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# The Maximum Jump Number of $(0,1)$-Matrices of Order $2k-2$ with Fixed Row and Column Sum $k$

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**Abstract:** In 1992, Brualdi and Jung first introduced the maximum jump number $M(n,k)$, that is, the maximum number of the jumps of all $(0,1)$-matrices of order $n$ with $k$ 1's in each row and column, and then gave a table about the values of $M(n,k)$ when $1 \leq k \leq n \leq 10$. They also put forward several conjectures, including the conjecture $M(2k-2,k) = 3k-4+\lfloor\frac{k-2}{2}\rfloor$. In this paper, we prove that $b(A) \geq 4$ for every $A \in \Lambda(2k-2,k)$ if $k \geq 11$, and find another counter-example to this conjecture .

**Key words:** $(0,1)$-matrices; jump number; stair number.
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## 1. Introduction and Lemmas

Let $P$ be a finite poset (partially ordered set) and its cardinality $|P| = n$. Let $\mathbf{n}_{\leq}$ denote the $n$-element poset formed by the set $\{1, 2, \cdots, n\}$ with its usual order. Then an order-preserving bijective mapping $L$: $P \longrightarrow \mathbf{n}_{\leq}$ is called a linear extension of $P$ to a totally ordered set. If $P = \{x_i \mid 1 \leq i \leq n\}$, then we can simply express a linear extension $L$ by $x_1 - - x_2 - - \cdots - - x_n$ with the property $x_i < x_j$ in $P$ implies $i < j$.

A consecutive pair $(x_i, x_{i+1})$ is called a jump(or setup) of $P$ in $L$ if $x_i$ is not comparable to $x_{i+1}$. If $x_i < x_{i+1}$ in $P$, then $(x_i, x_{i+1})$ is called a stair(or bump) of $P$ in $L$. Let $s(L,P)[b(L,P)]$ be the number of jumps[stairs] of $P$ in $L$, and let $s(P)[b(P)]$ be the minimum[maximum] of $s(L,P)[b(L,P)]$ over all linear extensions $L$ in $P$. The number $s(P)[b(P)]$ is called the jump [stair] number of $P$.

Let $A = [a_{ij}]$ be an $m \times n$ (0, 1)-matrix. Let $\{x_1, \cdots, x_m\}$ and $\{y_1, \cdots, y_n\}$ be disjoint sets of $m$ and $n$ elements, respectively, and define the order as $x_i < x_j$ iff $a_{ij} = 1$. Then the set $P_A = \{x_1, \cdots, x_m, y_1, \cdots, y_n\}$ with the defined order becomes a poset. For simplicity, $s(A)[b(A)]$ is used for the jump [stair] number of $P_A$. More discussions about jump numbers and the (0, 1)-matrices with fixed row and column sum are given in [1–3].

Let $\Lambda(n,k)$ denote the set of all $(0,1)$-matrices of order $n$ with $k$ 1's in each row and column and $M(n,k) = \max\{s(A) : A \in \Lambda(n,k)\}$. In [4], Brualdi and Jung first studied the maximum

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jump number $M(n,k)$ and gave out its values when $1 \le k \le n \le 10$. They also put forward several conjectures, including the conjecture that $M(2k-2,k) = 3k-4+\lfloor \frac{k-2}{2} \rfloor$.

In this paper, we prove that $M(2k-2,k) = 3k-4+\lfloor \frac{k-2}{2} \rfloor$ does not always hold, and find another counter-example (Corollary 2) to this conjecture .

Let $J_{a,b}$ denote the $a \times b$ matrix with all 1's, and let $J$ denote any matrix with all 1's of an appropriate size. The following lemmas obviously hold or come from [4] and [6].

**Lemma 1.1** *Let $A$ and $B$ be two $m \times n$ matrices. Then*

1) $s(A) + b(A) = m + n - 1$, *and hence* $M(n,k) \le 2n - 1 - \min\{b(A) : A \in \Lambda(n,k)\}$;

2) $s(A \oplus B) = s(A) + s(B) + 1$;

3) *If there exist two permutation matrices $R$ and $S$ such that $B = RAS$, that is, $A$ can be permuted to $B$, expressed $A \sim B$, then*

   (a) *$A$ and $B$ have the same row sum and column sum. And in this case, we call that $A$ can be permuted to $B$ and expressed as $A \sim B$.*

   (b) $b(A) = b(B)$ *and* $s(A) = s(B)$.

**Lemma 1.2** $b(A) \ge b(B)$ *holds for every sub-matrix $B$ of $A$.*

**Lemma 1.3** *Let $A$ be a $(0,1)$-matrix with no zero row or column. Let $b(A) = p$. Then there exist permutation matrices $R$ and $S$ and integers $m_1, \ldots, m_p$ and $n_1, \ldots, n_p$ such that $RAS$ equals*

$$\begin{bmatrix} J_{m_1,n_1} & A_{1,2} & \cdots & A_{1,p} \\ O & J_{m_2,n_2} & \cdots & A_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & J_{m_p,n_p} \end{bmatrix}.$$

**Lemma 1.4** *Let $A$ be a $(0,1)$-matrix with stair number $b(A) = 1$. Then $A$ can be permuted to the following matrix*

$$J \text{ or } [J\ O] \text{ or } \begin{bmatrix} J \\ O \end{bmatrix} \text{ or } \begin{bmatrix} J & O \\ O & O \end{bmatrix}.$$

**Lemma 1.5** *Let $A$ be a (0,1)-matrix having no rows or columns consisting of all 0's or all 1's. Then $b(A) = 2$ if and only if the rows and columns of $A$ can be permuted to an oblique direct sum*

$$O \overline{\oplus} \ldots \overline{\oplus} O$$

*of zero matrices.*

**Lemma 1.6** $M(2k-2,k) \ge 3k-4+\lfloor \frac{k-2}{2} \rfloor$ *holds for every* $k \ge 2$.

**Lemma 1.7** *If $A$ is an $m \times n$ (0,1) matrix without zero row [column] and there are at most $l$ 1's in each column [row], then* $b(A) \ge \lceil \frac{m}{l} \rceil [b(A) \ge \lceil \frac{n}{l} \rceil]$.

**Lemma 1.8** *If $k \nmid n$ and $n$ mod $k \nmid k$ for $1 \le k \le n$, then $M(n,k) \le 2n-2-\lceil \frac{n}{k} \rceil$.*

For a matrix $X$ in block form, we use $X[i_1, i_2, \cdots, i_s | j_1, j_2, \cdots, j_t]$ to denote the submatrix composed of the $i_1$-th, $i_2$-th, $\cdots$, $i_s$-th block-rows and the $j_1$-th, $j_2$-th,$\cdots$, $j_t$-th block-columns from $X$.

## 2. Main theorem

**Theorem 2.1** *If $k \ge 11$, then $b(A) \ge 4$ holds for every $A \in \Lambda(2k-2, k)$.*

**Proof** If there exists a matrix $A \in \Lambda(2k-2,k)$ such that $b(A)=3$, then by Lemma 1.3 we may suppose

$$A = \begin{bmatrix} J_{k,k-q-2} & B_{12} & B_{13} \\ O & J_{p,q} & B_{23} \\ O & O & J_{k-p-2,k} \end{bmatrix},$$

where $B_{ij} (i = 1 \text{ or } 2, j = 2 \text{ or } 3)$ are block-matrices and $1 \le p,\ q \le k-3$.

Since $b(A) = b(A^T) = 3$, we may assume $p \le q$, $1 \le b(B_{12}) \le b(B_{23}) \le 2$. First, we have the following lemmas.

**Lemma 2.1** $b(B_{12}) = 2$.

**Proof** Suppose $b(B_{12}) = 1$. Since $B_{12}$ has evidently no zero or all 1's column, by Lemma 1.4 we have $B_{12} \sim \begin{bmatrix} J_{k-p,q} \\ O \end{bmatrix}$, Thus

$$A \sim A_1 = \begin{bmatrix} J_{k-p,k-q-2} & J_{k-p,q} & B_{131} \\ J_{p,k-q-2} & O & B_{132} \\ O & J_{p,q} & B_{23} \\ O & O & J_{k-p-2,k} \end{bmatrix}.$$

The proof will be complete by the following Propositions 2.1, 2.2 and 2.3. □

**Proposition 2.1** *$B_{131}$ has zero columns and $1 \le b(B_{131}) \le 2$.*

**Proof** Trivial. □

**Proposition 2.2** $b(B_{131}) \ne 1$.

**Proof** If $b(B_{131}) = 1$, then by Lemma 1.4 we have $B_{131} \sim [J_{k-p,2}\ O]$. Thus

$$A_1 \sim A_2 = \begin{bmatrix} J_{k-p,k-q-2} & J_{k-p,q} & J_{k-p,2} & O \\ J_{p,k-q-2} & O & C_1 & C_2 \\ O & J_{p,q} & C_3 & C_4 \\ O & O & J_{k-p-2,2} & J_{k-p-2,k-2} \end{bmatrix}.$$

Obviously, $p \ge \lceil \frac{k-2}{2} \rceil \ge 5$ for $k \ge 11$.

It is clear that both $C_2$ and $C_4$ have no zero row or column and $b\left(\begin{bmatrix} C_2 \\ C_4 \end{bmatrix}\right) = b(C_2) =$

$b(C_4) = 2$. Hence we may suppose

$$A_2 \sim A_3 = \begin{bmatrix} J_{k-p,k-q-2} & J_{k-p,q} & J_{k-p,2} & O & O \\ J_{p_1,k-q-2} & O & C_{11} & J_{p_1,l} & * \\ J_{p-p_1,k-q-2} & O & C_{12} & O & J_{p-p_1,k-l-2} \\ O & J_{p+2-p_1,q} & C_{31} & J_{p+2-p_1,l} & * \\ O & J_{p_1-2,q} & C_{32} & O & J_{p_1-2,k-l-2} \\ O & O & J_{k-p-2,2} & J_{k-p-2,l} & J_{k-p-2,k-l-2} \end{bmatrix}.$$

Obviously, $0 \le b(C_{12}),\ b(C_{32}) \le 1$. If $C_{12} = O$ then $C_{32} = J_{p_1-2,2}$, or if $C_{32} = O$ then $C_{12} = J_{p-p_1,2}$. Without loss of generality, we suppose that the latter holds. If $C_{31}$ has zeroes, then $b([3,4,5,6|1,3,4,5]) = 4$, a contradiction. Hence $C_{31} = J_{p+2-p_1,2}$. Therefore, in the third block-column of $A_3$, the column sum $k \ge (k-p)+(p-p_1)+(p+2-p_1)+(k-p-2) = k+(k-2p_1) > k+(k-2p)$, that is, $k < 2p$. On the other hand, we have $(k-q-2)+2+(k-l-2) = k$ and $q+(k-l-2) = k$, and so $k = 2q$ which contradicts $k < 2p$. It follows that Proposition 2.2 holds. □

**Proposition 2.3** $b(B_{131}) \ne 2$.

**Proof** Suppose $b(B_{131}) = 2$. Then $B_{131} \sim \begin{bmatrix} J_{s,1} & * & O \\ O & J_{t,2} & O \end{bmatrix}$ $(s+t = k-p)$, and hence

$$A_1 \sim A_4 = \begin{bmatrix} J_{s,k-q-2} & J_{s,q} & J_{s,1} & * & O \\ J_{t,k-q-2} & J_{t,q} & O & J_{t,2} & O \\ J_{p,k-q-2} & O & * & * & D_1 \\ O & J_{p,q} & * & * & D_2 \\ O & O & J_{k-p-2,1} & J_{k-p-2,2} & J_{k-p-2,k-3} \end{bmatrix},$$

where $*$ denotes any matrix of an appropriate size.

It is clear that $b\left(\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}\right) = 1$ and both $D_1$ and $D_2$ have no zero column, and hence both $D_1$ and $D_2$ have all 1's rows. It induces $(k-q-2)+q+2(k-3) \le 2k$, that is, $k \le 8$, contradicting $k \ge 11$. Therefore, Proposition 2.3 holds. □

By Lemma 2.1 and the hypothesis we have $b(B_{12}) = b(B_{23}) = 2$ and the following Lemma 2.2.

**Lemma 2.2** *$B_{12}$ has neither zero row nor zero column.*

**Lemma 2.3** *$B_{12}$ has neither all 1's column nor all 1's row.*

**Proof** It is clear that $B_{12}$ has no all 1's column. Suppose $B_{12}$ has $t$ all 1's rows, then $B_{12} \sim \begin{bmatrix} J_{t,q} \\ E \end{bmatrix}$, where

$$E = O_{p,q_1} \overline{\oplus} \cdots \overline{\oplus} O_{p,q_m},\ mp = k-t,\ q_1 + \cdots + q_m = q,\ m \ge 2.$$

Thus

$$A \sim A_5 = \begin{bmatrix} J_{t,k-q-2} & J_{t,q} & E_1 \\ J_{k-t,k-q-2} & E & E_2 \\ O & J_{p,q} & B_{23} \\ O & O & J_{k-p-2,k} \end{bmatrix}.$$

Obviously, $E_1$ has zero columns and $1 \leq b(E_1) \leq 2$.

The proof of Lemma 2.3 will be complete by the following Proposition 2.4 and Proposition 2.5. □

**Proposition 2.4** $b(E_1) \neq 2$.

**Proof** Suppose $b(E_1) = 2$. Then $E_1 \sim \begin{bmatrix} J_{t_1,u} & * & O \\ O & J_{t-t_1,2} & O \end{bmatrix} (1 \leq u \leq 2)$, and hence

$$A_5 \sim A_6 = \begin{bmatrix} J_{t_1,k-q-2} & J_{t_1,q} & J_{t_1,u} & * & O \\ J_{t-t_1,k-q-2} & J_{t-t_1,q} & O & J_{t-t_1,2} & O \\ J_{k-t,k-q-2} & E & E_{21} & E_{22} & E_{23} \\ O & J_{p,q} & B_{231} & B_{232} & B_{233} \\ O & O & J_{k-p-2,u} & J_{k-p-2,2} & J_{k-p-2,k-4} \end{bmatrix}.$$

Obviously, $b\left(\begin{bmatrix} E_{23} \\ B_{233} \end{bmatrix}\right) = 1$ and $\begin{bmatrix} E_{23} \\ B_{233} \end{bmatrix} \sim \begin{bmatrix} J_{p+2,k-4} \\ O \end{bmatrix} (k-t > 2)$ or $J_{p+2,k-4} (k-t = 2)$.

If both $E_{23}$ and $B_{233}$ have all 1's rows, then $2k \geq (k-q-2) + 1 + q + 2(k-4)$, that is, $k \leq 9$, contradicting $k \geq 11$, and hence $B_{233} = O$. Since $b([B_{231} \quad B_{232} \quad B_{233}]) = b(B_{23}) = 2$, we have $b(A_6[1,4,5|1,3,4,5]) = 4$, a contradiction. □

**Proposition 2.5** $b(E_1) \neq 1$.

**Proof** Suppose $b(E_1) = 1$. Then $E_1 \sim [J_{t,2} \quad O]$ and

$$A_5 \sim A_7 = \begin{bmatrix} J_{t,k-q-2} & J_{t,q} & J_{t,2} & O \\ J_{k-t,k-q-2} & E & F_3 & F_1 \\ O & J_{p,q} & F_4 & F_2 \\ O & O & J_{k-p-2,2} & J_{k-p-2,k-2} \end{bmatrix},$$

where $F_1$ has obviously no zero column or zero row, and $F_2$ has no zero row.

By Lemma 1.7 we have

$$b\left(\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\right) \geq \left\lceil \frac{k-t+p}{p+2} \right\rceil = \left\lceil \frac{(m+1)p}{p+2} \right\rceil \geq \begin{cases} \lceil \frac{m+1}{2} \rceil \geq 3 & \text{if } m \geq 4 \\ \lceil \frac{3(m+1)}{5} \rceil \geq 3 & \text{if } m \geq 3 \text{ and } p \geq 3 \\ \lceil \frac{(2+1)5}{7} \rceil = 3 & \text{if } m = 2 \text{ and } p \geq 5, \end{cases}$$

which implies $b(A_7) \geq 4$, a contradiction. Hence $m = 3$ and $p = 2$, or $m = 2$ and $3 \leq p \geq 4$. If $m = 3$ and $p = 2$, then $k = mp + t \leq (m+1)p + 2 \leq 10$, which contradicts $k \geq 11$, and hence $m = 2$ and $3 \leq p \leq 4$.

The proof of Proposition 2.5 will be complete by the following claims. □

**Claim 2.1** $b(F_2) \neq 1$.

**Proof** Suppose $b(F_2) = 1$. Then $F_2 \sim [J_{p,w} \quad O]$, and hence

$$A_7 \sim A_8 = \begin{bmatrix} J_{t,k-q-2} & J_{t,q} & J_{t,2} & O & O \\ J_{p+2,k-q-2} & E' & K_1 & K_2 & J_{p+2,k-w-2} \\ J_{p-2,k-q-2} & E'' & K_3 & J_{p-2,w} & O \\ O & J_{p,q} & F_4 & J_{p,w} & O \\ O & O & J_{k-p-2,2} & J_{k-p-2,w} & J_{k-p-2,k-w-2} \end{bmatrix}.$$

It is clear that $b(F_4) = 0$ or 1.

If $b(F_4) = 1$, then $F_4$ has all 1's columns, otherwise, $b(A_8[1,4,5|1,2,3,5]) = 4$. Hence $t \leq 2$. Thus $k = 2p + t \leq 8 + 2 = 10$, contradicting $k \geq 11$.

If $b(F_4) = 0$, then $K_3 = J_{p-2,2}$, and hence $t \leq 4$. If $t \leq 2$, or $t \leq 4$ and $p \leq 3$, then $k = 2p + t \leq 10$, contradicting $k \geq 11$, and hence $p = 4$ and $t = 3$ or 4. It follows $K_2 = O$ and $K_1 \sim O$ or $\begin{bmatrix} J_{1,2} \\ O \end{bmatrix}$. If $K_1 = O$, then $b(E') = 1$. Since $E'$ has no zero row, $E'$ has all 1's column. Thus $A_8$ has a column which column sum $\geq t + (p+2) + p = k + 2$, impossible. If $K_1 \sim \begin{bmatrix} J_{1,2} \\ O \end{bmatrix}$, then $E'$ has a sub-matrix permuted to $J_{p+1,v}$, and hence $k \geq t + (p+1) + p$, that is, $k \geq k + 1$, a contradiction. □

**Claim 2.2** $b(F_2) \neq 2$.

**Proof** Suppose $b(F_2) = 2$. Then $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ has no zero row and $[F_4 \ \ F_2]$ has no zero column, and we may suppose

$$A_7 \sim A_9 = \begin{bmatrix} J_{t,k-q-2} & J_{t,q_1} & J_{t,q_2} & J_{t,2} & O & O \\ J_{p,k-q-2} & J_{p,q_1} & O & F_{31} & F_{111} & F_{121} \\ J_{p,k-q-2} & O & J_{p,q_2} & F_{32} & F_{112} & F_{122} \\ O & J_{p_1,q_1} & J_{p_1,q_2} & F_{41} & J_{p_1,l} & F_{21} \\ O & J_{p-p_1,q_1} & J_{p-p_1,q_2} & F_{42} & O & J_{p-p_1,k-l-2} \\ O & O & O & J_{k-p-2,2} & J_{k-p-2,l} & J_{k-p-2,k-l-2} \end{bmatrix}.$$

Obviously, we can assume $0 \leq b(F_{111}) \leq b(F_{112}) \leq 1$.

**Case 1** If $F_{111} = O$, then $F_{121} = J_{p,k-l-2}$ and $b(F_{31}) = 0$ or 1. If $b(F_{31}) = 1$, then $t \leq 2$, and hence $k \leq 8 + t \leq 10$, contradicting $k \geq 11$. Thus $F_{31} = O$, which induces $F_{42} = J_{p-p_1,2}$ and $l = q$. It follows $q_2 \leq 2$.

If $F_{41}$ or $F_2$ has zero column, then $b([F_{41}\ F_{21}]) = 0$ or 1. If the former holds, then $[F_{32}|F_{122}] \sim [J_{p,2}|O](p = p_1 + 2)$ or $\left[\begin{array}{c|c} * & J_{1,k-l-2} \\ J_{p-1,2} & O \end{array}\right]$ $(p = p_1 + 1)$. Hence $(p-1) + t \leq p_1 + 2$, and it follows $t \leq 2$. So $k \leq 8 + t \leq 10$, contradicting $k \geq 11$. If the latter holds, then $p \leq 2$ (impossible for $3 \leq p \leq 4$) or $t \leq 2$ (which induces $k \leq 10$). Hence $[F_{41}\ F_{21}]$ is a $p \times (k-q)$ matrix without zero column and there are $k - 2q$ 1's in its each row. By Lemma 1.7, we have $b([F_{41}\ F_{21}]) \geq \lceil \frac{k-q}{k-2q} \rceil = \lceil \frac{q_2+4}{4-q_1} \rceil \geq \lceil \frac{5}{4-q_1} \rceil$. If $q_1 = 1$, then $q = q_1 + q_2 \leq 3$. Since $t \leq p + 1$ we have $k = 2p + t \leq 3p + 1 \leq 3q + 1 \leq 10$, contradicting $k \geq 11$. Thus $q_1 = 2$ or 3, which induces $b[F_{41}\ F_{21}] \geq 3$, and hence $b(A_9) \geq 4$, a contradiction.

**Case 2** If $b(F_{111})=b(F_{112})=1$, then we have

$$A_7\sim A_{10}=\begin{bmatrix} J_{t,k-q-2} & J_{t,q_1} & J_{t,q_2} & J_{t,2} & O & O \\ J_{s,k-q-2} & J_{s,q_1} & O & G_1 & J_{s,l} & O \\ J_{p-s,k-q-2} & J_{p-s,q_1} & O & G_2 & O & J_{p-s,k-l-2} \\ J_{r,k-q-2} & O & J_{r,q_2} & G_3 & J_{r,l} & O \\ J_{p-r,k-q-2} & O & J_{p-r,q_2} & G_4 & O & J_{p-r,k-l-2} \\ O & J_{p_1,q_1} & J_{p_1,q_2} & G_5 & J_{p_1,l} & O \\ O & J_{p-p_1,q_1} & J_{p-p_1,q_2} & G_6 & O & J_{p-p_1,k-l-2} \\ O & O & O & J_{k-p-2,2} & J_{k-p-2,l} & J_{k-p-2,k-l-2} \end{bmatrix}$$

with $s+r+p_1=2p-2$.

If $G_5=O$, then $G_6=J_{p-p_1,2}$, $G_1=J_{s,2}$ and $G_3=J_{r,2}$. It follows $t+s+r+(p-p_1)\le p+2$, and so $t\le 2p_1-4\le 2$(since $p_1\le 3$). Similarly, we will have $t\le 2$ if $G_i=O$ for some $i=1,2,3,4,6$. Thus $k=2p+t\le 8+2=10$, contradicting $k\ge 11$.

Suppose $t\ge 3$ and $G_i\ne O$ for all $i=1,2,3,4,5,6$. Then $\begin{bmatrix} G_1\\ G_2\\ G_3\\ G_4\\ G_5\\ G_6\end{bmatrix}$ is a $3p\times 2$-matrix without zero rows and in its each column 1's number equals $p+2-t\le p+2-3=p-1$, and hence by Lemma 1.7, $b\left(\begin{bmatrix} G_1\\ G_2\\ G_3\\ G_4\\ G_5\\ G_6\end{bmatrix}\right)\ge\lceil\frac{3p}{p-1}\rceil=4$, a contradiction. Therefore, Claim 2.2 holds. □

Now we continue the proof of Theorem 2.1 as follows.

By lemmas 2.1, 2.2 and 2.3, we have

$$B_{12}\sim O_{p,q_1}\overline{\oplus}\cdots\overline{\oplus}O_{p,q_l},\quad q_1+\cdots+q_l=q,\quad lp=k(l\ge 2).$$

Similarly, we may suppose

$$B_{23}\sim O_{p_1,q}\overline{\oplus}\cdots\overline{\oplus}O_{p_s,q},\quad p_1+\cdots+p_s=p,\quad sq=k(s\ge 2).$$

Obviously, $B_{13}$ has neither zero row nor zero column. If $s\ge 4$, then $B_{13}$ has at most $p-1$ 1's in its each column and so by Lemma 1.7 $b(B_{13})\ge\lceil\frac{k}{p-1}\rceil=\lceil\frac{sq}{p-1}\rceil>s\ge 4$, a contradiction. Thus, the following Proposition 2.6 holds.

**Proposition 2.6** $s<4$.

**Proposition 2.7** *If $s=3$, then $p=q$ and $l=3$.*

**Proof** Suppose $q>p$. Then by Lemma 1.7 $b(B_{13})\ge\lceil\frac{k}{p}\rceil=\lceil\frac{sq}{p}\rceil\ge s+1=4$, a contradiction. Hence $q=p$ and so $l=s=3$. □

**Proposition 2.8** $l\le 4$.

**Proof** Suppose $l \geq 5$. Then by Lemma 1.7 $b(B_{13}) \geq \lceil \frac{k}{p+1} \rceil = \lceil \frac{lp}{p+1} \rceil \geq \lceil \frac{2l}{3} \rceil \geq \lceil \frac{10}{3} \rceil = 4$, a contradiction. □

Therefore we have $(s, l) = (3,3)$ or (2,2) or (2,3) or (2,4).

**Proposition 2.9** $(s, l) \neq (3, 3)$.

**Proof** Suppose $(s, l) = (3,3)$. Then $k = 3p = 3q$ and

$$A \sim \bar{A}_1 = \begin{bmatrix} J_{p,k-q-2} & J_{p,q_1} & J_{p,q_2} & O & H_1 & H_2 & H_3 \\ J_{p,k-q-2} & J_{p,q_1} & O & J_{p,q_3} & H_4 & H_5 & H_6 \\ J_{p,k-q-2} & O & J_{p,q_2} & J_{p,q_3} & H_7 & H_8 & H_9 \\ O & J_{p_1,q_1} & J_{p_1,q_2} & J_{p_1,q_3} & J_{p_1,p} & J_{p_1,p} & O \\ O & J_{p_2,q_1} & J_{p_2,q_2} & J_{p_2,q_3} & J_{p_2,p} & O & J_{p_2,p} \\ O & J_{p_3,q_1} & J_{p_3,q_2} & J_{p_3,q_3} & O & J_{p_3,p} & J_{p_3,p} \\ O & O & O & O & J_{k-p-2,p} & J_{k-p-2,p} & J_{k-p-2,p} \end{bmatrix}$$

with $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = p$.

Without loss of generality, we assume

$$0 \leq b(H_1) \leq b(H_2) \leq b(H_3) \leq 2, \quad 0 \leq b(H_4) \leq b(H_7) \leq 2.$$

If $b(H_1) = 0$. Then $b([H_2 \;\; H_3]) = 1$ and $b\left(\begin{bmatrix} H_4 \\ H_7 \end{bmatrix}\right) = 1$, and hence $[H_2 \;\; H_3] \sim [J_{p,q_3+2} \;\; O]$ and $\begin{bmatrix} H_4 \\ H_7 \end{bmatrix} \sim \begin{bmatrix} J_{p_3+2,p} \\ O \end{bmatrix}$, which implies $p_2 = p_3 = q_2 = q_3 = 1$. If $b(H_2) = b(H_3) = 1$ or $b(H_4) = b(H_7) = 1$, then $p_1 = p_2 = p_3 = 1$ or $q_1 = q_2 = q_3 = 1$, which implies $k = 3p = 3q = 9$, contradicting $k \geq 11$. Hence $H_2 = H_4 = O$ and so $H_5 = J_{p,p}$. It follows $q_1 = p_1 = 1$. Therefore, $k = 3p = 3(p_1 + p_2 + p_3) = 9$, contradicting $k \geq 11$.

If $b(H_1) = 1$. Then $H_1 \sim [J \;\; O]$ or $\begin{bmatrix} J \\ O \end{bmatrix}$, and it follows $p_1 = p_2 = p_3 = 1$ or $q_1 = q_2 = q_3 = 1$. Hence $k = 3p = 3q = 9$, contradicting $k \geq 11$.

If $b(H_1) = 2$. Then $[H_1 \;\; H_2 \;\; H_3]$ is a $p \times 3p$ (0,1)-matrix without zero column and there are just $q_3 + 2$ 1's in its each row. Hence by Lemma 1.7 $b([H_1 \;\; H_2 \;\; H_3]) \geq \lceil \frac{3p}{q_3+2} \rceil \geq 3$, which induces $b(\bar{A}_1) \geq 4$, a contradiction. Therefore, Proposition 2.9 holds. □

**Proposition 2.10** $(s, l) \neq (2, 2)$.

**Proof** Suppose $(s, l) = (2, 2)$. Then $k = 2p$ and $p = q = p_1 + p_2 = q_1 + q_2$ and

$$A \sim \bar{A}_2 = \begin{bmatrix} J_{p,p-2} & J_{p,q_1} & O & L_1 & L_2 \\ J_{p,p-2} & O & J_{p,q_2} & L_3 & L_4 \\ O & J_{p_1,q_1} & J_{p_1,q_2} & J_{p_1,q} & O \\ O & J_{p_2,q_1} & J_{p_2,q_2} & O & J_{p_2,q} \\ O & O & O & J_{p-2,q} & J_{p-2,q} \end{bmatrix}.$$

If $1 \leq q_1,\ q_2 \leq 2$, then $k = 2q = 2(q_1 + q_2) \leq 8$, contradicting $k \geq 11$. Without loss of generality, suppose $q_1 \geq \max\{q_2, 3\}$ and $b(L_1) \leq b(L_2)$.

If $b([L_1 \;\; L_2]) \geq 2$, then $[L_1 \;\; L_2]$ is a $p \times 2p$ -(0,1) matrix without zero column and there are just $p + 2 - q_1$ 1's in its each row. Hence by Lemma 1.7 $b([L_1 \;\; L_2]) \geq \lceil \frac{2p}{p+2-q_1} \rceil \geq \lceil \frac{2p}{p-1} \rceil = 3$, which induces $b(\bar{A}_2) \geq 4$, a contradiction. Hence $b([L_1 \;\; L_2]) = 1$ or $[L_1 \;\; L_2] \sim [J_{p,q_2+2} \;\; O]$.

If $L_1 = O$. Then $L_2 \sim [J_{p,q_2+2}\ \ O]\,(1 \le u < p)$ and so

$$\bar{A}_2 \sim \bar{A}_3 = \begin{bmatrix} J_{p,p-2} & J_{p,q_1} & O & O & O & J_{p,q_2+2} \\ J_{p,p-2} & O & J_{p,q_2} & L_3 & L_{41} & L_{42} \\ O & J_{p_1,q_1} & J_{p_1,q_2} & J_{p_1,q} & O & O \\ O & J_{p_2,q_1} & J_{p_2,q_2} & O & J_{p_2,q_1-2} & J_{p_2,q_2+2} \\ O & O & O & J_{p-2,q} & J_{p-2,q_1-2} & J_{p-2,q_2+2} \end{bmatrix}.$$

It is clear that $L_3 \sim \begin{bmatrix} J \\ O \end{bmatrix}$ and $L_{41} \sim \begin{bmatrix} J \\ O \end{bmatrix}$.

If $[L_3\ \ L_{41}]$ has zero row, then $b(\bar{A}_3[1,2,3,4|2,3,4,5]) = 4$, a contradiction. If $[L_3\ \ L_{41}]$ has all 1's row, then $(p-2)+q_2+p+(q_1-2) \le 2p$ or $q_1+q_2 \le 4$. Hence $k = 2q = 2(q_1+q_2) \le 8$, contradicting $k \ge 11$. Thus $[L_3 \mid L_{41}] \sim \left[\begin{array}{c|c} J_{u,p} & O \\ O & J_{v,p} \end{array}\right]$ with $u+v=p$, which implies $(p+2-p_1)+(p+2-p_2) = p$ or 4=0, a contradiction.

If $b(L_1) = b(L_2) = 1$, then $L_i \sim [J\ \ O]\,(i=1,2)$, and so $p_1 \le 2$ and $p_2 \le 2$. Therefore $k = 2(p_1+p_2) \le 8$, contradicting $k \ge 11$. Therefore Proposition 2.10 holds. □

**Proposition 2.11** $(s,l) \ne (2,3)$.

**Proof** Suppose $(s,l) = (2,3)$. Then $k = 3p = 2q$ and so

$$A \sim \bar{A}_4 = \begin{bmatrix} J_{p,k-q-2} & J_{p,q_1} & J_{p,q_2} & O & N_1 & N_2 \\ J_{p,k-q-2} & J_{p,q_1} & O & J_{p,q_3} & N_3 & N_4 \\ J_{p,k-q-2} & O & J_{p,q_2} & J_{p,q_3} & N_5 & N_6 \\ O & J_{p_1,q_1} & J_{p_1,q_2} & J_{p_1,q_3} & J_{p_1,q} & O \\ O & J_{p_2,q_1} & J_{p_2,q_2} & J_{p_2,q_3} & O & J_{p_2,q} \\ O & O & O & O & J_{k-p-2,q} & J_{k-p-2,q} \end{bmatrix}.$$

If $b(N_i) = 2$ for some $N_i$, assume $b(N_1) = 2$, then $\begin{bmatrix} N_1 \\ N_3 \\ N_5 \end{bmatrix}$ is a $3p \times q$ (0,1)- matrix without zero row and there are just $p+2-p_1$ 1's in its each column. Hence by Lemma 1.7 $b(\begin{bmatrix} N_1 \\ N_3 \\ N_5 \end{bmatrix}) \ge \lceil \frac{3p}{p+2-p_1} \rceil \ge 3$ since $p = \frac{k}{3} \ge \lceil \frac{k}{3} \rceil \ge 4$. Therefore, $b(N_i) \le 1$ for every $i$: $1 \le i \le 6$. Without loss of generality, assume $b(N_1) \le b(N_2)$ and $b(N_3) \le b(N_5)$ and $q_1 \ge q_2 \ge q_3$.

If $N_1 = O$. Then $N_5 \sim J$ or $\begin{bmatrix} J \\ O \end{bmatrix}$, and hence $q_2 = q_3 = 1$ and $N_6$ has zero rows. It follows that $N_2 = J$ and $q_1 = 1$, which implies $k = 2q = 2(q_1+q_2+q_3) = 6$, contradicting $k \ge 11$.

If $b(N_1) = b(N_2) = 1$. Then $N_1 \sim J$ or $\begin{bmatrix} J \\ O \end{bmatrix}$ or $[J\ \ O]$. If the previous two hold, then $q_1 = q_2 = 1$ and so $q_3 = 1$. Thus $k = 2(q_1+q_2+q_3) = 6$, contradicting $k \ge 11$. Hence $N_1 \sim [J\ \ O]$ and so $p_1 \le 2$.

Similarly, $N_2 \sim [J\ \ O]$ and $p_2 \le 2$.

If $p_1 < 2$ or $p_2 < 2$, then $k = 3p = 3(p_1+p_2) \le 9$, contradicting $k \ge 11$.

Suppose $p_1 = p_2 = 2$, then

$$\begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \\ N_5 & N_6 \end{bmatrix} \sim \begin{bmatrix} J & O & J & O \\ O & * & O & * \\ O & * & O & * \end{bmatrix}.$$

It follows $b(\bar{A}_4[1,2,3,4|3,4,5,6]) = 4$, a contradiction. Therefore Proposition 2.11 holds. □

**Proposition 2.12** $(s,l) \neq (2,4)$.

**Proof** Suppose $(s,l) = (2,4)$. Then $k = 4p = 2q$ and

$$A \sim \bar{A}_5 = \begin{bmatrix} J_{p,k-q-2} & J_{p,q_1} & J_{p,q_2} & J_{p,q_3} & O & Q_1 & Q_2 \\ J_{p,k-q-2} & J_{p,q_1} & J_{p,q_2} & O & J_{p,q_4} & Q_3 & Q_4 \\ J_{p,k-q-2} & J_{p,q_1} & O & J_{p,q_3} & J_{p,q_4} & Q_5 & Q_6 \\ J_{p,k-q-2} & O & J_{p,q_2} & J_{p,q_3} & J_{p,q_4} & Q_7 & Q_8 \\ O & J_{p_1,q_1} & J_{p_1,q_2} & J_{p_1,q_3} & J_{p_1,q_4} & J_{p_1,q} & O \\ O & J_{p_2,q_1} & J_{p_2,q_2} & J_{p_2,q_3} & J_{p_2,q_4} & O & J_{p_2,q} \\ O & O & O & O & O & J_{k-p-2,q} & J_{k-p-2,q} \end{bmatrix},$$

where $q = q_1+q_2+q_3+q_4$ and $p = p_1+p_2$. If $b(Q_i) = 2$ for some $i (1 \le i \le 8)$, assume $b(Q_2) = 2$, then $[Q_1 \ \ Q_2]$ is a $p \times 2q$ (0,1)-matrix without zero column and there are just $q_4 + 2$ 1's in its each row. Hence by Lemma 1.7 $b([Q_1 \ \ Q_2]) \ge \lceil \frac{2q}{q_3+2} \rceil = \lceil \frac{2q}{q+2-q_1-q_2-q_4} \rceil \ge \lceil \frac{2q}{q-1} \rceil = 3$, which induces $b(\bar{A}_5) \ge 4$, a contradiction. Hence $0 \le b(Q_i) \le 1$: $1 \le i \le 8$. Without loss of generality, suppose $b(Q_1) \le b(Q_2)$ and $q_1 \ge q_2 \ge q_3 \ge q_4$.

If $Q_1 = O$, then $b\left(\begin{bmatrix} Q_3 \\ Q_5 \\ Q_7 \end{bmatrix}\right) = 1$ and $\begin{bmatrix} Q_3 \\ Q_5 \\ Q_7 \end{bmatrix}$ has no zero column. Hence $\begin{bmatrix} Q_3 \\ Q_5 \\ Q_7 \end{bmatrix} \sim \begin{bmatrix} J_{p_2+2,q} \\ O \end{bmatrix}$, which implies $q_1+q_2+q_4 = 2$ or $q_1+q_3+q_4 = 2$ or $q_2+q_3+q_4 = 2$, impossible.

If $b(Q_1) = 1$, then $Q_1$, $Q_2 \sim [J \ \ O]$, and so $p_1 \le 2$, $p_2 \le 2$.

If $p_1 = 2$, then $\begin{bmatrix} Q_1 \\ Q_3 \\ Q_5 \\ Q_7 \end{bmatrix} \sim \begin{bmatrix} J_{p,v} & O \\ O & Q_{31} \\ O & Q_{51} \\ O & Q_{71} \end{bmatrix}$. If $Q_{31}$, $Q_{51}$ or $Q_{71}$ has zeroes, say, $Q_{31}$ has zeroes , then $b(\bar{A}_5[1,2,6,7|4,5,6]) = 4$, a contradiction. Hence $Q_{31}$, $Q_{51}$ and $Q_{71}$has no zeroes. Thus $Q_{31} = Q_{51} = Q_{71} = J_{p,q-v}$, which induces $3p = p+2-p_1$ or $2p = 0$, impossible.

Similarly, we will get a contradiction if $p_2 = 2$. Thus $p_1 = p_2 = 1$, which means $k = 4(p_1+p_2) = 8$, contradicting $k \ge 11$. Therefore Proposition 2.12 holds. □

The all above shows that $b(A) \ge 4$ for every $A \in \Lambda(2k-2,k)$ if $k \ge 11$. □

## 3. Corollaries

**Corollary 3.1** $3k-4+\lfloor \frac{k-2}{2} \rfloor \le M(2k-2,k) \le 4k-9$ *holds for* $k \ge 11$.

**Proof** By Lemma 1.6 we have $M(2k-2,k) \ge 3k-4+\lfloor \frac{k-2}{2} \rfloor$. On the other hand, by Theorem 2.1 and Lemma 1.1 we have $M(2k-2,k) \le 2(2k-2)-1-4 = 4k-9$. Thus Corollary 3.1 holds. □

**Corollary 3.2** $M(16,9) = 28$.

**Proof** From the proof of Proposition 2.9, we obtain a matrix

$$B = \begin{bmatrix} J_{3,4} & J_{3,1} & J_{3,1} & O & O & O & J_{3,3} \\ J_{3,4} & J_{3,1} & O & J_{3,1} & O & J_{3,3} & O \\ J_{3,4} & O & J_{3,1} & J_{3,1} & J_{3,3} & O & O \\ O & J_{1,1} & J_{1,1} & J_{1,1} & J_{1,3} & J_{1,3} & O \\ O & J_{1,1} & J_{1,1} & J_{1,1} & J_{1,3} & O & J_{1,3} \\ O & J_{1,1} & J_{1,1} & J_{1,1} & O & J_{1,3} & J_{1,3} \\ O & O & O & O & J_{4,3} & J_{4,3} & J_{4,3} \end{bmatrix},$$

with stair number $b(B) = 3$. Hence $s(B) = 32 - 1 - b(B) = 28$, and hence $M(16, 9) \geq 28$. On the other hand, by Lemma 1.8, $M(16, 9) \leq 32 - 2 - \lceil \frac{16}{9} \rceil = 28$. Thus $M(16, 9) = 28$. □

Therefore, the conjecture $M(2k-2, k) = 3k - 4 + \lfloor \frac{k-2}{2} \rfloor$ does not hold for $k = 9$.

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# 具有固定行列和 $k$ 的阶为 $2k-2$ 的 (0,1)- 矩阵的最大跳跃数


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**摘要**: 1992 年 Brualdi 与 Jung 首次引出了最大跳跃数 $M(n,k)$, 即每行每列均含 $k$ 个 1 的阶为 $n$ 的 (0,1)- 矩阵的跳跃数的极大数, 给出了满足条件 $1 \leq k \leq n \leq 10$ 的 (0, 1)- 矩阵的最大跳跃数 $M(n,k)$ 的一个表, 并提出了几个猜想, 其中包括猜想 $M(2k-2,k) = 3k-4+\lfloor \frac{k-2}{2} \rfloor$. 本文证明了当 $k \geq 11$ 时, 对每个 $A \in \Lambda(2k-2,k)$ 有 $b(A) \geq 4$. 还得到了该猜想的另一个反例.



**关键词**: (0,1)- 矩阵; 跳跃数.