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Global convergence of a solution to p-Ginzburg-Landau type equations

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Abstract: The author studies the global convergence of a solution of p-Ginzburg-Landau equations when the parameter tends to zero. The convergence is in C^α sense, which is derived by establishing a uniform gradient estimate for some solution of a regularized p-Ginzburg-Landau equations.

Key words: global convergence; regularizable solution; p-energy minimizer.
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1. Introduction

Let $G \subset R^2$ be a bounded and simply connected domain with smooth boundary ∂G, $g : \partial G \to S^1 = \{x \in R^2; |x| = 1\}$ be a smooth map satisfying $\deg(g, \partial G) = 0$. Consider the asymptotic behavior of some weak solution u_ε of the problem

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \frac{1}{\varepsilon^p} u(1 - |u|^2), \quad \text{on} \quad G, \tag{1.1}$$

$$u|_{\partial G} = g, \tag{1.2}$$

(where $\varepsilon \in (0,1)$ and $p > 2$) as $\varepsilon \to 0$.

We know that the minimizer of p-Ginzburg-Landau functional

$$E(u,G) = \frac{1}{p}\int_G |\nabla u|^p + \frac{1}{4\varepsilon^p}\int_G (1 - |u|^2)^2$$

on the function class $W = \{u \in W^{1,p}(G, R^2); u|_{\partial G} = g\}$ exists and satisfies (1.1) and (1.2) in the weak sense. Maybe the weak solution of (1.1) and (1.2) is not unique. We will consider the solution which is the limit of the minimizer u_ε^τ of the regularized functional

$$E^\tau(u,G) = \frac{1}{p}\int_G (|\nabla u|^2 + \tau)^{p/2} + \frac{1}{4\varepsilon^p}\int_G (1 - |u|^2)^2, \quad \tau \in (0,1)$$

on W as $\tau \to 0$. Namely, there exists a subsequence $u_\varepsilon^{\tau_k}$ of the minimizer u_ε^τ such that as $\tau_k \to 0$,

$$u_\varepsilon^{\tau_k} \to \tilde{u}_\varepsilon, \quad \text{in} \quad W^{1,p}(G, R^2), \tag{1.3}$$

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where $\tilde{u}_\varepsilon$ is a minimizer of $E(u,G)$ on W. Obviously, the minimizer $\tilde{u}_\varepsilon$ is a weak solution of (1.1) and (1.2). It is named the regularizable solution. In this paper, we will set up the global C^α convergence for this regularizable solution $\tilde{u}_\varepsilon$.

It is not difficult to prove that the minimizer u_ε^τ exists and solves the system

$$-\operatorname{div}(v^{(p-2)/2}\nabla u) = \frac{1}{\varepsilon^p}u(1-|u|^2), \quad \text{on} \quad G, \tag{1.4}$$

where $v = |\nabla u|^2 + \tau$. Moreover, it also satisfies the maximum principle: $|u_\varepsilon^\tau| \le 1$ on $\overline{G}$.

The limit u_p may be introduced as follows. In virtue of $\deg(g,\partial G) = 0$, we can see that there exists a function $\varphi_0 \in C^\infty(\partial G, R)$ such that $g = (\cos\varphi_0, \sin\varphi_0)$. It is easily obtained that the problem

$$-\operatorname{div}(|\nabla\varphi|^{p-2}\nabla\varphi) = 0 \quad \text{on} \quad G; \quad \varphi|_{\partial G} = \varphi_0 \tag{1.5}$$

has a unique solution $\Phi \in W^{1,p}(G,R)$. Let $u_p = (\cos\Phi, \sin\Phi)$ on $\overline{G}$. Thus $u_p \in W^{1,p}(G,S^1)$ and $u_p|_{\partial G} = g$. On the other hand, from the uniqueness of the solution of (1.5) it follows that the solution of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = u|\nabla u|^p, \quad \text{on} \quad G \tag{1.6}$$

is also unique in $W_g^{1,p}(G,S^1)$[4]. Noticing that the p-energy minimizer, i.e., the solution of $\min\{\int_G |\nabla u|^p; u \in W_g^{1,p}(G,S^1)\}$ satisfies (1.6), we know that it is precisely u_p.

When $p=2$, the $C^{1,\alpha}(\overline{G})$ convergence of the minimizer of $E(u,G)$ was given in [1]. If $p>2$, it was shown that[3], as $\varepsilon \to 0$,

$$E^\tau(u_\varepsilon^\tau, G) \le E^\tau(u_p, G) \le C, \tag{1.7}$$

$$\lim_{\varepsilon\to 0} |u_\varepsilon^\tau| = 1, \quad \text{in} \quad C(\overline{G}, R^2), \tag{1.8}$$

$$\tilde{u}_\varepsilon \to u_p, \quad \text{in} \quad W^{1,p}(G,R^2) \quad \text{and} \quad C_{\rm loc}^{1,\alpha}(G,R^2), \tag{1.9}$$

for some $\alpha \in (0,1)$. Here u_p is the p-energy minimizer on G with the boundary value g. We also want to know whether the global convergence can be derived at least in C^α sense. In this paper, we will prove

Theorem 1.1 *Assume that* $\deg(g,\partial G) = 0$, *and that* u_ε *is a regularizable solution of (1.1) and (1.2). Then as* $\varepsilon \to 0$, $u_\varepsilon \to u_p$, *in* $C^\alpha(\overline{G}, R^2)$, $\forall\alpha \in (0,1)$.

We shall set up a series of the gradient estimates of u_ε^τ near the boundary ∂G in Sections 2 and 3, then based on the results we shall complete the proof of Theorem 1.1 in §4.

2. Estimate for $\|\nabla u_\varepsilon^\tau\|_{L^l(\Omega_R)}$

Both ∂G and the value g are smooth, so we try to set up the gradient estimate of $\nabla u_\varepsilon^\tau$ near boundary. Denote $\Omega_{3R} = \cup_{x_0\in\partial G} B(x_0, 3R) \cap G$ where R is a sufficiently small constant. In view of $\deg(g,\partial G) = 0$, we know that $\varphi_0 \in C^\infty(\partial G, S^1)$ is a function such that $g = (\cos\varphi_0, \sin\varphi_0)$, and it is easily seen that the problem $\Delta\varphi = 0$ on G; $\varphi|_{\partial G} = \varphi_0$ has a unique solution $\Psi \in C^\infty(\overline{G})$. Let $U = (\cos\Psi, \sin\Psi)$ on $\overline{G}$. Thus $U \in C^\infty(\overline{G}, \partial B)$ and $U|_{\partial G} = g$. Extend the domain G to a domain G_0 such that $G \subset\subset G_0$ and U can be defined smoothly on G_0. Denote $w = u_\varepsilon^\tau - U$, then $w = 0$ on ∂G. Define $\tilde{w}$ as the odd shunt of w on $G_0 \setminus G$, which implies u_ε^τ solves (1.4) on G_0.

Theorem 2.1 *Assume $u = u_\varepsilon^\tau$ is a minimizer of $E^\tau(u, G)$, $x_0 \in \partial G$ and $R > 0$. Then for any $l > 1$, there exists a constant $C = C(\Omega_R)$ (independent of ε, τ), such that $\|\nabla u_\varepsilon^\tau\|_{L^l(\Omega_R)} \le C$.*

Proof Since u is defined on G_0 and it is smooth on $G_0 \setminus G$, we may differentiate (1.4) with respect to $x_j (j = 1, 2)$,

$$-(v^{(p-2)/2} u_{x_i})_{x_i x_j} = \frac{1}{\varepsilon^p}(u(1 - |u|^2))_{x_j}. \tag{2.1}$$

Here and in the sequel, double indices indicate summation.

Let $\zeta \in C_0^\infty(B(x_0, 3R))$ be a function such that $\zeta = 1$ on $B(x_0, R)$, $\zeta = 0$ on $G \setminus B(x_0, 2R), 0 \le \zeta \le 1, |\nabla\zeta| \le C = C(R)$ on $B(x_0, 3R)$. Denote $\Gamma_R = G \cap B(x_0, R)$. Integrating over Γ_{3R} the inner product of the both sides of (2.1) with $(u - U)_{x_j}\zeta^2$ we obtain

$$\begin{aligned}
&-\int_{\partial\Gamma_{3R}} \operatorname{div}(v^{(p-2)/2}\nabla u)\frac{\partial(u - U)}{\partial\nu}\zeta^2 \mathrm{d}s + \int_{\Gamma_{3R}} (v^{(p-2)/2} u_{x_i})_{x_j}(\zeta^2(u - U)_{x_j})_{x_i} \\
&= \frac{1}{\varepsilon^p}\int_{\Gamma_{3R}} (u(1 - |u|^2))_{x_j}\zeta^2(u - U)_{x_j}.
\end{aligned}$$

Summing up for $j = 1, 2$ and computing the term of the left side yield

$$-\int_{\partial\Gamma_{3R}} \operatorname{div}(v^{(p-2)/2}\nabla u)\frac{\partial(u - U)}{\partial\nu}\zeta^2 \mathrm{d}s = 0$$

since $\zeta = 0$ on $\partial B(x_0, 3R)$ and $\operatorname{div}(v^{(p-2)/2}\nabla u) = \frac{1}{\varepsilon^p}u(|u|^2 - 1) = 0$ on ∂G, where ν is the unit outside norm vector on $\partial\Gamma_{3R}$; and

$$\begin{aligned}
&\int_{\Gamma_{3R}} \zeta^2 v^{(p-2)/2}\sum_{j=1}^{2}|\nabla u_{x_j}|^2 + \frac{p-2}{4}\int_{\Gamma_{3R}} \zeta^2 v^{(p-4)/2}|\nabla v|^2 \\
&\le \frac{1}{\varepsilon^p}\Big|\int_{\Gamma_{3R}} \sum_{j=1}^{2}(u(1 - |u|^2))_{x_j}(\zeta^2(u - U)_{x_j})\Big| + \\
&\quad 2\Big|\int_{\Gamma_{3R}} (v^{(p-2)/2}\sum_{j=1}^{2} u_{x_i})_{x_j}(u - U)_{x_j}\zeta\zeta_{x_i}\Big| + \\
&\quad \Big|\int_{\Gamma_{3R}} \zeta^2\sum_{j=1}^{2} U_{x_i x_j}(v^{(p-2)/2}|\nabla u_{x_j}| + \nabla u(v^{(p-2)/2})_{x_j})\Big| := \sum_{k=1}^{3} J_k.
\end{aligned} \tag{2.2}$$

Noticing U is smooth, and applying the Young inequality we have that for any $\delta \in (0, 1)$,

$$\begin{aligned}
J_3 &\le \delta\Big[\int_{\Gamma_{3R}} \zeta^2 v^{(p-2)/2}\sum_{j=1}^{2}|\nabla u_{x_j}|^2 + \frac{p-2}{4}\int_{\Gamma_{3R}} \zeta^2 v^{(p-4)/2}|\nabla v|^2\Big] + \\
&\quad \delta\int_{\Gamma_{3R}} \zeta^2 v^{p/2} + C(\delta).
\end{aligned} \tag{2.3}$$

And moreover,

$$J_2 = \int_{\Gamma_{3R}} |\zeta\zeta_{x_i}[\frac{1}{2}v^{(p-2)/2}v_{x_i} + \frac{p-2}{2}v^{(p-4)/2}v_{x_j}u_{x_i}u_{x_j}\zeta\zeta_{x_i} +$$
$$v^{(p-2)/2}u_{x_ix_j}U_{x_j} + \frac{p-2}{2}v^{(p-4)/2}v_{x_j}u_{x_i}U_{x_j}\zeta\zeta_{x_i}]|$$
$$\leq \delta[\int_{\Gamma_{3R}} v^{(p-4)/2}|\nabla v|^2\zeta^2 + \int_{\Gamma_{3R}} \zeta^2 v^{(p-2)/2}\sum_{j=1}^{2}|\nabla u_{x_j}|^2] + C(\delta)[\int_{\Gamma_{3R}} v^{p/2} + 1]. \tag{2.4}$$

Finally, we will prove the following conclusion

$$J_1 \leq \delta[\int_{\Gamma_{3R}} v^{(p-4)/2}|\nabla v|^2\zeta^2 + \int_{\Gamma_{3R}} \zeta^2 v^{(p-2)/2}\sum_{j=1}^{2}|\nabla u_{x_j}|^2] +$$
$$C(\delta)\int_{\Gamma_{3R}} v^{(p+2)/2} + C. \tag{2.5}$$

In fact,

$$J_1 = \frac{1}{\varepsilon^p}\int_{\Gamma_{3R}}(1-|u|^2)\zeta^2|\nabla u|^2 - \frac{1}{2\varepsilon^p}\int_{\Gamma_{3R}}\zeta^2[(|u|^2)_{x_j}]^2 -$$
$$\frac{1}{\varepsilon^p}\int_{\Gamma_{3R}}(1-|u|^2)u_{x_j}U_{x_j}\zeta^2 - \frac{1}{\varepsilon^p}\int_{\Gamma_{3R}}(1-|u|^2)_{x_j}uU_{x_j}\zeta^2 = \sum_{k=1}^{4} I_k. \tag{2.6}$$

Clearly,

$$I_2 \leq 0 \quad \text{and} \quad I_3 \leq \frac{1}{\varepsilon^p}\int_{\Gamma_{3R}}(1-|u|^2)\zeta^2(|\nabla u|^2+1). \tag{2.7}$$

For I_4, noting $1-|u|^2=0$ on ∂G and $\zeta = 0$ on $\partial B(x_0, 3R)$, we have

$$I_4 = \frac{1}{\varepsilon^p}\int_{\partial\Gamma_{3R}}(1-|u|^2)(uU_{x_j}\zeta^2) - \frac{1}{\varepsilon^p}\int_{\Gamma_{3R}}(1-|u|^2)(uU_{x_j}\zeta^2)_{x_j}$$
$$= -\frac{1}{\varepsilon^p}\int_{\Gamma_{3R}}(1-|u|^2)(uU_{x_j}\zeta^2)_{x_j}. \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.6) yields

$$J_1 \leq C[\frac{1}{\varepsilon^p}\int_{\Gamma_{3R}}(1-|u|^2)\zeta^2(|\nabla u|^2+1) + \frac{1}{\varepsilon^p}\int_G(1-|u|^2)] := C\sum_{k=5}^{6} I_k. \tag{2.9}$$

To estimate I_5, using (1.4) and (1.7) we obtain that for any $\delta \in (0,1)$,

$$\frac{1}{\varepsilon^p}\int_{\Gamma_{3R}}(1-|u|^2)\zeta^2|\nabla u|^2 \leq \int_{\Gamma_{3R}}|u|^{-1}|\nabla u|^2\zeta^2|\mathrm{div}(v^{(p-2)/2}\nabla u)|$$
$$\leq \delta[\int_{\Gamma_{3R}} v^{(p-4)/2}|\nabla v|^2\zeta^2 + \int_{\Gamma_{3R}}\zeta^2 v^{(p-2)/2}\sum_{j=1}^{2}|\nabla u_{x_j}|^2] + C\int_{\Gamma_{3R}} v^{(p+2)/2}.$$

Substituting this and $I_6 \le C$ (which is implied by Proposition 2.3 in [3]) into (2.9) we derive (2.5). Substituting (2.3)–(2.5) into (2.2) and choosing δ small enough yield

$$\int_{\Gamma_{3R}} \zeta^2 v^{(p-2)/2} \sum_{j=1}^{2} |\nabla u_{x_j}|^2 + \int_{\Gamma_{3R}} \zeta^2 v^{(p-4)/2} |\nabla v|^2 \le C \int_{\Gamma_{3R}} \zeta^2 v^{(p+2)/2} + C \int_{\Gamma_{3R}} v^{p/2} |\nabla \zeta|^2 + C. \tag{2.10}$$

Now we estimate $\int_G \zeta^2 v^{(p+2)/2}$. Take $\varphi = \zeta^{2/q} v^{(p+2)/2}$ in the interpolation inequality

$$\|\varphi\|_{L^q} \le C_*(\|\nabla\varphi\|_{L^1} + \|\varphi\|_{L^1})^{\alpha} \|\varphi\|_{L^1}^{1-\alpha}, \quad q \in (1,2), \alpha = 2(1-1/q). \tag{2.11}$$

Thus,

$$\int_{\Gamma_{3R}} \zeta^2 v^{(p+2)/2} \le C\Big(\int_{\Gamma_{3R}} \zeta^{2/q} v^{(p+2)/2q}\Big)^{q(1-\alpha)} \Big(\int_{\Gamma_{3R}} \zeta^{2/q} v^{(p+2)/2q} + \int_{\Gamma_{3R}} \zeta^{2/q-1} |\nabla\zeta| v^{(p+2)/2q} + \int_{\Gamma_{3R}} \zeta^{2/q} v^{(p+2)/2q-1} |\nabla v|\Big)^{q\alpha}. \tag{2.12}$$

Since $p > 2$, we can choose $q \in (1 + 2/p, 2)$ and hence $\frac{p+2}{2q} < \frac{p}{2}$. By using (1.7) we have that both $\int_{\Gamma_{3R}} \zeta^{2/q} v^{(p+2)/2q}$ and $\int_{\Gamma_{3R}} \zeta^{2/q-1} |\nabla\zeta| v^{(p+2)/2q}$ are bounded. Thus, (2.12) gives

$$\int_{\Gamma_{3R}} \zeta^2 v^{(p+2)/2} \le C\Big(1 + \int_{\Gamma_{3R}} \zeta^{2/q} v^{(p+2)/2q-1} |\nabla v|\Big)^{q\alpha} \le C + C\Big(\int_{\Gamma_{3R}} \zeta^2 v^{(p-4)/2} |\nabla v|^2\Big)^{q\alpha/2} \Big(\int_{\Gamma_{3R}} \zeta^{4/q-2} v^{(p+2)/q-p/2}\Big)^{q\alpha/2}. \tag{2.13}$$

Since $q \in (1 + \frac{2}{p}, 2)$, we have $\frac{q\alpha}{2} < 1$, $\frac{p+2}{q} - \frac{p}{2} < \frac{p}{2}$. Thus, using the Holder inequality and (1.7), we obtain $\int_{\Gamma_{3R}} \zeta^{4/q-2} v^{(p+2)/q-p/2} \le C(\int_{\Gamma_{3R}} v^{p/2})^{2(p+2)/pq-1} \le C$. Hence, from (2.13),we have for any $\delta \in (0,1)$,

$$\int_{\Gamma_{3R}} \zeta^2 v^{(p+2)/2} \le C(\delta) + \delta \int_{\Gamma_{3R}} \zeta^2 v^{(p-4)/2} |\nabla v|^2,$$

since $\frac{q\alpha}{2} < 1$. Combining the inequality above with (2.10) we derive

$$\int_{\Gamma_{3R}} \zeta^2 v^{(p-2)/2} \sum_{j=1}^{2} |\nabla u_{x_j}|^2 + \int_{\Gamma_{3R}} \zeta^2 v^{(p-4)/2} |\nabla v|^2 \le C \tag{2.14}$$

or $\int_{\Gamma_{3R}} \zeta^2 |\nabla w|^2 \le C$, where $w = v^{p/4}$. Since (1.7) implies $\int_{\Gamma_{3R}} \zeta^2 |w|^2 \le C$ we have $\zeta w \in W^{1,2}(\Gamma_{3R}, R)$, and thus the embedding inequality gives $\int_{\Gamma_{3R}} (\zeta w)^l \le C(l)$ for any $l > 1$, which implies (2.1) since $\zeta = 1$ on Γ_R.

Remark Using the embedding theorem and Theorem 2.1 we can see that there exists $\alpha_0 \in (0,1)$ which depends on R such that for any $\alpha \in (0, \alpha_0]$, $\|u_\varepsilon^\tau\|_{C^\alpha(\overline{\Omega_R})} \le C$. This is not sufficient to derive the C^α convergence for all $\alpha \in (0,1)$. To prove that the unform estimate above holds for any $\alpha \in (0,1)$, we have to establish the upper bound of $\|\nabla u_\varepsilon^\tau\|_{L^\infty(\Omega_R)}$ and $\|\frac{1}{\varepsilon^p}(1 - |u_\varepsilon^\tau|)\|_{L^\infty(\Omega_R)}$.

3. Estimate for $\|\nabla u_\varepsilon^\tau\|_{L^\infty(\Omega_R)}$

Theorem 3.1 *Assume u_ε^τ is a minimizer of $E^\tau(u,G)$. Then for any $R\in(0,1)$, there exists a constant $C>0$ (independent of ε,τ), such that*

$$\|\nabla u_\varepsilon^\tau\|_{L^\infty(\Omega_R)}\le C. \tag{3.1}$$

Proof Given any $x_0\in\partial G$. $r>0$ is sufficiently small. Denote $Q_m=G\cap B(x_0,r_m), r_m=r+\frac{r}{2^m}$. We may choose $r,\sigma\in(0,1)$ such that $\sigma\le|Q_m|\le 1$. Choose $\zeta_m\in C_0^\infty(B(x_0,r_m),R)$ such that $\zeta_m=1$ on $B(x_0,r_{m+1})$, $|\nabla\zeta_m|\le Cr^{-1}2^m,(m=1,2,\cdots)$. Integrate over Q_m the inner product of the both sides of (2.1) with $\zeta_m^2v^b(u-U)_{x_j}(b\ge 1)$. Then

$$\begin{aligned}&-\int_{\partial Q_m}\mathrm{div}(v^{(p-2)/2}\nabla u)\frac{\partial(u-U)}{\partial\nu}\zeta_m^2\mathrm{d}s+\int_{Q_m}(v^{(p-2)/2}u_{x_i})_{x_j}(\zeta_m^2v^b(u-U)_{x_j})_{x_i}\\&=\frac{1}{\varepsilon^p}\int_{Q_m}(u(1-|u|^2))_{x_j}\zeta_m^2v^b(u-U)_{x_j}.\end{aligned}$$

Here ν is the unit outside norm vector on ∂Q_m. We also have $\int_{\partial Q_m}\mathrm{div}(v^{(p-2)/2}\nabla u)\frac{\partial(u-U)}{\partial\nu}\zeta_m^2\mathrm{d}s=0$. Noting that

$$\begin{aligned}(v^bu_{x_j})_{x_i}(v^{(p-2)/2}u_{x_i})_{x_j}&=v^{(p+2b-2)/2}u_{x_ix_j}^2+\\&\frac{p+2b-2}{4}v^{(p+2b-4)/2}|\nabla v|^2+\frac{b(p-2)}{2}v^{(p+2b-6)/2}(\nabla u\nabla v)^2,\end{aligned}$$

and for any $\delta\in(0,1)$, $|(v^bU_{x_j})_{x_i}(v^{(p-2)/2}u_{x_i})_{x_j}|\le\delta I(b)+C(\delta)[v^{(p+2b-2)/2}+I(b-1)]$, where

$$I(b):=\int_{Q_m}\zeta_m^2v^{(p+2b-2)/2}\sum_j|\nabla u_{x_j}|^2+\frac{p+2b-2}{4}\int_{Q_m}\zeta_m^2v^{(p+2b-4)/2}|\nabla v|^2,$$

we also obtain

$$\begin{aligned}I(b)\le&\delta I(b)+CI(b-1)+C\int_{Q_m}|\nabla\zeta_m|^2v^{(p+2b)/2}+C\int_{Q_m}v^{(p+2b-2)/2}\zeta_m^2+\\&\frac{1}{\varepsilon^p}\int_{Q_m}(1-|u|^2)\zeta_m^2(v^{b+1}+1)+\frac{1}{\varepsilon^p}\left|\int_{Q_m}u(1-|u|^2)_{x_j}v^bU_{x_j}\zeta_m^2\right|,\end{aligned}$$

by the similar argument in the derivation of (2.2) and the dealing with J_2,J_3. Combining this with

$$\begin{aligned}&\frac{1}{\varepsilon^p}\int_{Q_m}(1-|u|^2)\zeta_m^2v^{b+1}\le\int_{Q_m}|u|^{-1}\zeta_m^2v^{b+1}|\mathrm{div}(v^{(p-2)/2}\nabla u)|\\&\le C(\delta)\int_{Q_m}\zeta_m^2v^{(p+2b+2)/2}+\delta\int_{Q_m}\zeta_m^2v^{(p+2b-2)/2}|\Delta u|^2+\\&\quad\frac{C(\delta)(p+2b-2)}{2}\int_{Q_m}\zeta_m^2v^{(p+2b+2)/2}+\frac{\delta(p+2b-2)}{2}\int_{Q_m}\zeta_m^2v^{(p+2b-4)/2}|\nabla v|^2,\end{aligned}$$

we have

$$\begin{aligned}I(b)=&\int_{Q_m}\zeta_m^2v^{(p+2b-2)/2}\sum_j|\nabla u_{x_j}|^2+\frac{p+2b-2}{4}\int_{Q_m}\zeta_m^2v^{(p+2b-4)/2}|\nabla v|^2\\\le&C\int_{Q_m}|\nabla\zeta_m|^2v^{(p+2b)/2}+C(p+2b-2)\int_{Q_m}v^{(p+2b+2)/2}\zeta_m^2+\\&\frac{1}{\varepsilon^p}\left|\int_{Q_m}u(1-|u|^2)_{x_j}v^bU_{x_j}\zeta_m^2\right|+CI(b-1),\end{aligned}\tag{3.2}$$

as long as $\delta > 0$ sufficiently small. Here the constants C is independent of b, m, ε and τ.

To estimate $\frac{1}{\varepsilon^p}|\int_{Q_m} u(1-|u|^2)_{x_j} v^b U_{x_j}\zeta_m^2|$, we will integrate by parts. Noticing $1-|u|^2=0$ on ∂G and $\zeta_m = 0$ on $\partial B(x_0, r_m)$, we have

$$\frac{1}{\varepsilon^p}|\int_{Q_m} u(1-|u|^2)_{x_j} v^b U_{x_j}\zeta_m^2| = \frac{1}{\varepsilon^p}|\int_{Q_m}(1-|u|^2)(uv^b U_{x_j}\zeta_m^2)_{x_j}|. \tag{3.3}$$

Substituting $\frac{1}{\varepsilon^p}|1-|u|^2| = \frac{1}{|u|}|\mathrm{div}(v^{(p-2)/2}\nabla u)|$ into (3.3), and using Holder inequality, we can see that for any $\delta \in (0,1)$,

$$\begin{aligned}&\frac{1}{\varepsilon^p}|\int_{Q_m} u(1-|u|^2)_{x_j} v^b U_{x_j}\zeta_m^2|\\ &\le \delta I(b) + C(\delta)(\int_{Q_m} v^{(p+2b)/2}\zeta_m^2 + 1 + \int_{Q_m} v^{(p+2b-2)/2}|\nabla\zeta_m|^2 + I(b-1)).\end{aligned}$$

Substituting this into (3.2) and choosing $\delta > 0$ sufficiently small, we obtain

$$\begin{aligned}I(b) \le& C\int_{Q_m}|\nabla\zeta_m|^2(v^{(p+2b)/2}+1)+\\ &C(p+2b-2)\int_{Q_m}(v^{(p+2b+2)/2}+1)\zeta_m^2 + CI(b-1) + C.\end{aligned} \tag{3.4}$$

If $b=1$, from (2.14) it is led to $I(0) \le C$. Hence

$$I(1) \le C\int_{Q_m}|\nabla\zeta_m|^2(v^{(p+2b)/2}+1) + C(p+2b-2)\int_{Q_m}(v^{(p+2b+2)/2}+1)\zeta_m^2 + C.$$

If for some $l > 1$,

$$I(l) \le C\int_{Q_m}|\nabla\zeta_m|^2(v^{(p+2l)/2}+1) + C(p+2l-2)\int_{Q_m}(v^{(p+2l+2)/2}+1)\zeta_m^2 + C,$$

then from (3.4) it follows that

$$I(l+1) \le C\int_{Q_m}|\nabla\zeta_m|^2(v^{(p+2(l+1))/2}+1) + C(p+2(l+1)-2)\int_{Q_m}(v^{(p+2(l+1)+2)/2}+1)\zeta_m^2 + C$$

by using Young inequality. The argument above means that for any $b \ge 1$,

$$I(b) \le C[\int_{Q_m}|\nabla\zeta_m|^2(v^{(p+2b)/2}+1) + (p+2b-2)\int_{Q_m}(v^{(p+2b+2)/2}+1)\zeta_m^2 + 1]. \tag{3.5}$$

To estimate $\int_{Q_m}\zeta_m^2 v^{(p+2b+2)/2}$, we take $\varphi = \zeta_m^{2/q} v^{(p+2b+2)/2q}$ in the interpolation inequality (2.11). Now, the constant C_* in (2.11) depends on the domain Q_m. Noticing the choosing of $r, \sigma \in (0,1)$ such that $\sigma \le |Q_m| \le 1$, where $Q_m = B(x_0, r_m), r_m = r + \frac{r}{2^m}$, we can see that

$C_* = C_*(\sigma)$ may be independent of m. Then we obtain

$$\begin{aligned}
&\int_{Q_m} \zeta_m^2 v^{(p+2b+2)/2} \le C\Big(\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q}\Big)^{q(1-\alpha)}\cdot \\
&\quad \Big(\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q} + \frac{2}{q}\int_{Q_m} \zeta_m^{2/q-1}|\nabla\zeta_m| v^{(p+2b+2)/2q} + \\
&\quad \frac{p+2b+2}{2q}\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q-1}|\nabla v|\Big)^{q\alpha} \\
&\le C\Big(\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q}\Big)^{q(1-\alpha)}\Big[\Big(\frac{2}{q}\Big)^{q\alpha}\Big(\int_{Q_m} \zeta_m^{2/q-1}|\nabla\zeta_m| v^{(p+2b+2)/2q}\Big)^{q\alpha}\Big] + \\
&\quad \Big(\frac{p+2b+2}{2q}\Big)^{q\alpha}\Big(\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q-1}|\nabla v|\Big)^{q\alpha} + C\Big(\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q}\Big)^{q}. \qquad (3.6)
\end{aligned}$$

Now we estimate all integrals on the right side of (3.6). In the computing we need to notice that $q \in (1+\frac{2}{p}, 2)$, which implies $q > 1 + \frac{2}{p+2b}$ or $\frac{p+2b}{2q} < \frac{p+2b}{2}$. We have

$$\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q} \le \int_{Q_m} v^{(p+2b+2)/2q} \le \Big(\int_{Q_m} v^{(p+2b)/2}\Big)^{(p+2b+2)/(q(p+2b))},$$

$$\int_{Q_m} \zeta_m^{2/q-1}|\nabla\zeta_m| v^{(p+2b+2)/2q} \le \frac{2^m}{r}\Big(\int_{Q_m} v^{(p+2b)/2}\Big)^{(p+2b+2)/(q(p+2b))},$$

$$\begin{aligned}
&\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q-1}|\nabla v| \\
&\le \Big(\int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2}|\nabla v|^2\Big)^{1/2}\Big(\int_{Q_m} v^{(p+2b)/2}\Big)^{(p+2b+2)/(q(p+2b))-1/2}.
\end{aligned}$$

Combining these inequalities with (3.5) and (3.6) yields

$$I_1 \le C\Big[\Big(\frac{2^m}{r}\Big)^2(I_2+1) + \Big(\frac{2^m}{r}\Big)^{q\alpha} I_2^{1+2/(p+2b)} + \Big(\frac{p+2b+2}{2q}\Big)^{q\alpha} I_1^{q\alpha/2} I_2^{1+2/(p+2b)-q\alpha/2}, \qquad (3.7)$$

where $I_1 = \int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2}|\nabla v|^2$, $I_2 = \int_{Q_m} v^{(p+2b)/2}$. Let $p+2b = s^m$, $w = v^{(p+2b)/4} = v^{s^m/4}$ with $s > 2$ to be determined later. Then (3.7) becomes

$$I_1 \le C\Big[\Big(\frac{2^m}{r}\Big)^2(I_2+1) + \Big(\frac{2^m}{r}\Big)^{q\alpha} I_2^{1+2/S^m} + \Big(\frac{s^m+2}{2q}\Big)^{q\alpha} I_1^{q\alpha/2} I_2^{1+2/s^m-q\alpha/2}\Big].$$

Using the Young inequality to treat the last term on the right side yields

$$\begin{aligned}
&C\Big(\frac{s^m+2}{2q}\Big)^{q\alpha} I_1^{q\alpha/2} I_2^{1+2/(s^m)-q\alpha/2} \\
&\quad \le \delta I_1 + C(\delta)\Big(\frac{s^m+2}{2q}\Big)^{2q\alpha/(2-q\alpha)} I_2^{2(1+2/(s^m)-q\alpha/2)/(2-q\alpha)}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
I_1 \le\, & C(\delta)\Big[\Big(\frac{2^m}{r}\Big)^2(I_2+1) + \Big(\frac{2^m}{r}\Big)^{q\alpha} I_2^{1+2/(s^m)} + \\
& \Big(\frac{s^m+2}{2q}\Big)^{2q\alpha/(2-q\alpha)} I_2^{2(1+2/(s^m)-q\alpha/2)/(2-q\alpha)}\Big]. \qquad (3.8)
\end{aligned}$$

By the embeding theorem, we have for any $s > 1$

$$\begin{aligned}\int_{Q_m} (\zeta_m w)^{2s} &\le C(s)[\int_{Q_m} (\zeta_m w)^2 + \int_{Q_m} |\nabla \zeta_m|^2 w^2 + \int_{Q_m} \zeta_m^2 |\nabla w|^2]^s \\ &\le C(s)[(1 + (\frac{2^m}{r})^2) I_2 + (\frac{s^m}{4})^2 I_1]^s\end{aligned}$$

which, by using (3.8) turns out to be

$$\begin{aligned}\int_{Q_m} (\zeta_m w)^{2s} \le & C(s)[(1 + (\frac{2^m}{r})^2 + (\frac{s^m}{4})^2 (\frac{2^m}{r})^2)(I_2 + 1) + (\frac{s^m}{4})^2 (\frac{2^m}{r})^{q\alpha} I_2^{1+\frac{2}{s^m}} + \\ & (\frac{s^m}{4})^2 (\frac{s^m + 2}{2q})^{\frac{2q\alpha}{2-q\alpha}} I_2^{(1+\frac{2}{s^m} - \frac{q\alpha}{2})\frac{2}{2-q\alpha}}]^s. \end{aligned} \tag{3.9}$$

If there exists a subsequence of positive integers $\{m_i\}$ with $m_i \to \infty$, such that $I_2 = \int_{Q_{m_i}} v^{\frac{s^{m_i}}{2}} < 1$, then letting $m_i \to \infty$ yields immediately

$$\|v\|_{L^\infty(Q_\infty, R)} \le C(r). \tag{3.10}$$

Otherwise, there must be a positive integer m_0 such that $I_2 = \int_{Q_m} v^{\frac{s^m}{2}} \ge 1$, for $m \ge m_0$. Since $(1 + \frac{2}{s^m} - \frac{q\alpha}{2})\frac{2}{2-q\alpha} = 1 + \frac{2}{s^m}\frac{1}{2-q} > 1 + \frac{2}{s^m} > 1$, the power of the last term in (3.9) is higher than those of the other terms. Now we compare the coefficients of the terms in (3.9). we have $(\frac{s^m}{r})^2 \ge 1, (\frac{2^m}{r})^2 \ge (\frac{2^m}{r})^{q\alpha}$ and, if we choose $s > 2q(\frac{2}{r})^{\frac{2(q-1)}{2-q}}$ and $r \le 1$, then $(\frac{s^m+2}{2q})^{\frac{2q\alpha}{2-q\alpha}} = (\frac{s^m+2}{2q})^{\frac{2(q-1)}{2-q}} \ge (\frac{2^m}{r})^2$. Therefore the coefficient of the last term in (3.9) is larger than those of the other terms. Hence we have

$$\int_{Q_m} (\zeta_m w)^{2s} \le C[(\frac{s^m}{4})^2 (\frac{s^m + 2}{2q})^{\frac{2(q-1)}{2-q}} I_2^{1+\frac{2}{s^m}\frac{1}{2-q}}]^s$$

or

$$\int_{Q_{m+1}} v^{s^{m+1}/2} \le (C_0 C_1^m)^s (\int_{Q_m} v^{s^m/2})^{(1+C_2/S^m)s}$$

with some constant $C_0 > 0, C_2 = \frac{2}{2-q}, C_1 = s^{(2+\frac{2(q-1)}{2-q})s}$. Using an iteration lemma in [3] (Proposition 4.2), we also reach the estimate (3.10) by applying the local estimate (Proposition 3.1 in [3]) and Theorem 3.1. Thus the proof of Theorem 3.1 is completed.

4. Completion of the proof

Theorem 4.1 *Let $\psi_\varepsilon^\tau = \frac{1}{\varepsilon^p}(1 - |u_\varepsilon^\tau|^2)$. Then there exists a constant $C > 0$ independent of $\varepsilon, \tau \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ small enough, such that*

$$\|\psi_\varepsilon^\tau\|_{L^\infty(\overline{G}, R)} \le C. \tag{4.1}$$

Proof Consider the inner product of the both sides of (1.4) with u

$$-\mathrm{div}(v^{(p-2)/2} \nabla u) u = \frac{1}{\varepsilon^p} |u|^2 (1 - |u|^2) = |u|^2 \psi,$$

where $u = u_\varepsilon^\tau, \psi = \psi_\varepsilon^\tau$. Combining this and $\nabla \psi = -\frac{2}{\varepsilon^p} u \cdot \nabla u$ with

$$-\mathrm{div}(v^{(p-2)/2} \nabla u) u = -\mathrm{div}(v^{(p-2)/2} u \cdot \nabla u) + v^{(p-2)/2} |\nabla u|^2$$

yields $|u|^2\psi = v^{(p-2)/2}|\nabla u|^2 + \frac{\varepsilon^p}{2}\mathrm{div}(v^{(p-2)/2}\nabla\psi)$. Using (1.8) we further obtain

$$\frac{1}{4}\psi \le v^{(p-2)/2}|\nabla u|^2 + \frac{\varepsilon^p}{2}div(v^{(p-2)/2}\nabla\psi), \quad \forall\varepsilon,\tau\in(0,\varepsilon_0). \tag{4.2}$$

Since $\psi(x)=0$ on ∂G, the point x_0 where ψ achieves its maximum must be in G. Hence, at x_0, $\nabla\psi=0, \Delta\psi\le 0$ and $\mathrm{div}(v^{(p-2)/2}\nabla\psi) = v^{(p-2)/2}\Delta\psi + \frac{p-2}{2}v^{(p-4)/2}\nabla v\nabla\psi \le 0$, so we derive (4.1) from (4.2) by using the local estimate (Proposition 4.1 in [3]) and Theorem 3.1.

To complete the proof of the theorem, we need to apply a result in [2]. Now according to Theorem 4.1 the right hand side of (1.4) is bounded on $\overline{G}$ uniformly in $\varepsilon,\tau\in(0,\varepsilon_0)$. Thus applying Theorem 1 and Line 19-21 in Page 104 of [2] yields that for any $\beta\in(0,1)$ one has

$$\|u_\varepsilon^\tau\|_{C^\beta(\overline{G})} \le C, \tag{4.3}$$

where the constant does not depend on $\varepsilon,\tau\in(0,\varepsilon_0)$. From this it follows that there exist a function u_* and a subsequence $u_{\varepsilon_k}^{\tau_k}$ $(\varepsilon_k,\tau_k\to 0$, as $k\to\infty)$ of u_ε^τ, such that $\lim_{k\to\infty}u_{\varepsilon_k}^{\tau_k}=u_*$ in $C^\alpha(\overline{G},R^2)$, $\alpha\in(0,\beta)$. (1.3) and (1.9) imply that $u_*=u_p$. By the fact that any subsequence of u_ε^τ contains a subsequence convergent in $C^\alpha(\overline{G},R^2)$ and the limit is the same function u_p, we may assert

$$\lim_{\varepsilon,\tau\to 0} u_\varepsilon^\tau = u_p, \quad \text{in } C^\alpha(\overline{G},R^2). \tag{4.4}$$

On the other hand, for any $\varepsilon\in(0,\varepsilon_0)$ as a regularizable solution of (1.1) and (1.2), $\tilde u_\varepsilon$ is the limit of some subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ in $W^{1,p}(G,R^2)$. For large k, $u_\varepsilon^{\tau_k}$ satisfies (4.3) and hence it contains a subsequence, where for simplicity we suppose it is $u_\varepsilon^{\tau_k}$ itself, such that $\lim_{k\to\infty}u_\varepsilon^{\tau_k}=w$ in $C^\alpha(\overline{G},R^2)$, where the function w must be $\tilde u_\varepsilon$ by (1.3). Combining this with (4.4) we finally obtain $\lim_{\varepsilon\to 0}\tilde u_\varepsilon = u_p$, in $C^\alpha(\overline{G},R^2)$ and complete the proof of the theorem.

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一类 p-Ginzburg-Landau 型方程解的整体收敛性

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摘要: 作者研究了一类 p-Ginzburg-Landau 型方程解的整体收敛性. 通过建立正则化方程解的梯度的一致估计, 最终证明了解在 C^α 意义下收敛.

关键词: 整体收敛性; 可正则化的解; p- 能量极小元.