathbf{T}_6, \end{array}\right. \tag{67}$$

$$\left\{\begin{array}{l} \bar{\mathbf{Q}}_8 + \bar{\mathbf{Q}}_7 = c_4\mathbf{T}_7, \\ 3\bar{\mathbf{Q}}_8 = 2c_4\mathbf{T}_8 + \frac{1}{2}c_5\mathbf{T}_7, \\ 3\bar{\mathbf{Q}}_9 = c_4\mathbf{T}_9 + +2c_5\mathbf{T}_8, \\ \bar{\mathbf{Q}}_{10} = c_5\mathbf{T}_9. \end{array}\right. \tag{68}$$

From (64,3), (65), (66), (67,1) and (67,2) we have

$$\left\{\begin{array}{l} c_0 + c_2 = 2c_1, \\ c_1 + c_3 = 2c_2, \\ c_2 + c_4 = 2c_3. \end{array}\right. \tag{69}$$

Therefore, (63)–(68) equal

$$\begin{cases} \bar{\mathbf{Q}}_0 = b_0\mathbf{T}_0, \\ 3\bar{\mathbf{Q}}_1 = 2b_0\mathbf{T}_1 + c_0\mathbf{T}_0, \\ 3\bar{\mathbf{Q}}_2 = \frac{1}{2}b_0\mathbf{T}_2 + 2c_0\mathbf{T}_1, \\ 3\bar{\mathbf{Q}}_3 = c_0\mathbf{T}_3 + \frac{1}{2}c_1\mathbf{T}_2, \\ 3\bar{\mathbf{Q}}_4 = \frac{1}{3}c_0\mathbf{T}_4 + (2c_1 - \frac{1}{2}c_0)\mathbf{T}_3 - \frac{1}{2}c_1\mathbf{T}_2, \\ 3\bar{\mathbf{Q}}_5 = \frac{1}{3}c_1\mathbf{T}_5 + (\frac{7}{6}c_2 - \frac{1}{3}c_1)\mathbf{T}_4 - \frac{1}{4}c_2\mathbf{T}_3, \\ 3\bar{\mathbf{Q}}_6 = \frac{1}{2}c_2\mathbf{T}_6 + (\frac{7}{6}c_3 - \frac{1}{3}c_2)\mathbf{T}_5 - \frac{1}{6}c_3\mathbf{T}_4, \\ 3\bar{\mathbf{Q}}_7 = \frac{1}{2}c_3\mathbf{T}_7 + c_4\mathbf{T}_6, \\ 3\bar{\mathbf{Q}}_8 = 2c_4\mathbf{T}_8 + \frac{1}{2}c_5\mathbf{T}_7, \\ 3\bar{\mathbf{Q}}_9 = c_4\mathbf{T}_9 + 2c_5\mathbf{T}_8, \\ \bar{\mathbf{Q}}_{10} = c_5\mathbf{T}_9, \end{cases} \tag{70}$$

and

$$\begin{cases} c_0\mathbf{T}_3 - (3c_0 - \frac{1}{2}(c_1 + b_0))\mathbf{T}_2 + 2c_0\mathbf{T}_1 = 0, \\ \frac{1}{6}c_1\mathbf{T}_5 - \frac{1}{2}c_1\mathbf{T}_4 + c_1\mathbf{T}_3 - \frac{1}{2}c_1\mathbf{T}_2 = 0, \\ \frac{1}{4}c_2\mathbf{T}_6 - \frac{1}{2}c_2\mathbf{T}_5 + \frac{1}{2}c_2\mathbf{T}_4 - \frac{1}{4}c_2\mathbf{T}_3 = 0, \\ \frac{1}{2}c_3\mathbf{T}_7 - c_3\mathbf{T}_6 + \frac{1}{2}c_3\mathbf{T}_5 - \frac{1}{6}c_3\mathbf{T}_4 = 0, \\ 2c_4\mathbf{T}_8 - (3c_4 - \frac{1}{2}(c_3 + c_5))\mathbf{T}_7 + c_4\mathbf{T}_6 = 0. \end{cases} \tag{71}$$

In (70), let

$$\begin{cases} 3c_0 - \frac{1}{2}(b_0 + c_1) = k_0c_0, \\ 3c_4 - \frac{1}{2}(c_3 + c_5) = k_4c_4, \end{cases} \tag{72}$$

where k_0, k_4 are constants. From (70) and $\mathbf{T}_j = \mathbf{P}_{0,j+1} + \mathbf{P}_{0,j}$, we have

$$\begin{cases} \mathbf{P}_{0,4} - (1 + k_0)\mathbf{P}_{0,3} + (2 + k_0)\mathbf{P}_{0,2} - 2\mathbf{P}_{0,1} = 0, \\ \mathbf{P}_{0,6} - 4\mathbf{P}_{0,5} + 9\mathbf{P}_{0,4} - 9\mathbf{P}_{0,3} + 3\mathbf{P}_{0,2} = 0, \\ \mathbf{P}_{0,7} - 3\mathbf{P}_{0,6} + 4\mathbf{P}_{0,5} - 3\mathbf{P}_{0,4} + \mathbf{P}_{0,3} = 0, \\ 3\mathbf{P}_{0,8} - 9\mathbf{P}_{0,7} + 9\mathbf{P}_{0,6} - 4\mathbf{P}_{0,5} + \mathbf{P}_{0,4} = 0, \\ 2\mathbf{P}_{0,9} - (2 + k_4)\mathbf{P}_{0,8} + (1 + k_4)\mathbf{P}_{0,7} - \mathbf{P}_{0,6} = 0. \end{cases} \tag{73}$$

We obtain from (73)

$$\begin{cases} \mathbf{P}_{0,3} = \frac{-4(-7+16k_4)\mathbf{P}_{0,1}+2(-11+23k_4+k_0(-7+16k_4))\mathbf{P}_{0,2}+(19+8k_4)\mathbf{P}_{0,8}-20\mathbf{P}_{0,9}}{3-14k_4+2k_0(-7+16k_4)}, \\ \mathbf{P}_{0,4} = \frac{(34-92k_4)\mathbf{P}_{0,1}+(-28+74k_4+k_0(-11+28k_4))\mathbf{P}_{0,2}-(1+k_0)(19+8k_4)\mathbf{P}_{0,8}-20(1+k_0)\mathbf{P}_{0,9}}{3-14k_4+2k_0(-7+16k_4)}, \\ \mathbf{P}_{0,5} = \frac{(34-272k_4)\mathbf{P}_{0,1}+(-28+227k_4+k_0(-11+64k_4))\mathbf{P}_{0,2}+(-32+257k_0-15k_4+112k_0k_4)\mathbf{P}_{0,8}+(34-272k_0)\mathbf{P}_{0,9}}{3-14k_4+2k_0(-7+16k_4)}, \\ \mathbf{P}_{0,6} = \frac{-20(1+k_4)\mathbf{P}_{0,1}+(17+4k_0)(1+k_4)\mathbf{P}_{0,2}+(-32+86k_0-15k_4+40k_0k_4))\mathbf{P}_{0,8}+(34-92k_0)\mathbf{P}_{0,9}}{3-14k_4+2k_0(-7+16k_4)}, \\ \mathbf{P}_{0,7} = \frac{-20\mathbf{P}_{0,1}+(17+4k_4)\mathbf{P}_{0,2}+2((-13-7k_4+k_0(29+16k_4))\mathbf{P}_{0,8}+2(7-16k_0)\mathbf{P}_{0,9}}{3-14k_4+2k_0(-7+16k_4)}. \end{cases} \tag{74}$$

If $n \geq 6$, substitution (20),$\widehat{\mathbf{Q}}_j$ in (??) and (21) into (25) and (26) for $\widehat{\mathbf{P}}_j$, $\widehat{\mathbf{Q}}_j$ and $\widehat{\mathbf{T}}_j$ yields

$$\begin{cases} \bar{\mathbf{Q}}_0 = b_0\mathbf{T}_0, \\ 3\bar{\mathbf{Q}}_1 = 2b_0\mathbf{T}_1 + c_0\mathbf{T}_0, \\ 3\bar{\mathbf{Q}}_2 = \frac{1}{2}b_0\mathbf{T}_2 + 2c_0\mathbf{T}_1, \end{cases} \tag{75}$$

$$\begin{cases} \bar{\mathbf{Q}}_3 + \bar{\mathbf{Q}}_2 = c_0\mathbf{T}_2, \\ 3\bar{\mathbf{Q}}_3 = c_0\mathbf{T}_3 + \frac{1}{2}c_1\mathbf{T}_2, \\ 3(\bar{\mathbf{Q}}_4 + \bar{\mathbf{Q}}_3) = 2c_0(\frac{1}{6}\mathbf{T}_4 + \frac{1}{4}\mathbf{T}_3) + 2c_1\mathbf{T}_3, \end{cases} \tag{76}$$

$$\begin{cases} \frac{1}{12}(2\bar{\mathbf{Q}}_5 + 7\bar{\mathbf{Q}}_4 + 3\bar{\mathbf{Q}}_3) = c_1(\frac{1}{6}\mathbf{T}_4 + \frac{1}{4}\mathbf{T}_3), \\ \bar{\mathbf{Q}}_5 + 2\bar{\mathbf{Q}}_4 = \frac{2}{3}c_1\mathbf{T}_4 + c_2(\frac{1}{6}\mathbf{T}_4 + \frac{1}{4}\mathbf{T}_3), \\ 2\bar{\mathbf{Q}}_5 + \bar{\mathbf{Q}}_4 = \frac{1}{6}c_1(\mathbf{T}_5 + \mathbf{T}_4) + \frac{2}{3}c_2\mathbf{T}_4, \end{cases} \tag{77}$$

$$\begin{cases} \bar{\mathbf{Q}}_{j+3} + 4\bar{\mathbf{Q}}_{j+2} + \bar{\mathbf{Q}}_{j+1} = c_{j-1}(\mathbf{T}_{j+2} + \mathbf{T}_{j+1}), \\ \bar{\mathbf{Q}}_{j+3} + 2\bar{\mathbf{Q}}_{j+2} = \frac{2}{3}c_{j-1}\mathbf{T}_{j+2} + \frac{1}{6}c_j(\mathbf{T}_{j+2} + \mathbf{T}_{j+1}), \quad j = 3, \ldots, n-3 \\ 2\bar{\mathbf{Q}}_{j+3} + \bar{\mathbf{Q}}_{j+2} = \frac{1}{6}c_{j-1}(\mathbf{T}_{j+3} + \mathbf{T}_{j+2}) + \frac{2}{3}c_j\mathbf{T}_{j+2}, \end{cases} \tag{78}$$

$$\begin{cases} \bar{\mathbf{Q}}_{n+1} + 4\bar{\mathbf{Q}}_n + \bar{\mathbf{Q}}_{n-1} = c_{n-3}(\mathbf{T}_n + \mathbf{T}_{n-1}), \\ \bar{\mathbf{Q}}_{n+1} + 2\bar{\mathbf{Q}}_n = \frac{2}{3}c_{n-3}\mathbf{T}_n + \frac{1}{6}c_{n-2}(\mathbf{T}_n + \mathbf{T}_{n-1}), \\ 2\bar{\mathbf{Q}}_{n+1} + \bar{\mathbf{Q}}_n = c_{n-3}(\frac{1}{4}\mathbf{T}_{n+1} + \frac{1}{6}\mathbf{T}_n) + \frac{2}{3}c_{n-2}\mathbf{T}_n, \end{cases} \tag{79}$$

$$\begin{cases} \frac{1}{12}(3\bar{\mathbf{Q}}_{n+2} + 7\bar{\mathbf{Q}}_{n+1} + 2\bar{\mathbf{Q}}_n) = c_{n-2}(\frac{1}{4}\mathbf{T}_{n+1} + \frac{1}{6}\mathbf{T}_n), \\ \frac{3}{2}(\bar{\mathbf{Q}}_{n+2} + \bar{\mathbf{Q}}_{n+1}) = c_{n-2}\mathbf{T}_{n+1} + c_{n-1}(\frac{1}{4}\mathbf{T}_{n+1} + \frac{1}{6}\mathbf{T}_n), \\ 3\bar{\mathbf{Q}}_{n+2} = \frac{1}{2}c_{n-2}\mathbf{T}_{n+2} + c_{n-1}\mathbf{T}_{n+1}, \end{cases} \tag{80}$$

$$\begin{cases} \bar{\mathbf{Q}}_{n+3} + \bar{\mathbf{Q}}_{n+2} = c_{n-1}\mathbf{T}_{n+2}, \\ 3\bar{\mathbf{Q}}_{n+3} = 2c_{n-1}\mathbf{T}_{n+3} + \frac{1}{2}c_n\mathbf{T}_{n+2}, \\ 3\bar{\mathbf{Q}}_{n+4} = c_{n-1}\mathbf{T}_{n+4} + 2c_n\mathbf{T}_{n+3}, \\ \bar{\mathbf{Q}}_{n+5} = c_n\mathbf{T}_{n+4}. \end{cases} \tag{81}$$

From (76,3), (77), (78), (79), (80,1) and (80,2) we have

$$c_{j-2} + c_j = 2c_{j-1}. \tag{82}$$

Therefore, (75)-(81) equal

$$\left\{\begin{array}{l}\bar{\mathbf{Q}}_0=b_0\mathbf{T}_0,\\ 3\bar{\mathbf{Q}}_1=2b_0\mathbf{T}_1+c_0\mathbf{T}_0,\\ 3\bar{\mathbf{Q}}_2=\frac{1}{2}b_0\mathbf{T}_2+2c_0\mathbf{T}_1,\\ 3\bar{\mathbf{Q}}_3=c_0\mathbf{T}_3+\frac{1}{2}c_1\mathbf{T}_2,\\ 3\bar{\mathbf{Q}}_4=\frac{1}{3}c_0\mathbf{T}_4+(2c_1-\frac{1}{2}c_0)\mathbf{T}_3-\frac{1}{2}c_1\mathbf{T}_2,\\ 3\bar{\mathbf{Q}}_5=\frac{1}{3}c_1\mathbf{T}_5+(\frac{7}{6}c_2-\frac{1}{3}c_1)\mathbf{T}_4-\frac{1}{4}c_2\mathbf{T}_3,\\ 3\bar{\mathbf{Q}}_{j+3}=\frac{1}{3}c_{j-1}\mathbf{T}_{j+3}+(\frac{7}{6}c_j-\frac{1}{3}c_{j-1})\mathbf{T}_{j+2}-\frac{1}{6}c_j\mathbf{T}_{j+1},\\ 3\bar{\mathbf{Q}}_{n+1}=\frac{1}{2}c_{n-3}\mathbf{T}_{n+1}+(\frac{7}{6}C_{n-2}-\frac{1}{3}c_{n-3})\mathbf{T}_n-\frac{1}{6}c_{n-2}\mathbf{T}_{n-1},\\ 3\bar{\mathbf{Q}}_{n+2}=\frac{1}{2}C_{n-2}\mathbf{T}_{n+2}+c_{n-1}\mathbf{T}_{n+1},\\ 3\bar{\mathbf{Q}}_{n+3}=2c_{n-1}\mathbf{T}_{n+3}+\frac{1}{2}c_n\mathbf{T}_{n+2},\\ 3\bar{\mathbf{Q}}_{n+4}=c_{n-1}\mathbf{T}_{n+4}+2c_n\mathbf{T}_{n+3},\\ \bar{\mathbf{Q}}_{n+5}=c_n\mathbf{T}_{n+4},\end{array}\right. \tag{83}$$

and

$$\left\{\begin{array}{l}c_0\mathbf{T}_3-(3c_0-\frac{1}{2}(c_1+b_0))\mathbf{T}_2+2c_0\mathbf{T}_1=0,\\ \frac{1}{6}c_1\mathbf{T}_5-\frac{1}{2}c_1\mathbf{T}_4+c_1\mathbf{T}_3-\frac{1}{2}c_1\mathbf{T}_2=0,\\ \frac{1}{6}c_2\mathbf{T}_6-\frac{1}{2}c_2\mathbf{T}_5+\frac{1}{2}c_2\mathbf{T}_4-\frac{1}{4}c_2\mathbf{T}_3=0,\\ \frac{1}{6}c_{j-1}\mathbf{T}_{j+3}-\frac{1}{2}c_{j-1}\mathbf{T}_{j+2}+\frac{1}{2}c_{j-1}\mathbf{T}_{j+1}-\frac{1}{6}c_{j-1}\mathbf{T}_j=0,\quad j=4,\ldots,n-3\\ \frac{1}{4}c_{n-3}\mathbf{T}_{n+1}-\frac{1}{2}c_{n-3}\mathbf{T}_n+\frac{1}{2}c_{n-3}\mathbf{T}_{n-1}-\frac{1}{6}c_{n-3}\mathbf{T}_{n-2}=0,\\ \frac{1}{2}c_{n-2}\mathbf{T}_{n+2}-c_{n-2}\mathbf{T}_{n+1}+\frac{1}{2}c_{n-2}\mathbf{T}_n-\frac{1}{6}c_{n-2}\mathbf{T}_{n-1}=0,\\ 2c_{n-1}\mathbf{T}_{n+3}-(3c_{n-1}-\frac{1}{2}(c_{n-2}+c_n))\mathbf{T}_{n+2}+c_{n-1}\mathbf{T}_{n+1}=0.\end{array}\right. \tag{84}$$

In (84), let

$$\left\{\begin{array}{l}3c_0-\frac{1}{2}(b_0+c_1)=k_0c_0,\\ 3c_{n-1}-\frac{1}{2}(c_{n-2}+c_n)=k_{n-1}c_{n-1},\end{array}\right. \tag{85}$$

where k_0, k_{n-1} are constants. From (85) and $\mathbf{T}_j=\mathbf{P}_{0,j+1}+\mathbf{P}_{0,j}$, we have

$$\left\{\begin{array}{l}\mathbf{P}_{0,4}-(1+k_0)\mathbf{P}_{0,3}+(2+k_0)\mathbf{P}_{0,2}-2\mathbf{P}_{0,1}=0,\\ \mathbf{P}_{0,6}-4\mathbf{P}_{0,5}+9\mathbf{P}_{0,4}-9\mathbf{P}_{0,3}+3\mathbf{P}_{0,2}=0,\\ 2\mathbf{P}_{0,7}-8\mathbf{P}_{0,6}+12\mathbf{P}_{0,5}-9\mathbf{P}_{0,4}+3\mathbf{P}_{0,3}=0,\\ \mathbf{P}_{0,j+4}-4\mathbf{P}_{0,j+3}+6\mathbf{P}_{0,j+2}-4\mathbf{P}_{0,j+1}+\mathbf{P}_{0,j}=0,\qquad j=4,\ldots,n-3\\ 3\mathbf{P}_{0,n+2}-9\mathbf{P}_{0,n+1}+12\mathbf{P}_{0,n}-8\mathbf{P}_{0,n-1}+2\mathbf{P}_{0,n-2}=0,\\ 3\mathbf{P}_{0,n+3}-9\mathbf{P}_{0,n+2}+9\mathbf{P}_{0,n+1}-4\mathbf{P}_{0,n}+\mathbf{P}_{0,n-1}=0,\\ 2\mathbf{P}_{0,n+4}-(2+k_{n-1})\mathbf{P}_{0,n+3}+(1+k_{n-1})\mathbf{P}_{0,n+2}-\mathbf{P}_{0,n+1}=0.\end{array}\right. \tag{86}$$

We obtain from (86)

$$
\left\{\begin{array}{l}
\mathbf{P}_{0,3} = \frac{6(23-36k_{n-1})\mathbf{P}_{0,1}+3(-31+47k_{n-1}+k_0(-23+36k_{n-1}))\mathbf{P}_{0,2}+(37-36k_{n-1})\mathbf{P}_{0,n+3}-42\mathbf{P}_{0,n+4}}{40-69k_{n-1}+3k_0(-23+36k_{n-1})}, \\
\mathbf{P}_{0,4} = \frac{(218-354k_{n-1})\mathbf{P}_{0,1}+(-173+279k_{n-1}+2k_0(-32+51k_{n-1}))\mathbf{P}_{0,2}-(1+k_0)(-37+36k_{n-1})\mathbf{P}_{0,n+3}+42\mathbf{P}_{0,n+4}}{40-69k_{n-1}+3k_0(-23+36k_{n-1})} \\
\mathbf{P}_{0,5} = \frac{(428-774k_{n-1})\mathbf{P}_{0,1}+(-358+649k_{n-1}+2k_0(-47+81k_{n-1}))\mathbf{P}_{0,2}+(-58+298k_0+55k_{n-1}-288k_0k_{n-1})\mathbf{P}_{0,n+3}}{80-138k_{n-1}+6k_0(-23+36k_{n-1})} \\
\qquad\qquad + \frac{(68-342k_0)\mathbf{P}_{0,n+4}}{80-138k_{n-1}+6k_0(-23+36k_{n-1})}, \\
\mathbf{P}_{0,j+4} = \frac{1}{6}(12+4j-15j^2-j^3)\mathbf{P}_{0,1} + (2+3j^2-j^3+\frac{1}{12}(12+8j-3j^2+6j^3)k_0)\mathbf{P}_{0,2} \\
\quad +(\frac{1}{6}(j-1)(-6+2j+5j^2+6j^2-9j-6)k_0)\mathbf{P}_{0,3} + \frac{1}{6}j(2+3j+j^2)\mathbf{P}_{0,5} \\
\qquad\qquad\qquad\qquad j = 2,\ldots,n-4 \\
\mathbf{P}_{0,n+1} = -\frac{1}{6}(132-193n+84n^2-11n^3)\mathbf{P}_{0,1} + \frac{1}{12}((-120+230n-114n^2+16n^3) \\
\quad +(-132+193n-84n^2+11n^3)k_0)\mathbf{P}_{0,2} + \frac{1}{12}((-120+134n-42n^2+4n^3) \\
\quad +(132-193n+84n^2-11n^3)k_0))\mathbf{P}_{0,3} + \frac{1}{6}(-6+11n-6n^2+n^3)\mathbf{P}_{0,5} \\
\mathbf{P}_{0,n+2} = \frac{1}{6}(52-103n+62n^2-11n^3)\mathbf{P}_{0,1} + (\frac{1}{6}(-16+55n-41n^2+8n^3) \\
\quad +\frac{1}{12}(-52+103n-62n^2-11n^3)k_0)\mathbf{P}_{0,2} + \frac{1}{12}(-56+86n-34n^2+4n^3 \\
\quad +(60-103n+62n^2-11n^3)k_0)\mathbf{P}_{0,3} + \frac{1}{12}(-12+10n-8n^2+2n^3)\mathbf{P}_{0,5}.
\end{array}\right.
\tag{87}
$$

5. Conclusion

In this paper, we obtain the conditions for G^1 continuity between two adjacent B-spline surface patches with double knots for two typical cases of the knot vectors. Using these conditions, we may achieve a local scheme of (true) G^1 continuity over an arbitrary B-spline surface network. In fact, the existence of a local scheme of constructing G^1 continuous B-spline surface models only requires two pairs of interior double knots which can be placed anywhere of the knot vectors.

References:

[1] DEROSE T. *Necessary and sufficient conditions for tangent plane continuity of Bézier surfaces* [J]. Computer Aided Geometric Design, 1990, **7**(1–4): 165–180.

[2] DEROSE T, BARSKY B A. *Geometric continuity, shape parameters, and geometric constructions for Catmull-Rom splines* [J]. ACM Transactions on Graphics, 1988, **7**(1): 1–41.

[3] FARIN G. *A construction for the visual C^1 continuity of polynomial surface patches* [J]. Computer Graphics and Image Processing, 1982, **20**: 272–282.

[4] HAHN J M. *Geometric continuous patch complexes* [J]. Computer Aided Geometric Design, 1989, **6**: 55–67.

[5] KAHMANN J. *Continuity of Curvature between Adjacent Bézier Patches* [M]. in: R.E. Barnhill and W. Boehm, eds., Surfaces in CAGD, North-Holland, Amsterdam, 1983, 65-75.

[6] LIU D, HOSCHEK J. *GC^1 continuity conditions between adjacent rectangular and triangular Bézier surface Patches* [J]. Computer Aided Design, 1989, **21**: 194–200.

[7] PETERS J. *Local smooth surface interpolation: a classification* [J]. Computer Aided Geometric Design, 1990, **7**: 191–195.

[8] PIEGL L, TILLER W. *The NURBS Book* [M], Springer, Second Edition, 1997.

[9] PIPER B. *Visually smooth interpolation with triangular Bézier patches* [J]. in G. Farin (ed.), Geometric Modeling: Algorithms and New Trends, SIAM, Philadelphia, 1987, 221-233.

[10] 施锡泉，赵岩. 双三次 B- 样条曲面的 G^1 连续条件 [J]. 计算机辅助设计与图形学学报， 2002, 7.
SHI Xi-quan, ZHAO Yan. *G^1 continuity conditions of bicubic B-sphine surfaces* [J]. Journal of Computer-Aided Design & Computer Graplics, 2002,7. (in Chinese)

[11] SHIRMAN Se-quin, SHIRMAN L, SÉQUIN C. *Local surface interpolation with shape parameters between adjoining Gregory patches* [J]. Computer Aided Geometric Design, 1990, **7**(5): 375–388.

[12] SHIRMAN L, SÉQUIN C. *Local surface interpolation with Bézier patches: errata and improvements* [J]. Computer Aided Geometric Design, 1991, **8**(3): 217–222.
[13] WATKINS M. *Problems in geometric continuity* [J]. Computer Aided Design, 1988, **20**: 499–502.
[14] YE X, LIANG Y, NOWACKI H. *Geometric continuity between adjacent Bézier patches and their constructions* [J]. Computer Aided Geometric Design, 1996, **13**: 521–548.

具有重结点的 B- 样条曲面 G^1 的连续条件

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摘要: 本文得到了关于两个双三次内部重结点 B- 样条曲面片 G^1 连续的充分必要条件和在公共边界线上控制向量的本征条件. 这些条件直接由两个 B- 样条曲面的控制向量表示. 文 [10] 证明了使用内部单结点的双三次 B- 样条曲面来构造 G^1 光滑曲面，局部格式不存在. 使用本文的这些条件就可以构造出具有局部各式的 G^1 光滑造型.

关键词: B- 样条曲面片；几何连续性；边界控制顶点的本征条件.