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On ∂-reducible Heegaard Splittings

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Abstract: In this paper, we shall prove that any Heegaard splitting of a ∂-reducible 3-manifold M, say $M = W \cup V$, can be obtained by doing connected sums, boundary connected sums and self-boundary connected sums from Heegaard splittings of n manifolds $M_1, \cdots, M_n$, where M_i is either a solid torus or an irreducible, ∂-irreducible manifold.

Key words: connected sum; boundary connected sum; self-boundary connected sum.
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1. Introduction

Let M be a compact 3-manifold such that each component of ∂M is not a 2-sphere. If there is a 2-sphere in M which does not bound any 3-ball, then we say M is reducible; otherwise, M is irreducible. If there is an essential disk D in M, then we say M is ∂-reducible; otherwise, M is ∂-irreducible.

Let M be a compact 3-manifold. If there is a closed surface S which separates M into two compression bodies W and V with $\partial_+ W = \partial_+ V = S$, then we say M has a Heegaard splitting, denoted by $M = W \cup_S V$ or $M = W \cup V$. In this case, S is called a Heegaard surface of M. We call $g(M) = g(S)$ the genus of M if $g(S)$ is minimal among all Heegaard surfaces of M.

A Heegaard splitting $M = W \cup_S V$ is said to be reducible if there are two essential disks $D_1 \subset W$ and $D_2 \subset V$ such that $\partial D_1 = \partial D_2$; otherwise, it is irreducible. A Heegaard splitting $M = W \cup_S V$ is said to be ∂-reducible if there is an essential disk D which intersects S in only one essential simple closed curve in S; otherwise, it is ∂-irreducible. A Heegaard splitting $M = W \cup_S V$ is said to be weakly reducible if there are two essential disks $D_1 \subset W$ and $D_2 \subset V$ such that $\partial D_1 \cap \partial D_2 = \emptyset$; otherwise, it is strongly irreducible.

Now there are some results on reducibilities of Heegaard splittings. For example, W. Haken proved that any Heegaard splitting of a reducible 3-manifold is reducible; A. Casson and C. Gordon gave a disk version of Haken's lemma, that say, any Heegaard splitting of a ∂-reducible 3-manifold is ∂-reducible, they also show that if M has a weakly reducible Heegaard splitting $W \cup V$ then either $W \cup V$ is reducible or M contains an essential closed surface of genus at least one.

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In this paper, we shall consider Heegaard splittings of ∂-reducible manifolds, and give a more fined disk version of Haken's lemma as follows:

Theorem 1 *1) Any Heegaard splitting of a ∂-reducible manifold M, say $M = W \cup V$, can be obtained by doing connected sums, boundary connected sums and self-boundary connected sums from Heegaard splittings of n manifolds $M_1, \ldots, M_n$, where M_i is either a solid torus or an irreducible, ∂-irreducible manifold.*

2) The set $\{M_1, \ldots, M_n\}$ is unique up to homeomorphism.

2. The proof of Theorem 1

The definitions of connected sum and boundary connected sum are standard. Now we define self-boundary connected sum.

Let M be a compact, ∂-reducible 3-manifold, and D be an essential disk in M. Suppose that D is non-separating in M, but ∂D is separating in ∂M. Now $M' = M - D \times (0,1)$ is a connected manifold such that $\partial M'$ contains at least two components F_1 and F_2. We may assume that $D \times \{0\} \subset F_1$ and $D \times \{1\} \subset F_2$. In this case, we say that M is a self-boundary connected sum of M', denoted by $M = M'\sharp_\partial$.

let $M' = W' \cup V'$ be a Heegaard splitting of M', such that $F_1, F_2 \subset \partial_- V'$. Now suppose that α_1, α_2 are two unknotted, properly embedded arcs in V' and β is a unknotted, properly embedded arc in $D \times [0,1]$ such that $\partial_1\alpha_1, \partial_2\alpha_2 \subset \partial_+ V'$, and $\partial_2\alpha_1 = \partial_1\beta$ and $\partial_1\alpha_2 = \partial_2\beta$. Then $\gamma = \alpha_1 \cup \beta \cup \alpha_2$ is a properly embedded arc in $V' \cup D \times [0,1]$. Let $N(\gamma)$ be a regulalr neighborhood of γ in $V' \cup D \times [0,1]$. It is easy to see that $W = W' \cup N(\gamma)$ is a compression body and the closure of $V' \cup D \times [0,1] - N(\gamma)$, denoted by V, is also a compression body. Hence $M = W \cup V$ is a Heegaard splitting of M. We say $W \cup V$ is a self-boundary connected sum of $W' \cup V'$, denoted by $W \cup V = (W' \cup V')\sharp_\partial$.

Lemma 2.1[1] *Any Heegaard splitting of a ∂-reducible 3-manifold is ∂-reducible.*

The proof of Theorem 1 We first prove Theorem 1(1).

Suppose that $M = W \cup_S V$ is a Heegaard splitting of a ∂-reducible 3-manifold. If the genus of $M = W \cup_S V$ is one, then M is a solid torus and $M = W \cup_S V$ is a trivial Heegaard splitting of M. So we may assume that the genus of $M = W \cup_S V$ is at least two.

By Lemma 2.1, there is an essential disk D such that D intersects S in an essential simple closed curve in S. We may assume that $D_W = D \cap W$ is a disk and $A_V = D \cap V$ is an annulus. That means that $\partial D \subset \partial_- V$. Now there are three cases:

Case 1 D is separating in M.

Now $M - D \times (0,1)$ contains two components M_1 and M_2, D_W separates W into two compression bodies W_1 and W_2 and A_V separates V into two components V_1 and V_2. We assume that $W_1, V_1 \subset M_1$, $W_2, V_2 \subset M_2$, $D \times \{0\} \subset \partial M_1$, $D \times \{1\} \subset \partial M_2$. Let $N((D \cap W) \times \{0\})$ be a regular neighborhood of $(D \cap W) \times \{0\}$ in W_1 and $N(D_W \times \{1\})$ be a regular neighborhood of $D_W \times \{1\}$ in W_2. Then $V' = V_1 \cup N((D \cap W) \times \{0\})$ and $V'' = V_2 \cup N(D_W \times \{1\})$ are two compression bodies. We denote by W' the closure of $W_1 - N(D_W \times \{0\})$ and W'' the closure of $W_2 - N(D_W \times \{1\})$. Then W' and W'' are two compression bodies. Hence $W' \cup V'$ is a Heegaard splitting of M_1 and $W'' \cup V''$ is a Heegaard splitting of M_2. By definition, $W \cup V$ is a boundary connected sum of $W' \cup V'$ and $W'' \cup V''$.

Case 2 D is non-separating in M, but ∂D is separating in ∂M.

Claim 1 D_W is non-separating in W.

Proof Suppose, otherwise, that D_W is separating in W. Then ∂D_W is separating in $\partial_+ W = \partial_+ V$. Since W and V are two compression bodies, D_W is separating in W and A_V is separating in V. Hence D is separating in M, a contradiction. □

Now $M - D \times (0,1)$ is a manifold M'. Since D_W is a non-separating disk in W, $W - D_W \times (0,1)$ is a compression body, say W^*. Let $N(D_W \times \{0\})$ be a regular neighborhood of $D_W \times \{0\}$ and $N(D_W \times \{1\})$ be a regular neighborhood of $D_W \times \{1\}$ in W^*. Then $(V - D \times (0,1)) \cup N(D_W \times \{0\}) \cup N(D_W \times \{1\})$ is a compression body, say V', in M'. Note that the closure of $W^* - (N(D_W \times \{0\}) \cup N(D_W \times \{1\}))$, say W', is also a compression body. By definition, $W \cup V$ is a self-boundary connected sum of $W' \cup V'$.

Case 3 D is non-separating in M, and ∂D is non-separating in ∂M.

Claim 2 ∂D_W is non-separating in $S = \partial_+ V = \partial_+ W$.

Proof Suppose, otherwise, that ∂D_W is separating in S. Let V^* be the manifold obtained by attaching a handlebody H to V along $\partial_- V$ such that ∂D bounds a disk D^* in H. Then V^* is also a compression body and $A_V \cup D^*$ is a disk in V^*. Since ∂D_W is separating in S, $A_V \cup D^*$ is separating in V^*, but D^* is non-separating in H, a contradiction. □

Claim 3 There is an annulus A such that

1) one boundary component of A lies in $\partial_+ V$ and the other lies in $\partial_- V$, and

2) A intersects the annulus A_V in only one essential arc in both A and A_V.

Proof Suppose that $\partial_1 A_V = \partial D$ and $\partial_2 A_V = \partial D_W$.

Now since $\partial_1 A_V$ in $\partial_- V$ is a non-separating curve, there is a curve in $\partial_- V$, say c, such that $|\partial_1 A_V \cap c| = 1$. Then c, together with a simple closed curve in $\partial_+ V$, cobound an annulus, say A such that $\partial_1 A = c$ and $\partial_2 A \subset S = \partial_+ V$. We may assume that $|A \cap A_V|$ is minimal among all such annuli. Now we prove $|A \cap A_V| = 1$.

Note that A and A_V are incompressible in V. Hence $A \cap A_V$ is a set of arcs. Since c intersects $\partial_1 A_V$ in one point, there is only one arc, say a, in $A \cap A_V$. which is essential in both A and A_V.

Suppose that $|A \cap A_V| > 1$. Let b be an arc in $A \cap A_V$ which is outermost in A_V, then it, with a sub-arc of $\partial_2 A_V$, cobound a disk E in A_V such that int$E$ is disjoint from A. Now b, with a sub-arc of $\partial_2 A$, cobound a disk E' in A. Thus $A' = (A - E) \cup E'$ is also an annulus, but A' can be isotoped so that $|A' \cap A_V| < |A \cap A_V|$, a contradiction. □

By Claim 3, there is an annulus A which intersects the annulus A_V in only one arc. We may assume that $\partial D \subset F \subset \partial_- V$. Now let $N = N(A \cup A_V)$ and A^* be the closure of $\partial N(A \cup A_V) - \partial_- V \cup \partial_+ V$. Then A^* is also an annulus in V. We may assume that $\partial_1 A^* \subset \partial_+ V$ and $\partial_2 A^* \subset F$. Since the genus of $M = W \cup_S V$ is at least two, $\partial_1 A^*$ is an essential, separating, simple closed curve in $\partial_+ V$. Since $\partial_1 A^*$ is coplanar to ∂D_W in S, $\partial_1 A^*$ bounds a disk B in W. Now there are two subcases:

Case 3.1 F is a torus.

In this case, $\partial_2 A^*$ bounds a disk B^* in F. Now let $P = B \cup A^* \cup B^*$. Then P is a 2-sphere which intersects $\partial_+ V$ in an essential simple closed curve. That means that $M = W \cup V$ is a connected sum of two Heegaard splittings $M_1 = W_1 \cup V_1$ and $M_2 = W_2 \cup V_2$.

Case 3.2 $g(F) \geq 2$.

Now $\partial_2 A^*$ is an essential, separating, simple closed curve in $\partial_- V$. $A^* \cup B$ is an essential disk which intersects $\partial_+ V$ in an essential, simple closed curve. By Case 1 and Case 2, $W \cup V$ is a boundary connected sum or a self-boundary connected sum of Heegaard splittings .

Now by induction, we can prove Theorem 1(1).

Now we prove that Theorem 1(2).

By Kneser-Milnor's theorem, we may assume that M is irreducible. By (1), $M = W \cup V$ can be obtained by doing boundary connected sums and self-boundary connected sums from Heegaard splittings of n manifolds $M_1, \ldots, M_n$ along n^* disks $D_1, \ldots, D_{n^*}$ where M_i is either a solid torus or an irreducible, ∂-irreducible manifold. In this case, n is the number of boundary connected sums and $n^* - n$ is the number of self-boundary connected sums. Note that $D_1, \ldots, D_{n^*}$ are essential disks in M. Furthermore, ∂D_i is separating in ∂M.

Without loss of generality, we may assume that $\partial_- V$ contains only one component F. Then $\partial D_i \subset F$ for $1 \leq i \leq n^*$ satisfying the following conditions:

1) ∂D_i is separating in F,

2) each component of $F - \cup \partial D_i$ is not a planar surface; otherwise, one component of $M - \cup D_i$ is a 3-ball, and

3) if c is a separating, simple closed curve in F such that c bounds a disk in M and $c \cap (\cup_{i=1}^{n^*} \partial D_i) = \emptyset$, then one component of $F - c \cup_i \partial D_i$ is a planar surface.

By Dehn's lemma, $V = V^* \cup_{\partial_- V^*} M^*$ where V^* is a compression body with $\partial_+ V^* = F$ and M^* is irreducible, ∂-irreducible. Furthermore, $D_i \subset V^*$. Hence D_i is separating in V^*. In this case, it is possible that M^* is not connected. Since $\partial_- V^*$ is incompressible in M, D_i can be isotoped so that D_i is disjoint from $\partial_- V^*$. Hence each component of $V^* - \cup_i D_i$ is either a solid torus or $F_j \times I$ where F_j is a component of $\partial_- V^*$.

Now if $M = W \cup V$ can be obtained by doing boundary connected sums and self-boundary connected sums from Heegaard splittings of m manifolds $M'_1, \ldots, M'_m$ along m^* disks $D'_1, \ldots, D'_{m^*}$ where M'_i is either a solid torus or an irreducible, ∂-irreducible manifold. By the above argument,

D_i' is separating in V^* such that

1) $\partial D_i'$ is separating in F,

2) each component of $F - \cup\partial D_i'$ is not a planar surface, and

3) if c is a separating, simple closed curve in F such that c bounds a disk in M and $c \cap (\cup_i \partial D_i') = \emptyset$, then one component of $F - c \cup_i \partial D_i'$ is a planar surface.

Since V^* is a compression body, $n = m$ and $n^* = m^*$ and $V^* - \cup D_i$ is homeomorphic to $V^* - \cup D_i'$. □

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边界可约的 Heegaard 分解

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摘要: 本文证明了任意边界可约流形的 Heegaard 分解都是 n 个不可约的、边界不可约的三维流形的 Heegaard 分解通过连通和、边界连通和及边界自连通和运算而得到.

关键词: 连通和；边界连通和；边界自连通和.