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On a New Mapping Related to Hadamard's Inequalities

YU Yong-xin, LIU Zheng

(Anshan University of Science and Technology, Liaoning 114002, China)

(E-mail: yyx8191258@163.com)

Abstract: In this paper, we introduce a new mapping in connection to a recent generalization of Hadamard's inequalities for convex functions which gives a continuous scale of refinements of the mentioned inequalities. Some applications are also mentioned.

Key words: Hadamard's inequalities; convex function; generalization; mapping.
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For refining the famous Hadamad's inequalities

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\le\frac{f(a)+f(b)}{2},$$

in which $f:[a,b]\to R$ be a convex function, Dragomir.S.S. established a mapping as follows[1]:

Let $f:[a,b]\to R$ be a convex function, $F:[0,1]\to R$ be defined by

$$F(t)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y.$$

Then

(i) $F(\tau+1/2)=F(1/2-\tau)$ for all τ in $[0,1/2]$;

(ii) F is convex on $[0,1]$;

(iii) We have

$$\sup_{t\in[0,1]}F(t)=F(0)=F(1)=\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x,$$

$$\inf_{t\in[0,1]}F(t)=F(1/2)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y;$$

(iv) The following inequality is valid:

$$f\left(\frac{a+b}{2}\right)\le F\left(\frac{1}{2}\right);$$

(v) F decreases monotonically on $[0,1/2]$ and increases monotonically on $[1/2,1]$;

(vi) For another mapping in [1]

$$H(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)\mathrm{d}x,\qquad t\in[0,1]\ ,$$

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we have the inequality $H(t) \le F(t)$ for all $t \in [0,1]$. By using this mapping F, Dragomir S.S. has proved some specific inequalities[1].

In a recent paper[2], Wang L.C. generalized the inequalities (1) as follows: Let $f : [a,b] \to R$ be a convex function, for $p, q \in (0,1)$, with $p+q=1$ and $\xi = pa+qb$, then

$$f(pa+qb) \le \frac{1}{b-a}\left(\frac{p}{q}\int_a^{\xi} f(x)\,\mathrm{d}x + \frac{q}{p}\int_{\xi}^{b} f(x)\,\mathrm{d}x\right) \le pf(a)+qf(b). \tag{3}$$

If f is a strictly convex function, the two inequalities in (3) are strict.

The main purpose of this paper is to introduce a new mapping in connection to the left inequality of (3) which is a generalization of the mapping F and gives a continuous scale of refinements of this inequality. Some applications are also mentioned.

Theorem *Let $f : [a,b] \to R$ be a convex function. Let mapping $\overline{F} : [0,1] \to R$ be defined by*

$$\begin{aligned}\overline{F}(t) = &\frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^{\xi}\int_a^{\xi} f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y + \int_{\xi}^{b}\int_a^{\xi} f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y + \right.\\ &\left.\int_a^{\xi}\int_{\xi}^{b} f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y + \frac{q^2}{p^2}\int_{\xi}^{b}\int_{\xi}^{b} f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y\right]\end{aligned} \tag{4}$$

in which p, q and ξ are as in (3). Then $\overline{F}$ has the following properties:

(i) $\overline{F}(\tau+1/2) = \overline{F}(1/2-\tau)$ for all τ in $[0,1/2]$;

(ii) $\overline{F}$ is convex on $[0,1]$. If f is strictly convex on $[a,b]$, $\overline{F}$ is strictly convex on $[0,1]$;

(iii) We have

$$\sup_{t\in[0,1]} \overline{F}(t) = \overline{F}(0) = \overline{F}(1) = \frac{1}{b-a}\left(\frac{p}{q}\int_a^{\xi} f(x)\,\mathrm{d}x + \frac{q}{p}\int_{\xi}^{b} f(x)\,\mathrm{d}x\right),$$

$$\begin{aligned}\inf_{t\in[0,1]} \overline{F}(t) = \overline{F}(1/2) = &\frac{1}{(b-a)^2}\left(\frac{p^2}{q^2}\int_a^{\xi}\int_a^{\xi} f\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y + \int_{\xi}^{b}\int_a^{\xi} f\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y + \right.\\ &\left.\int_a^{\xi}\int_{\xi}^{b} f\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y + \frac{q^2}{p^2}\int_{\xi}^{b}\int_{\xi}^{b} f\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y\right);\end{aligned}$$

(iv) The following inequality is valid:

$$f(pa+qb) \le \overline{F}(1/2).$$

If f is strictly convex on $[a,b]$, the inequality is strict.

(v) $\overline{F}$ decreases monotonically on $[0,1/2]$ and increases monotonically on $[1/2,1]$. If f is strictly convex on $[a,b]$, the corresponding monotony of $\overline{F}$ is strict.

(vi) For another mapping in connection to the inequalities (3)

$$\overline{H}(t) = \frac{1}{b-a}\left(p\int_{\frac{a}{q}}^{\frac{\xi}{q}} f(qtx+(1-t)\xi)\,\mathrm{d}x + q\int_{\frac{\xi}{p}}^{\frac{b}{p}} f(ptx+(1-t)\xi)\,\mathrm{d}x\right),$$

we have the inequality $\overline{H}(t) \leq \overline{F}(t)$, *for all* $t \in [0,1]$.

If f is strictly convex on $[a,b]$, the inequality for all t in $[0,1)$ is strict.

Proof (i) Let $\tau \in [0,1/2]$. By exchanging the integral variable x and y, we have

$$\begin{aligned}
\overline{F}\left(\tau+\frac{1}{2}\right) =& \frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^\xi\int_a^\xi f\left(\left(\tau+\frac{1}{2}\right)x+\left(\frac{1}{2}-\tau\right)y\right)\mathrm{d}x\mathrm{d}y+\right.\\
&\int_\xi^b\int_a^\xi f\left(\left(\tau+\frac{1}{2}\right)x+\left(\frac{1}{2}-\tau\right)y\right)\mathrm{d}x\mathrm{d}y+\\
&\int_a^\xi\int_\xi^b f\left(\left(\tau+\frac{1}{2}\right)x+\left(\frac{1}{2}-\tau\right)y\right)\mathrm{d}x\mathrm{d}y+\\
&\left.\frac{q^2}{p^2}\int_\xi^b\int_\xi^b f\left(\left(\tau+\frac{1}{2}\right)x+\left(\frac{1}{2}-\tau\right)y\right)\mathrm{d}x\mathrm{d}y\right]\\
=& \frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^\xi\int_a^\xi f\left(\left(\frac{1}{2}-\tau\right)x+\left(\tau+\frac{1}{2}\right)y\right)\mathrm{d}x\mathrm{d}y+\right.\\
&\int_a^\xi\int_\xi^b f\left(\left(\frac{1}{2}-\tau\right)x+\left(\tau+\frac{1}{2}\right)y\right)\mathrm{d}x\mathrm{d}y+\\
&\int_\xi^b\int_a^\xi f\left(\left(\frac{1}{2}-\tau\right)x+\left(\tau+\frac{1}{2}\right)y\right)\mathrm{d}x\mathrm{d}y+\\
&\left.\frac{q^2}{p^2}\int_\xi^b\int_\xi^b f\left(\left(\frac{1}{2}-\tau\right)x+\left(\tau+\frac{1}{2}\right)y\right)\mathrm{d}x\mathrm{d}y\right]\\
=& \overline{F}\left(\frac{1}{2}-\tau\right).
\end{aligned}$$

(ii) For $\alpha,\beta \in (0,1)$ with $\alpha+\beta=1$, $t_1,t_2 \in [0,1]$ and $x,y \in [a,b]$, by the convexity of f,

$$\begin{aligned}
&f\left((\alpha t_1+\beta t_2)x+(1-(\alpha t_1+\beta t_2))y\right)=f\left(\alpha(t_1x+(1-t_1)y)+\beta(t_2x+(1-t_2)y)\right)\\
&\quad\leq \alpha f(t_1x+(1-t_1)y)+\beta f(t_2x+(1-t_2)y). \qquad (5)
\end{aligned}$$

Substituting this relation in (4), we have

$$\overline{F}(\alpha t_1+\beta t_2) \leq \alpha\overline{F}(t_1)+\beta\overline{F}(t_2).$$

So $\overline{F}$ is convex on $[0,1]$.

If f is strictly convex on $[a,b]$, the inequalities (5) and (6) are strict, so $\overline{F}$ is strictly convex on $[0,1]$.

(iii) For all x,y on $[a,b]$ and t on $[0,1]$, we have

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y).$$

Then by (4), we get

$$\overline{F}(t) \leq \frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^\xi\int_a^\xi (tf(x)+(1-t)f(y))\,\mathrm{d}x\mathrm{d}y+\right.$$

$$\int_\xi^b\int_a^\xi (tf(x)+(1-t)f(y))\,\mathrm{d}x\mathrm{d}y+$$
$$\int_a^\xi\int_\xi^b (tf(x)+(1-t)f(y))\,\mathrm{d}x\mathrm{d}y+$$
$$\left.\frac{q^2}{p^2}\int_\xi^b\int_\xi^b (tf(x)+(1-t)f(y))\,\mathrm{d}x\mathrm{d}y\right]$$
$$=\frac{1}{b-a}\left[\frac{p^2}{q}\int_a^\xi f(x)\,\mathrm{d}x+qt\int_\xi^b f(x)\,\mathrm{d}x+\right.$$
$$(1-t)p\int_a^\xi f(x)\,\mathrm{d}x+tp\int_a^\xi f(x)\,\mathrm{d}x+$$
$$\left.q(1-t)\int_\xi^b f(x)\,\mathrm{d}x+\frac{q^2}{p}\int_\xi^b f(x)\,\mathrm{d}x\right]$$
$$=\frac{1}{b-a}\left(\frac{p}{q}\int_a^\xi f(x)\,\mathrm{d}x+\frac{q}{p}\int_\xi^b f(x)\,\mathrm{d}x\right)=\overline{F}(0)=\overline{F}(1).$$

Since f is convex on $[a,b]$, for all t on $[0,1]$ and x,y on $[a,b]$, we have

$$f\left(\frac{x+y}{2}\right)\le\frac{1}{2}\left(f(tx+(1-t)y)+f((1-t)x+ty)\right).$$

Then by (4), we get

$$\overline{F}\left(\frac{1}{2}\right)=\frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^\xi\int_a^\xi f\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y+\int_\xi^b\int_a^\xi f\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y+\right.$$
$$\left.\int_a^\xi\int_\xi^b f\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y+\frac{q^2}{p^2}\int_\xi^b\int_\xi^b f\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y\right]$$
$$\le\frac{1}{2(b-a)^2}\left[\frac{p^2}{q^2}\int_a^\xi\int_a^\xi f(tx+(1-t)y)\mathrm{d}x\mathrm{d}y+\int_\xi^b\int_a^\xi f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y+\right.$$
$$\int_a^\xi\int_\xi^b f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y+\frac{q^2}{p^2}\int_\xi^b\int_\xi^b f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y+$$
$$\frac{p^2}{q^2}\int_a^\xi\int_a^\xi f((1-t)x+ty)\,\mathrm{d}x\mathrm{d}y+\int_\xi^b\int_a^\xi f((1-t)x+ty)\mathrm{d}x\mathrm{d}y+$$
$$\left.\int_a^\xi\int_\xi^b f((1-t)x+ty)\,\mathrm{d}x\mathrm{d}y+\frac{q^2}{p^2}\int_\xi^b\int_\xi^b f((1-t)x+ty)\,\mathrm{d}x\mathrm{d}y\right]$$
$$=\frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^\xi\int_a^\xi f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y+\int_\xi^b\int_a^\xi f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y+\right.$$
$$\left.\int_a^\xi\int_\xi^b f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y+\frac{q^2}{p^2}\int_\xi^b\int_\xi^b f(tx+(1-t)y)\,\mathrm{d}x\mathrm{d}y\right]$$
$$=\overline{F}(t).$$

(iv) By using Jensen's inequality for double integrals, with simple computations, we have

$$\int_a^\xi \int_a^\xi f\left(\frac{x+y}{2}\right) \mathrm{d}x\mathrm{d}y \geq (\xi-a)^2 f\left(\frac{1}{(\xi-a)^2}\int_a^\xi \int_a^\xi \left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y\right)$$
$$= q^2(b-a)^2 f\left(\frac{\xi+a}{2}\right),$$
$$\int_\xi^b \int_a^\xi f\left(\frac{x+y}{2}\right) \mathrm{d}x\mathrm{d}y \geq (b-\xi)(\xi-a) f\left(\frac{1}{(b-\xi)(\xi-a)}\int_\xi^b \int_a^\xi \left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y\right)$$
$$= pq(b-a)^2 f\left(\frac{2\xi+a+b}{4}\right),$$
$$\int_a^\xi \int_\xi^b f\left(\frac{x+y}{2}\right) \mathrm{d}x\mathrm{d}y \geq (\xi-a)(b-\xi) f\left(\frac{1}{(\xi-a)(b-\xi)}\int_a^\xi \int_\xi^b \left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y\right)$$
$$= pq(b-a)^2 f\left(\frac{2\xi+a+b}{4}\right),$$
$$\int_\xi^b \int_\xi^b f\left(\frac{x+y}{2}\right) \mathrm{d}x\mathrm{d}y \geq (b-\xi)^2 f\left(\frac{1}{(b-\xi)^2}\int_\xi^b \int_\xi^b \left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y\right)$$
$$= p^2(b-a)^2 f\left(\frac{\xi+b}{2}\right),$$

therefore

$$\overline{F}\left(\frac{1}{2}\right) \geq p^2 f\left(\frac{\xi+a}{2}\right) + pqf\left(\frac{2\xi+a+b}{4}\right) + pqf\left(\frac{2\xi+a+b}{4}\right) + q^2 f\left(\frac{\xi+b}{2}\right)$$
$$\geq pf\left(\frac{3}{4}\xi+\frac{a}{4}\right) + qf\left(\frac{3}{4}\xi+\frac{b}{4}\right) \geq f(pa+qb). \tag{7}$$

If f is strictly convex on $[a,b]$, the inequalities (7) are strict.

(v) Since $\overline{F}(t)$ is convex on $[0,1]$, we have for $t_1, t_2 \in (1/2, 1]$, $t_1 < t_2$,

$$\frac{\overline{F}(t_2)-\overline{F}(t_1)}{t_2-t_1} \geq \overline{F}_+(t_1) = \frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^\xi \int_a^\xi f_+(t_1x+(1-t_1)y)(x-y)\,\mathrm{d}x\mathrm{d}y + \right.$$
$$\int_\xi^b \int_a^\xi f_+(t_1x+(1-t_1)y)(x-y)\,\mathrm{d}x\mathrm{d}y +$$
$$\int_a^\xi \int_\xi^b f_+(t_1x+(1-t_1)y)(x-y)\,\mathrm{d}x\mathrm{d}y +$$
$$\left. +\frac{q^2}{p^2}\int_\xi^b \int_\xi^b f_+(tx+(1-t)y)(x-y)\,\mathrm{d}x\mathrm{d}y\right]. \tag{8}$$

By the convexity of f on $[a,b]$, we deduce

$$f\left(\frac{x+y}{2}\right) - f(t_1x+(1-t_1)y) \geq f_+(t_1x+(1-t_1)y)\frac{(x-y)(1-2t_1)}{2}$$

for all x, y on $[a,b]$ and $t_1 \in (1/2, 1)$, which is equivalent to

$$(x-y)f_+(t_1x+(1-t_1)y) \geq \frac{2}{2t_1-1}\left(f(t_1x+(1-t_1)y) - f\left(\frac{x+y}{2}\right)\right). \tag{9}$$

Integrating the inequality (9) on $[a,\xi]\times[a,\xi]$,$[\xi,b]\times[a,\xi]$,$[a,\xi]\times[\xi,b]$ and $[\xi,b]\times[\xi,b]$ respectively, and substituting them in (8), we can obtain

$$\frac{\overline{F}(t_2)-\overline{F}(t_1)}{t_2-t_1}\geq\overline{F}_+(t_1)\geq\frac{2}{2t_1-1}\left[\overline{F}(t_1)-\overline{F}\left(\frac{1}{2}\right)\right]\geq 0,\quad t_1\in(1/2,1)\ , \tag{10}$$

which shows that $\overline{F}$ increases monotonically on $[1/2,1]$.

The fact that $\overline{F}$ decreases monotonically on $[0,1/2]$ follows from the above conclusion by using statement (i).

If f is strictly convex on $[a,b]$, because of the monotony of $\overline{F}$ on $[1/2,1]$, the right inequality in (10) is strict, and so $\overline{F}$ increases monotonically strictly on $(1/2,1)$. Therefore, $\overline{F}$ increases monotonically strictly on $[1/2,1]$, and decreases monotonically strictly on $[0,1/2]$ correspondingly. So the monotonies of $\overline{F}$ on $[0,1/2]$ or on $[1/2,1]$ are both strict.

(vi) By using Jensen's integral inequality, after simple computations, for all t in $(0,1)$ we have

$$\begin{aligned}\overline{F}(t)\geq&\frac{1}{b-a}\left[\frac{p^2}{q}\int_a^\xi f\left(\frac{1}{q(b-a)}\int_a^\xi(tx+(1-t)y)\,\mathrm{d}y\right)\mathrm{d}x+\right.\\&q\int_\xi^b f\left(\frac{1}{q(b-a)}\int_a^\xi(tx+(1-t)y)\,\mathrm{d}y\right)\mathrm{d}x+\\&p\int_a^\xi f\left(\frac{1}{p(b-a)}\int_\xi^b(tx+(1-t)y)\,\mathrm{d}y\right)\mathrm{d}x+\\&\left.\frac{q^2}{p}\int_\xi^b f\left(\frac{1}{p(b-a)}\int_\xi^b(tx+(1-t)y)\mathrm{d}y\right)\mathrm{d}x\right]\\=&\frac{1}{b-a}\left[\frac{p^2}{q}\int_a^\xi f\left(tx+(1-t)\frac{\xi+a}{2}\right)\mathrm{d}x+q\int_\xi^b f\left(tx+(1-t)\frac{\xi+a}{2}\right)\mathrm{d}x+\right.\\&\left.p\int_a^\xi f\left(tx+(1-t)\frac{\xi+b}{2}\right)\mathrm{d}x+\frac{q^2}{p}\int_\xi^b f\left(tx+(1-t)\frac{\xi+b}{2}\right)\mathrm{d}x\right]\\\geq&\frac{1}{b-a}\left[\frac{p}{q}\int_a^\xi f(tx+(1-t)\xi)\,\mathrm{d}x+\frac{q}{p}\int_\xi^b f(tx+(1-t)\xi)\,\mathrm{d}x\right]\\=&\frac{1}{b-a}\left(p\int_{\frac{a}{q}}^{\frac{\xi}{q}} f(qtx+(1-t)\xi)\,\mathrm{d}x+q\int_{\frac{\xi}{p}}^{\frac{b}{p}} f(ptx+(1-t)\xi)\,\mathrm{d}x\right)\\=&\overline{H}(t).\end{aligned}$$

If f is strictly convex on $[a,b]$, for $t\in[0,1)$ the above inequalities are strict, so we have

$$\overline{F}(t)>\overline{H}(t),\quad t\in[0,1).$$

The proof is complete.

Remark It is clear that the mapping F is just a special case of the mapping $\overline{F}$ for $p=q=1/2$.

Corollary *Let the assumptions of Theorem hold. Then for all $t \in [0,1]$, we have*

$$f(pa+qb) \le \overline{F}\left(\frac{1}{2}\right) \le \overline{F}(t) \le \frac{1}{b-a}\left(\frac{p}{q}\int_a^{\xi} f(x)\,\mathrm{d}x + \frac{q}{p}\int_{\xi}^{b} f(x)\,\mathrm{d}x\right). \tag{11}$$

If f is strictly convex on $[a,b]$, then the left inequality and the right inequality in (11) are strict.

Let $f(x)=x^r$ $(r>1)$, we have

$$\begin{aligned}(pa+qb)^r <& \frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^{\xi}\int_a^{\xi}\left(\frac{x+y}{2}\right)^r \mathrm{d}x\mathrm{d}y + 2\int_{\xi}^{b}\int_a^{\xi}\left(\frac{x+y}{2}\right)^r \mathrm{d}x\mathrm{d}y + \right.\\ &\left.\frac{q^2}{p^2}\int_{\xi}^{b}\int_{\xi}^{b}\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y\right]\\ \le& \frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^{\xi}\int_a^{\xi}(tx+(1-t)y)^r\,\mathrm{d}x\mathrm{d}y + \int_{\xi}^{b}\int_a^{\xi}(tx+(1-t)y)^r\,\mathrm{d}x\mathrm{d}y\right.\\ \le& \frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^{\xi}\int_a^{\xi}(tx+(1-t)y)^r\,\mathrm{d}x\mathrm{d}y + \int_{\xi}^{b}\int_a^{\xi}(tx+(1-t)y)^r\,\mathrm{d}x\mathrm{d}y + \right.\\ &\left.\int_a^{\xi}\int_{\xi}^{b}(tx+(1-t)y)^r + \frac{q^2}{p^2}\int_{\xi}^{b}\int_{\xi}^{b}(tx+(1-t)y)^r\,\mathrm{d}x\mathrm{d}y\right]\\ <& \frac{1}{(r+1)(b-a)}\left(\frac{q}{p}b^{r+1} + \left(\frac{p}{q}-\frac{q}{p}\right)\xi^{r+1} - \frac{p}{q}a^{r+1}\right)^r.\end{aligned}$$

Which has generalized and refined the inequality

$$\left(\frac{a+b}{2}\right)^r < \frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)} \quad (r>1).$$

Let $f(x)=e^x$ in (11), we have

$$\begin{aligned}e^{pa+qb} <& \frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^{\xi}\int_a^{\xi} e^{\frac{x+y}{2}}\mathrm{d}x\mathrm{d}y + 2\int_{\xi}^{b}\int_a^{\xi} e^{\frac{x+y}{2}}\mathrm{d}x\mathrm{d}y + \frac{q^2}{p^2}\int_{\xi}^{b}\int_{\xi}^{b} e^{\frac{x+y}{2}}\mathrm{d}x\mathrm{d}y\right]\\ \le& \frac{1}{(b-a)^2}\left[\frac{p^2}{q^2}\int_a^{\xi}\int_a^{\xi} e^{tx+(1-t)y}\mathrm{d}x\mathrm{d}y + \int_{\xi}^{b}\int_a^{\xi} e^{tx+(1-t)y}\mathrm{d}x\mathrm{d}y + \right.\\ &\left.\int_a^{\xi}\int_{\xi}^{b} e^{tx+(1-t)y}\mathrm{d}x\mathrm{d}y + \frac{q^2}{p^2}\int_{\xi}^{b}\int_{\xi}^{b} e^{tx+(1-t)y}\mathrm{d}x\mathrm{d}y\right]\\ <& \frac{1}{b-a}\left(\frac{q}{p}e^b + \left(\frac{p}{q}-\frac{q}{p}\right)e^{pa+qb} - \frac{p}{q}e^a\right).\end{aligned}$$

Which has generalized and refined the inequality

$$e^{\frac{a+b}{2}} < \frac{e^b-e^a}{b-a}.$$

Let $f(x)=\frac{1}{x}$, $0<a<b$ in (11), we have

$$\frac{1}{pa+qb} < \frac{1}{(b-a)^2}\left(\frac{p^2}{q^2}\int_a^{\xi}\int_a^{\xi}\frac{2}{x+y}\mathrm{d}x\mathrm{d}y + 2\int_{\xi}^{b}\int_a^{\xi}\frac{2}{x+y}\mathrm{d}x\mathrm{d}y + \right.$$

$$\frac{q^2}{p^2}\int_{\xi}^{b}\int_{\xi}^{b}\frac{2}{x+y}\mathrm{d}x\mathrm{d}y\Bigg) \le \frac{1}{(b-a)^2}\left(\frac{p^2}{q^2}\int_a^{\xi}\int_a^{\xi}\frac{\mathrm{d}x\mathrm{d}y}{tx+(1-t)y}+\right.$$

$$\int_{\xi}^{b}\int_a^{\xi}\frac{\mathrm{d}x\mathrm{d}y}{tx+(1-t)y}+\int_a^{\xi}\int_{\xi}^{b}\frac{\mathrm{d}x\mathrm{d}y}{tx+(1-t)y}+$$

$$\left.\frac{q^2}{p^2}\int_{\xi}^{b}\int_{\xi}^{b}\frac{\mathrm{d}x\mathrm{d}y}{tx+(1-t)y}\right) < \frac{1}{b-a}\left(\frac{q}{p}\ln b+\left(\frac{p}{q}-\frac{q}{p}\right)\ln\xi-\frac{p}{q}\ln a\right).$$

Which has generalized and refined the inequality

$$\frac{2}{a+b} < \frac{\ln b-\ln a}{b-a}.$$

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一个新的与 Hadamard 不等式相关的映射

于永新， 刘 证

(鞍山科技大学数学系，辽宁 鞍山 114002)

摘要: 对于一个最近发表的凸函数的 Hadamard 不等式的推广形式的不等式，本文引进了一个与这个不等式相关的的映射，从而给出了该不等式连续的加细．同时提及了它的某些应用．

关键词: Hadamard 不等式；凸函数；推广；映射．