第25卷第2期 数学研究与评论 Vol.25, No.2
2005年2月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION May, 2005

Article ID: 1000-341X(2005)02-0255-12 **Document code**: A

On the Adjacent Strong Edge Coloring of Outer Plane Graphs

LIU Lin-zhong[1], *ZHANG Zhong-fu*[1], *WANG Jian-fang*[2]

(1. School of Traffic & Transportation, Lanzhou Jiaotong University, Gansu 730070, China;
2. Institute of Applied Mathematics, Chinese Academy of Science, Beijing 100080, China)
(E-mail: lzliu@orsc.edu.cn, llzh@public.lz.gs.cn)

Abstract: A k-adjacent strong edge coloring of graph $G(V,E)$ is defined as a proper k-edge coloring f of graph $G(V,E)$ such that $f[u] \neq f[v]$ for every $uv \in E(G)$, where $f[u] = \{f(uw)|uw \in E(G)\}$ and $f(uw)$ denotes the color of uw, and the adjacent strong edge chromatic number is defined as $\chi'_{as}(G) = \min\{k|$ there is a k-adjacent strong edge coloring of $G\}$. In this paper, it has been proved that $\Delta \leq \chi'_{as}(G) \leq \Delta + 1$ for outer plane graphs with $\Delta(G) \geq 5$, and $\chi'_{as}(G) = \Delta + 1$ if and only if there exist adjacent vertices with maximum degree.

Key words: outer plane graph; vertex distinguishing edge coloring; adjacent strong edge coloring.
MSC(2000): 05C15
CLC number: O157.5

1. Introduction

The coloring problem is one of the most important problems of graph theory. As an extension to classical coloring problem, the strong coloring problems, which were first presented by M.Aigner, et al.[1,2] and F.Harary[10], is of significance in both theory and practice. Although it is more difficult than the classical coloring problem, some meaningful results about vertex distinguishing edge coloring were obtained recently. For example, M.Aigner, et al.[1,2], O.Favaron, et al.[9] and A.C.Burns[7] studied the strong edge-coloring for general graphs and obtained some results. C.Bazgan, et al.[4] studied the vertex-distinguishing proper coloring of graphs with large minimum degree and [5] of general graphs. P.N.Balister, et al.[3] studied the vertex distinguishing colorings of graphs with $\Delta(G) = 2$. P.Wittmann[12] studied the vertex-distinguishing edge-colorings of 2-regular graphs.

As an extension of vertex distinguishing edge coloring of graphs, Z.Zhang and L.Liu[13] have studied the adjacent vertex distinguishing edge coloring (also says the adjacent strong edge coloring) of graphs, presented the adjacent vertex distinguishing edge coloring conjecture and finally obtained the adjacent vertex distinguishing edge chromatic number of some special graphs. Furthermore, graphs such as fan graph, wheel graph, tree, Halin-graph, 1-tree, 1-outer plane graphs, series-parallel graph, plane graph with high-degree and so on, have also been

Received date: 2003-02-09
Foundation item: National Natural Science Foundation of China (No.19871036) and Qinglan talent Funds of Lanzhou Jiaotong University.

obtained in other papers. In this paper, we study the adjacent strong edge coloring of outer plane graphs.

Let $G(V,E)$ be a graph with vertices set $V(G)$ and edges set $E(G)$. $G[S]$ denotes the subgraph of graph $G(V,E)$ induced by set $S \subset V(G)$ or $S \subset E(G)$. $N(v)$ denotes the adjacent vertices set of vertex $v \in V(G)$, $N[v] = N(v) \bigcup \{v\}$. V_k is the vertices set of graph $G(V,E)$ with k-degree, where k is a positive integer.

A k-proper edge coloring f is an assignment of k colors to the edges set $E(G)$ such that arbitrary adjacent edges are assigned with distinct colors.

Definition 1.1[7,9] *A k-vertex distinguishing edge coloring, which is abbreviated to k-VDEC, is a proper k-edge coloring f of graph $G(V,E)$ such $f[u] \neq f[v]$ for every $u, v \in V(G)$, where $f[u]$, says meeting colors set of vertex u, is defined as a set $\{f(uw)|uw \in E(G)\}$ and $f(uw)$ denotes the color of uw. The vertex distinguishing edge chromatic number of graph $G(V,E)$ is defined as $\chi'_s(G) = \min\{k|$ there is a k-VDEC of $G\}$.*

Definition 1.2[13] *A k-adjacent strong edge coloring, which is abbreviated to k-ASEC, is a proper k-edge coloring f of graph $G(V,E)$ such that $f[u] \neq f[v]$ for every $uv \in E(G)$, where $f[u]$, says meeting colors set of vertex u, is defined as a set $\{f(uw)|uw \in E(G)\}$ and $f(uw)$ denotes the color of edge uw. The the adjacent strong edge chromatic number of graph $G(V,E)$ is defined as $\chi'_{as}(G) = \min\{k|$ there is a k-ASEC of $G\}$.*

For outer plane graph $G(V,E)$, we have proved $\Delta(G) \leq \chi'_{as}(G) \leq \Delta(G)+1$, and $\chi'_{as}(G) = \Delta(G)+1$ if and only if there are at least two adjacent vertices with maximum degree. In this paper, we only prove the conclusion for $\Delta(G) \geq 5$.

Definition 1.3 *Let $G(V,E)$ be a plane graph, if all vertices of G are on the boundary of one face f_0, then G is called outer plane graph, and the face f_0 is called the outer face (the others interior face).*

About the $\chi'_{as}(G)$, based on some results about some special graphs, we have proposed a conjecture as follows:

Conjecture 1.1[13] *If $G(V,E)$ is a connected graph with $|V(G)| \geq 3$ and $G \neq C_5$(5-cycle). Then*

$$\Delta(G) \leq \chi'_{as}(G) \leq \Delta(G)+2.$$

The other terminologies we refer to [14].

2. Main results

Lemma 2.1[15] *Let $G(V,E)$ be a 2-connected outer plane graph with $\Delta(G) \geq 4$. Then at least one of the following statements holds in graph $G(V,E)$,*

1. *There exist two adjacent vertices u and v of degree 2.*
2. *There exist two vertices u and v of degree 2 adjacent to one vertex w of degree 4.*

3. There exists one vertex u of degree 2 adjacent to one vertex v of degree 3.

Lemma 2.2 *Let $G(V, E)$ be an outer plane graph with $\Delta(G) = 5$. If the statement 1 of Lemma 2.1 does not hold in $G(V, E)$, and all statements 2 and 3 of Lemma 2.1 which hold in graph $G(V, E)$ satisfy following two conditions:*

(i) all such vertices u, v, w, w_1, w_2 of statement 2 of Lemma 2.1 satisfy that uw_1, $vw_2 \in E(G)$, $d(w_1) = d(w_2) = 5$, where u, v and $w \in V(G)$ are the vertices in the statement 2 of Lemma 2.1, and $d(u) = d(v) = 2$, $d(w) = 4$, $wu, wv \in E(G)$, $\{w_1, w_2\} = N(w) \setminus \{u, v\}$;

(ii) all such vertices u, v and w of statement 3 of Lemma 2.1 satisfy that $d(w) = 5$, $wv \in E(G)$, $d(v_1) \geq 4$, where u , v and $w \in V(G)$ are the vertices in the statement 3 of Lemma 2.1, $d(u) = 2$, $d(v) = 3$, $uv \in E(G)$, $w \neq v$ is another adjacent vertex of u, and $v_1 = N(v) \setminus \{w, u\}$. Then at least one of the following statements will hold in $G(V, E)$:

1. *There is at least a group of such vertices u, v, w, w_1 and w_2, which satisfy the condition (i), that there is a vertex $w' \in N(w_1) \setminus \{u, w\}$ or $w' \in N(w_2) \setminus \{v, w\}$ with degree $d(w') \leq 4$.*

2. *There is at least a group of such vertices u, v, w and v_1, which satisfy the condition (ii), that there is a vertex $w' \in N(w) \setminus \{u, v\} = \{w_1, w_2, w_3\}$ with degree $d(w') \leq 4$.*

3. *If the statements 1 and 2 of Lemma 2.1 do not hold. Then there is at least a group of such vertices u, v, w and v_1 satisfy the condition (ii) that $d(v_1) = 4$ and there is a vertex $v_0 \in N(v_1) \setminus \{v\}$ with degree $d(v_0) = 2$, or $d(v_1) = 5$ and there exist $x, y \in N(v_1) \setminus \{v\}$ with degree $d(x) = 2$, $d(y) = 3$ and $xy \in E(G)$.*

Proof Let G' be a graph obtained from $G(V, E)$ by deleting all such u and v which satisfy the condition (i) of Lemma 2.1 and all such u which satisfy the condition (ii) of Lemma 2.1. Then G' is also an outer plane graph. If all statements 1,2 and 3 of Lemma 2.2 do not hold, then it follows from the condition of Lemma 2.1 that the statements 1,2 and 3 of Lemma 2.1 do not hold in graph G'. That is a contradiction. Thus, the lemma is proved. □

We quote $V_\Delta = \{v | v \in V \text{ and } d(v) = \Delta(G)\}$, $V'_\Delta = \{v | v \in V(G_0) \text{ and } d_{G_0}(v) = \Delta(G)\}$.

Lemma 2.3[11] *Let $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) \leq 4$ and $|V(G)| \geq 3$, then*

$$4 \leq \chi'_{as}(G) \leq 5$$

and $\chi'_{as}(G) = 5$ if and only if $E(G[V_\Delta]) \neq \emptyset$.

Theorem 2.1 *Let $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) = 3$, then $\chi'_{as}(G) = 4$.*

Lemma 2.4 *Let $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) = 5$. If $E(G[V_\Delta]) = \emptyset$, then $\chi'_{as}(G) = \Delta(G) = 5$.*

Proof It is obvious that $\chi'_{as}(G) \geq 5$. We now prove $\chi'_{as}(G) \leq 5$ by using induction method on $p = |V(G)|$. Let $C = \{1, 2, 3, 4, 5\}$ be a set of five colors.

If $|V(G)| = 6$, then $G(V, E)$ is a fan graph F_6. By enumeration, the conclusion is true. Assume that the conclusion is true when $|V(G)| < p$. We prove the conclusion is true for $|V(G)| = p$. In the following, we denote $f_0[x] = \{f_0(xy)|xy \in E(G_0)$, where f_0 is a 5-ASEC of $G_0\}$.

Case 1. Assume that the statement 1 of Lemma 2.1 holds, and $d(u) = d(v) = 2$, $uv \in E(G)$, u_0, v_0 are the another adjacent vertex of u and v, respectively. We define a new graph as

$$G_0 = G - \{u\} + \{u_0v\}.$$

It is clear that G_0 is also a 2-connected outer plane graph, where $|V(G_0)| = |V(G)| - 1 < p$, $\Delta(G_0) = 5$ and $E(G[V'_\Delta]) = \emptyset$. By the induction hypothesis, there exists a 5-ASEC f_0 of G_0. we now prove there exists a 5-ASEC f of G on the basis of f_0. Let $f(uu_0) = f(u_0v)$.

Subcase 1.1. If $d(u_0) \geq 3$ and $d(v_0) \geq 3$, then let $f(uv) \in C \setminus \{f(uu_0), f(vv_0)\}$. Obviously, f is a 5-ASEC of G.

Subcase 1.2. If $d(u_0) = d(v_0) = 2$, then let $f(uv) \in C \setminus (f_0[u_0] \bigcup f_0[v_0])$, where $|f_0[u_0] \bigcup f_0[v_0]| \leq 4$, obviously, f is a 5-ASEC of G.

The cases of $d(u_0) = 2$ and $d(v_0) \geq 3$ or $d(u_0) \geq 3$ and $d(v_0) = 2$ can be proved with the same method, and the proof is omitted.

Case 2. If statement 1 of Lemma 2.1 does not hold, but statement 2 of Lemma 2.1 holds. Assume that $d(u) = d(v) = 2$, $d(w) = 4$, $uw, vw \in E(G)$, $u_1 \neq w$ and $v_1 \neq w$ are another one adjacent vertex of u and v, respectively, and $w_1, w_2 \notin \{u, v\}$ another two adjacent vertices of w. Then it follows from the assumption that $d(u_1) \geq 3$, $d(v_1) \geq 3$.

Subcase 2.1. If $\{u_1, v_1\} \bigcap \{w_1, w_2\} = \emptyset$, we define a new graph as

$$G_0 = G - \{u, v\} + \{wu_1, wv_1\}.$$

Then G_0 is also a 2-connected outer plane graph, where $|V(G_0)| = |V(G)| - 2 < p$, $\Delta(G_0) = 5$ and $E(G[V'_\Delta]) = \emptyset$. By the hypothesis of induction, there exists a 5-ASEC f_0 of G_0. We now define a 5-ASEC f of G on the basis of f_0. Firstly, let $f(uu_1) = f_0(wu_1)$, $f(vv_1) = f_0(wv_1)$, $f(uw) = f_0(wv_1)$, $f(wv) = f_0(wu_1)$. It is obvious that f is a 5-ASEC of G.

Subcase 2.2. If $u_1 \notin \{w_1, w_2\}$ and $v_1 \in \{w_1, w_2\}$, without lose of generality, assume that $v_1 = w_2$. We define a new graph as

$$G_0 = G - \{u\} + \{u_1w\}.$$

Then G_0 is also a 2-connected outer plane graph, where $|V(G_0)| = |V(G)| - 1 < p$, $\Delta(G_0) = 5$ and $E(G[V'_\Delta]) = \emptyset$. By the hypothesis of induction, there exists a 5-ASEC f_0 of G_0. We prove there exists a 5-ASEC of G on the basis of f_0. Firstly, let $f(uu_1) = f_0(wu_1)$.

Subcase 2.2.1. If $f_0(wu_1) \neq f_0(vw_2)$, then let $f(uw) = f_0(wv), f(wv) = f_0(wu_1)$. Obviously,

f is a 5-ASEC of G.

Subcase 2.2.2. If $f_0(wu_1) = f_0(vw_2)$, we firstly let $f(vw_2) = f_0(ww_2)$, $f(ww_2) = f_0(w_2v)$ (i.e. to exchange the colors of vw_2 and ww_2). Then let $f(uw) = f_0(ww_2)$. Obviously, f is a 5-ASEC of G.

Subcase 2.3. If $\{u_1, v_1\} = \{w_1, w_2\}$, without lose of generality, we assume that $u_1 = w_1$, $v_1 = w_2$.

Subcase 2.3.1. If $w_1w_2 \notin E(G)$. The proof is easy for the case $d(w_1) = d(w_2) = 3$. Hence we can assume that $d(w_1) \geq 4$ and $d(w_2) \geq 4$.

Subcase 2.3.1.1. If $d(w_1) = 4$ or $d(w_2) = 4$, we define a new graph as

$$G_0 = G - \{u, v\} + \{w_1w_2\}.$$

Then G_0 is also a 2-connected outer plane graph, where $|V(G_0)| = |V(G)| - 2 < p$, $\Delta(G_0) = 5$ and $E(G[V_\Delta']) = \emptyset$. By the hypothesis of induction, there exists a 5-ASEC f_0 of G_0. We now prove there exists a 5-ASEC f of G on the basis of f_0. Firstly, let $f(w_1u) = f(w_2v) = f_0(w_1w_2)$. Then we have

$$f_0(w_1w_2) \in ((f_0[w_1] \setminus \{f_0(ww_1\}) \bigcup \{f(w_1u)\}) \bigcap ((f_0[w_2] \setminus \{f_0(w_1w_2)\}) \bigcup \{f(w_2v\})$$
$$f_0(ww_1) \neq f(uw_1) = f(vw_2) = f_0(w_1w_2), f_0(ww_2) \neq f(uw_1) = f(vw_2) = f_0(w_1w_2)$$

If $f_0(ww_2) \in (f_0[w_1] \setminus \{f_0(w_1w_2)\}) \bigcup \{f(w_1u)\}$, then let

$$f(wu) \in C \setminus (f_0[w_1] \setminus \{f_0(w_1w_2)\}) \bigcup \{f(w_1u)\}.$$

If $f_0(ww_2) \notin (f_0[w_1] \setminus \{f_0(w_1w_2)\}) \bigcup \{f(w_1u)\}$, then let

$$f(wu) \in C \setminus \{f(w_1u), f_0(ww_1), f_0(ww_2)\}.$$

1. If $\{f(uw), f_0(ww_1)\} \subset (f_0[w_2] \setminus \{f_0(w_1w_2)\}) \bigcup \{f(vw_2)\}$, then let

$$f(wv) \in C \setminus (f_0[w_2] \setminus \{f_0(w_1w_2)\}) \bigcup \{f(vw_2)\}.$$

2. If $\{f(uw), f_0(ww_1)\} \not\subset (f_0[w_2] \setminus \{f_0(w_1w_2)\}) \bigcup \{f(vw_2)\}$, then let

$$f(wv) \in C \setminus \{f(uw), f_0(ww_1), f_0(ww_2), f(vw_2)\}.$$

The colors of other elements are the same as f_0. Obviously, f is a 5-ASEC of G.

Subcase 2.3.1.2. If $d(w_1) = 5, d(w_2) \neq 5$ (the proof of $d(w_1) \neq 5$ and $d(w_2) = 5$ is same), we define a new graph as

$$G_0 = G - \{u, v\} + \{w_1w_2\}.$$

We can easily redefine a 5-ASEC of G with similar method as Subcase 2.3.1.1, and the proof is omitted here.

Subcase 2.3.1.3. If $d(w_1) = d(w_2) = 5$, we define a new graph

$$G_0 = G - \{u\}.$$

Since $E(G[V_\Delta]) = \emptyset$, hence we can let

$$f(uw_1) \in C \setminus f_0[w_1], f(uw) \in C \setminus \{f(uw_1), f_0(ww_1), f_0(ww_2), f_0(wv)\}.$$

Obviously, f is a 5-ASEC of G.

Subcase 2.3.2. If $w_1w_2 \in E(G)$, then we have $d(w_1) \neq 5$ or $d(w_2) \neq 5$, and $d(w_1) \geq 4$, $d(w_2) \geq 4$.

Subcase 2.3.2.1. If $d(w_1) = 4, d(w_2) = 4$, we define a new graph as

$$G_0 = G - \{u, w, v\}.$$

It is easy to redefine a 5-ASEC f of G, and the proof is omitted.

Subcase 2.3.2.2. If $d(w_1) = 5, d(w_2) = 4$, assume that $\{y_1\} = N(w_2) \setminus \{w_1, w, v\}$. We define a new graph as

$$G_0 = G - \{v\}.$$

Then G_0 is also a 2-connected outer plane graph, where $|V(G_0)| = |V(G)| - 1 < p$, $\Delta(G_0) = 5$ and $E(G[V'_\Delta]) = \emptyset$. By the hypothesis of induction, there exists a 5-ASEC f_0 of G_0. We now prove there exists a 5-ASEC f of G on the basis of f_0.

Subcase 2.3.2.2.1. If $f_0[w_2] \subset f_0[y_1]$, then let $f(vw_2) \in C \setminus f_0[y_1]$.

Subcase 2.3.2.2.1.1. If $\{f_0(ww_1), f_0(ww_2), f_0(uw)\} \subset \{f_0(ww_2), f_0(w_1w_2), f_0(w_2y_1), f(vw_2)\}$, then let $f(wv) \in C \setminus f_0[w_2]$. Obviously, f is a 5-ASEC of G.

Subcase 2.3.2.2.1.2. If $\{f_0(ww_1), f_0(ww_2), f_0(uw)\} \not\subset \{f_0(ww_2), f_0(w_1w_2), f_0(w_2y_1), f(vw_2)\}$, then let $f(wv) \in C \setminus (\{f_0(ww_1), f_0(ww_2), f_0(uw)\} \bigcup \{f(vw_2)\})$. Obviously, f is a 5-ASEC of G.

Subcase 2.3.2.3. Since $E(G[V_\Delta]) = \emptyset$, hence the case of $d(w_1) = d(w_2) = 5$ will not occur.

Case 3. If both statements 1 and 2 of Lemma 2.1 do not hold, then 3 of Lemma 2.1 must hold. Assume that $d(u) = 2$ and $d(v) = 3$, $uv \in E(G)$, $wu \in E(G)$ ($w \neq v$, and $d(w) \geq 3$), and $\{v_1, v_2\} = N(v) \setminus \{u\}$. For this case, we can prove that there is a group of vertices u, v, w such that $wv \in E(G)$, that is $w \in \{v_1, v_2\}$. Otherwise we define a new graph G' by deleting all such vertex u from G and adding edge wv into G. Obviously, G' is also an outer plane graph, by the assumption of Case 3, all statements 1, 2 and 3 of Lemma 2.1 do not occur in G', that is a contradiction.

Hence we assume that all such u, v, w satisfy $wv \in E(G)$, that is $w \in \{v_1, v_2\}$. Without lose of generality, we assume that $w = v_2$, that is all such w, u and v satisfying $wv \in E(G)$. We define a new graph as

$$G_0 = G - \{u\}.$$

If $V_\Delta = |w|$, then $\Delta(G_0) = 4$, by Lemma 2.1, there exists a 5-ASEC f_0 of G. Otherwise G_0 is a 2-connected outer plane graph, and where $|V(G_0)| = |V(G)| - 1 < p$, $\Delta(G_0) = 5$, $E(G[V'_\Delta]) = \emptyset$. By the hypothesis of induction, there exists a 5-ASEC f_0 of G_0. We now prove that there exists a 5-ASEC f of G.

Subcase 3.1. When $d(w) = 3$, the proof is easy, and omitted here.

Subcase 3.2. If $d(w) = 4$, $w_1, w_2 \notin \{u, v\}$ are another two adjacent vertices of w and satisfy $d(w_1) = d(w_2) = 4$ (the proof for the case of $d(w_1) \neq 4$ or $d(w_2) \neq 4$ is more easy).

Subcase 3.2.1. If $f_0[w] \subset f_0[w_1] \bigcap f_0[w_2]$ and $f_0[w_1] \neq f_0[w_2]$, then $|f_0[w_1] \bigcap f_0[w_2]| = 3$. Let

$$f(wv) \in (C \setminus (f_0[w_1] \bigcap f_0[w_2])) \setminus \{f_0(vv_1)\},$$

$$f(wu) \in (C \setminus (f_0[w_1] \bigcap f_0[w_2])) \setminus \{f(wv)\}.$$

Subcase 3.2.1.1. If $d(v_1) \neq 3$, then let $f(uv) \in C \setminus \{f(wu), f(wv), f_0(vv_1)\}$. Obviously, f is a 5-ASEC of G.

Subcase 3.2.1.2. When $d(v_1) = 3$.

1. If $\{f(wv), f_0(vv_1)\} \not\subset f_0[v_1]$, then let $f(uv) \in C \setminus \{f(wu), f(wv), f_0(vv_1)\}$.
2. If $\{f(wv), f_0(vv_1)\} \subset f_0[v_1]$, then let

$$f(uv) \in C \setminus (\{f(wu)\} \bigcup f_0[v_1]), \quad \text{(because } |\{f(wu)\} \bigcup f_0[v_1]| \leq 4).$$

Obviously, f is a 5-ASEC of G.

Subcase 3.2.2. If $f_0[w] \subset f_0[w_1] \bigcap f_0[w_2]$ and $f_0[w_1] = f_0[w_2]$, then let $f(wu) \in C \setminus f_0[w_1]$. The colorings of uv is similar to Subcases 3.2.1.1 and 3.2.1.2.

Subcase 3.2.3. If $f_0[w] \not\subset f_0[w_1] \bigcap f_0[w_2]$, without lose of generality, we assume that $f_0[w] \not\subset f_0[w_2]$ and $f_0[w] \subset f_0[w_1]$. Letting $f(wu) \in C \setminus f_0[w_1]$, and the colorings of uv is similar to Subcases 3.2.1.1 and 3.2.1.2.

For the cases of $f_0[w] \not\subset f_0[w_1]$ and $f_0[w] \not\subset f_0[w_2]$, the proof is easy, and omitted here.

Subcase 3.3. If all such w satisfy $d(w) = 5$, because $E(G[V_\Delta]) = \emptyset$, we can let

$$f(wu) \in C \setminus f_0[w].$$

The edge uv can be easily colored with the same method as Subcases 3.2.1.1 and 3.2.1.2. Thus, the conclusion is true.

From what discussed above, we known that if $E(G[V_\Delta]) = \emptyset$, then $\chi'_{as}(G) = \Delta(G) = 5$. □

Theorem 2.2 *Let G be an outer plane graph with $\Delta(G) = 5$. If $E(G[V_\Delta]) \neq \emptyset$, then $\chi'_{as}(G) = \Delta(G) + 1 = 6$.*

Proof It is obvious that $\chi'_{as}(G) \geq \Delta(G) + 1 = 6$. We now prove $\chi'_{as}(G) \leq \Delta(G) + 1 = 6$ by using induction on $p = |V(G)|$. By enumeration, the conclusion is true for the outer plane graph with order $|V(G)| = 8$ and $E(G[V_\Delta]) \neq \emptyset$. The proofs except the Subcase 2.3.2.3 and Subcase 3.3 are the same as that of Theorem 2.1, hence we only prove the Subcase 2.3.2.3 and Subcase 3.3. $C = \{1, 2, 3, 4, 5, 6\}$ denotes a set of six colors, and the same notations as the Subcases 2.3.2.3 and 3.3 of Theorem 2.1 are used.

Subcase 2.3.2.3. If $d(w_1) = d(w_2) = 5$, by the Lemma 2.2, we assume there is at least one vertex in $N(w_1) \setminus \{u, w\}$ which the degree is less than 5. Without lose of generality, we assume $N(w_1) \setminus \{u, w\} = \{x, y, z\}$ and $d(x) \leq 4$. We define a new graph

$$G_0 = G - \{u\}.$$

If $|V_\Delta| = 2$ and $w \in V_\Delta$, then $\Delta(G_0) = 4$, by Theorem 2.1, there exists a 6-ASEC f_0 of G. Otherwise G_0 is a 2-connected outer plane graph, $|V(G_0)| = |V(G)| - 1 < p$, $\Delta(G_0) = 5$ and $E(G[V_\Delta]) = \emptyset$. By the hypothesis of induction, there exists a 6-ASEC f_0 of G_0. We define a 6-ASEC f of G on the basis of f_0.

1. If $f_0[z] = f_0[y]$,

   (a) If $f_0[w_1] \subset f_0[y] = f_0[z]$, then let

   $$f(w_1u) \in C \setminus f_0[z], f(uw) \in C \setminus \{f(w_1u), f_0(w_1w), f_0(ww_2), f_0(wv)\}.$$

   Obviously, f is a 6-ASEC of G.

   (b) If $f_0[w_1] \not\subset f_0[y] = f_0[z]$, then let

   $$f(w_1u) \in C \setminus f_0[w_1], f(uw) \in C \setminus \{f(w_1u), f_0(w_1w), f_0(ww_2), f_0(wv)\}.$$

   Obviously, f is a 6-ASEC of G.

2. If $f_0[z] \neq f_0[y]$,

   (a) If $f_0[w_1] \subseteq f_0[y] \bigcap f_0[z]$, then $|f_0[z] \bigcap f_0[y]| = 4$. Let

   $$\begin{aligned} &f(ww_1) \in (C \setminus (f_0[z] \textstyle\bigcap f_0[y])) \setminus \{f_0(ww_2)\}, \\ &f(w_1u) \in (C \setminus (f_0[z] \textstyle\bigcap f_0[y])) \setminus \{f_0(ww_1)\}, \\ &f(uw) \in C \setminus \{f(w_1u), f(w_1w), f_0(ww_2)\}, \\ &f(wv) \in C \setminus \{f(uw), f(w_1w), f(ww_2), f_0(vw_2)\}. \end{aligned}$$

   Obviously, f is a 6-ASEC of G.

   (b) If $f_0[w_1] \not\subseteq f_0[y] \bigcap f_0[z]$, then $f_0[w_1] \not\subset f_0[z]$ or $f_0[w_1] \not\subset f_0[y]$.

      i. If $f_0[w_1] \not\subset f_0[z]$ and $f_0[w_1] \not\subset f_0[y]$, then let

      $$f(uw_1) \in C \setminus f_0[w_1], f(wu) \in C \setminus \{f(w_1u), f_0(w_1w), f_0(ww_2), f_0(wv)\}.$$

      Obviously, f is a 6-ASEC of G.

ii. If $f_0[w_1] \subset f_0[z]$ and $f_0[w_1] \not\subset f_0[y]$ (the proof of $f_0[w_1] \not\subset f_0[z]$ and $f_0[w_1] \subset f_0[y]$ is similar), then let

$$f(uw_1) \in C \setminus f_0[z], f(wu) \in C \setminus \{f(w_1u), f_0(w_1w), f_0(ww_2), f_0(wv)\}.$$

Obviously, f is a 6-ASEC of G.

Subcase 3.3 If the statements 1 and 2 of Lemma 2.1 do not occur in G, we assume that all such vertices w satisfies $d(w) = 5$. The other notations are the same as in Subcase 3.3 of Theorem 2.1.

Subcase 3.3.1. If there exists at least one vertex w' in $N(w) \setminus \{u, v\} = \{w_1, w_2, w_3\}$ such that $d(w') \leq 4$, without lose of generality, we assume that $w' = w_3$. We consider graph

$$G_0 = G - \{u\}.$$

If $|V_\Delta| = 2$ and $w \in V_\Delta$, then $\Delta(G_0) = 4$. By Theorem 2.1, there exists a 6-ASEC f_0 of G. Otherwise G_0 is a 2-connected outer plane graph, $|V(G_0)| = |V(G)| - 1 < p$, $\Delta(G_0) = 5$ and $E(G[V_\Delta]) = \emptyset$. By the hypothesis of induction, there exists a 6-ASEC f_0 of G_0. We define a a 6-ASEC f of G on the basis of f_0.

Subcase 3.3.1.1. When $\{f_0(ww_1), f_0(ww_2), f_0(ww_3), f_0(wv)\} \subset f_0[w_1] \bigcap f_0[w_2]$.

1. If $f_0[w_1] \neq f_0[w_2]$, then let

$$f(wv) \in (C \setminus (f_0[w_1] \bigcap f_0[w_2])) \setminus \{f_0(vv_1)\}, f(wu) \in (C \setminus (f_0[w_1] \bigcap f_0[w_2])) \setminus \{f_0(wv)\},$$
$$f(uv) \in C \setminus \{f(wu), f(wv), f_0(vv_1), f_0(v_1v_1'), f_0(v_1v_2')\}.$$

2. If $f_0[w_1] = f_0[w_2]$, then let

$$f(wu) \in C \setminus f_0[w_1], f(uv) \in C \setminus \{f(wu), f(wv), f_0(vv_1), f_0(v_1v_1'), f_0(v_1v_2')\}.$$

Obviously, f is a 6-ASEC of G.

Subcase 3.3.1.2. When $\{f_0(ww_1), f_0(ww_2), f_0(ww_3), f_0(wv)\} \not\subset f_0[w_1] \bigcap f_0[w_2]$.

1. If $\{f_0(ww_1), f_0(ww_2), f_0(ww_3), f_0(wv)\} \not\subset f_0[w_2]$, $\{f_0(ww_1), f_0(ww_2), f_0(ww_3), f_0(wv)\} \not\subset f_0[w_2]$, the proof is easy.

2. If $\{f_0(ww_1), f_0(ww_2), f_0(ww_3), f_0(wv)\} \subset f_0[w_1]$, $\{f_0(ww_1), f_0(ww_2), f_0(ww_3), f_0(wv)\} \not\subset f_0[w_2]$, then let

$$f(wu) \in C \setminus f_0[w_1], f(uv) \in C \setminus \{f(wu), f(wv), f_0(vv_1), f_0(v_1v_1'), f_0(v_1v_2')\}.$$

Obviously, f is a 6-ASEC of G.

Hence we can assume that all the adjacent vertices of w, except u and v, are degree 5.

Subcase 3.3.2. All the adjacent vertices of w, except u and v, are degree 5.

Subcase 3.3.2.1. If $d(v_1) = 3$, it follows from above assumption that $wv_1 \notin E(G)$ (otherwise there is an adjacent vertex of w which distinct to u,v and the degree is less than 5). Define a graph as

$$G_0 = G - \{u\} + \{wv_1\}$$

where $|V(G_0)| = |V(G)| - 1 < p$, $\Delta(G_0) = 5$. By the hypothesis of induction, there exists a 6-ASEC f_0 of G_0. We now prove that there exists a 6-ASEC f of G. Firstly, let $f(wu) = f_0(wv_1)$. Assume that v_1' and v_2' are another adjacent vertices of v_1.

Subcase 3.3.2.1.1. If $\{f_0(vv_1),\ f_0(v_1v_1'),\ f_0(v_1v_2')\} \neq f_0[v_1']$, $\{f_0(vv_1),\ f_0(v_1v_1'),\ f_0(v_1v_2')\} \neq f_0[v_2']$, then let $f(uv) \in C \setminus (\{f(wu), f_0(wv)\} \bigcup f_0[v_1])$. Obviously. f is a 6-ASEC of G.

Subcase 3.3.2.1.2. The other subcases can be proved as follows

1. if $\{f_0(v_1v_1'), f_0(v_1v_2')\} \not\subset f_0[v_1']$, $\{f_0(v_1v_1'), f_0(v_1v_2')\} \not\subset f_0[v_2']$, easily to prove.
2. if $\{f_0(v_1v_1'), f_0(v_1v_2')\} \not\subset f_0[v_1']$, $\{f_0(v_1v_1'), f_0(v_1v_2')\} \subset f_0[v_2']$, then let $f(vv_1) \in C \setminus (f_0[v_2'] \bigcup \{f_0(wv_1)\}$, $f(uv) \in C \setminus \{f(wu), f_0(wv), f(vv_1),\ f_0(v_1v_1'), f_0(v_1v_2')\}$. Obviously, f is a 6-ASEC of G;
3. if $\{f_0(v_1v_1'), f_0(v_1v_2')\} \subset f_0[v_1']$, $\{f_0(v_1v_1'), f_0(v_1v_2')\} \subset f_0[v_2']$, then let

$$f(vv_1) \in C \setminus (f_0[v_1'] \bigcup f_0[v_2'] \bigcup \{f_0(wv_1)\}), \text{where} |(f_0[v_1'] \bigcup f_0[v_2']| \leq 4$$
$$f(uv) \in C \setminus \{f(wu), f_0(wv), f(vv_1), f_0(v_1v_1'), f_0(v_1v_2')\}.$$

Obviously, f is a 6-ASEC of G.

Subcase 3.3.2.2. If $d(v_1) = 4$, the edges of G can be colored as follows.

Subcase 3.3.2.2.1. If $wv_1 \in E(G)$, the proof is similar as that of Subcase 3.3.1 (i.e. the case of there are at most two adjacent vertices of degree 5 of w). Hence we can assume that all such w, u, v and v_1 satisfy $wv_1 \notin E(G)$.

Subcase 3.3.2.2.2. If all such w, u, v and v_1 satisfy $wv_1 \notin E(G)$, and $\{w_1, w_2, w_3\} = N(w) \setminus \{u, v\}$, it follows from the Lemma 2.1, the statements 2 and 3 of Lemma 2.1 must occur.

Subcase 3.3.2.2.2.1. If the statement 2 of Lemma 2.1 occurs, the proof is similar as that of Subcase 3.3.1.

Subcase 3.3.2.2.2.2. If statement 2 of Lemma 2.1 does not occurs, and statement 2 and 3 of Lemma 2.1 occur. We assume that $\{x, y, z\} = N(v_1) \setminus \{v\}$, and $d(x) = 2$. We define a graph as

$$G_0 = G - \{u\} + \{wv_1\}.$$

Then G_0 is also an outer plane graph, where $|V(G_0)| = |V(G)| - 1 < p$, $\Delta(G_0) = 5$ and $E(G[V'_\Delta]) \neq \emptyset$. By the hypothesis of induction, there exists a 6-ASEC f_0 of G_0. We now prove there exists a 6-ASEC of G. Let $f(wu) = f_0(wv_1)$. The other elements can be colored with similar method as Subcase 3.3.1.

Subcase 3.3.2.3. If statement 3 of Lemma 2.1 occurs and $d(v_1) = 5$. We assume that $x \neq v_1, y \neq v_2$ are two adjacent vertices of v_1 and $d(x) = 2, d(y) = 3, xy \in E(G)$. We consider graph

$$G_0 = G - \{x\},$$

where the definition of x, y and z is the same as that in Subcase 3.3.2.2.2.2, the proof is similar to that of Subcase 3.3.1 (i.e. there exists at most two adjacent vertices of w are degree 5).

From what stated above, the proof is completed. □

Using the same method, we can prove the following theorems.

Theorem 2.3 *For 2-connected outer plane graph with $\Delta(G) \geq 6$,*

$$\chi'_{as}(G) = \begin{cases} \Delta(G), & E(G[V_\Delta]) = \emptyset, \\ \Delta(G) + 1, & E(G[V_\Delta]) \neq \emptyset. \end{cases}$$

Combining Lemmas 2.3 and 2.4, and Theorems 2.1, 2.2 and 2.3, we can prove the following theorem.

Theorem 2.4 *For 2-connected outer plane graph $G(V, E)$ $(G \neq C_5)$,*

$$\chi'_{as}(G) = \begin{cases} \Delta(G), & E(G[V_\Delta]) = \emptyset, \\ \Delta(G) + 1, & E(G[V_\Delta]) \neq \emptyset. \end{cases}$$

References:

[1] AIGNER M, TRIESCH E, TUZA Z. *Irregular assignments and vertex-distinguishing edge-colorings of graphs* [J]. Ann. Discrete Math., 1990, **52**: 1–9.
[2] AIGNER M, TRIESCH E. *Irregular assignments of trees and forests* [J]. SIAM J. Discrete Math., 1990, **3**: 439–449.
[3] BALISTER P N, BOLLOBÁS B, SCHELP R H. *Vertex distinguishing colorings of graphs with* $\Delta(G) = 2$ [J]. Discrete Math., 2002, **252**: 17–29.
[4] BAZGAN C, AMEL H B, et al. *A note on the vertex-distinguishing proper coloring of graphs with large minimum degree* [J], Discrete Math., 2001, **236**: 37–42.
[5] BAZGAN C, AMEL H B, et al. *On the vertex-distinguishing proper edge-colorings of graphs* [J]. J. Combin. Theory, Ser. B, 1999, **75**: 288–301.
[6] RICHARD A B, JENNIFER J, et al. *Incidence and strong edge colorings of graphs* [J]. Discrete Math., 1993, **122**: 51–58.
[7] BURNS A C, SCHELP R H. *Vertex-distinguishing proper edge-colorings* [J]. J Graph Theory, 1997, **22**: 73–82.
[8] CHARTRAND G, LESNIAK L. *Graphs and Digraphs* [M]. Second edition. The Wadsworth & Brooks/Cole. Mathematics Series, 1986.
[9] FAVARON O, LI H, SCHELP R H. *Strong edge colorings of graphs* [J]. Discrete Math., 1996, **159**: 103–109.
[10] HARARY F. *Conditional Colorability in Craphs, in: Graph and Applications, Proc. First Colorado Symp, Graph Theory* [M]. John & Wiley Inc., New York, 1985.

[11] LIU Lin-zhong, ZHANG Zhong-fu, WANG Jian-fang. *On the adjacent strong edge chromatic number of outer plane graphs with* $\Delta(G) \leq 4$ [J]. Appl. Math. J. Chinese Univ., 2000, **15**(2): 140–146.
[12] WITTMANN P. *Vertex-distinguishing edge-colorings of 2-regular graphs* [J]. Discrete Math., 1997, **79**: 265–277.
[13] ZHANG Zhong-fu, LIU Lin-zhong, WANG Jian-fang. *On the adjacent strong edge coloring of graphs* [J]. Appl. Math. Lett., 2002, **15**: 623–626.
[14] ZHANG Zhong-fu, ZHANG Jian-xun. *On some sufficient conditions of first kind graph* [J]. J. Math. (Wuhan), 1985, **5**: 161–165.
[15] ZHANG Zhong-fu, ZHANG Jian-xun, WANG Jian-fang. *The total chromatic number of some graphs* [J]. Sci. Sinica, Ser.A, 1988, **31**(12): 1434-1441.

最大度不小于 5 的外平面图的邻强边染色

刘林忠[1], 张忠辅[1], 王建方[2]
(1. 兰州交通大学交通运输学院, 甘肃 兰州 730070; 2. 中国科学院应用数学研究所, 北京 100080)

摘要: 图 $G(V,E)$ 的一 k- 正常边染色叫做 k- 邻强边染色当且仅当对任意 $uv \in E(G)$ 有 $f[u] \neq f[v]$, 其中 $f[u] = \{f(uw)|uw \in E(G)\}$, $f(uw)$ 表示边 uw 的染色. 并且 $\chi'_{as}(G) = \min\{k|$ 存在 k- 图 G 的邻强边染色 $\}$ 叫做图 G 的图的邻强边色数. 本文证明了对最大度不小于 5 的外平面图有 $\Delta \leq \chi'_{as}(G) \leq \Delta + 1$, 且 $\chi'_{as}(G) = \Delta + 1$ 当且仅当存在相邻的最大度点.

关键词: 外平面图; 点可区分边染色; 邻强边染色.