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Some Applications of $\mathcal{L}$-injective Envelopes

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Abstract: As applications of $\mathcal{L}$-injective envelopes, we study some properties of the homomorphism of two modules which have isomorphic $\mathcal{L}$-injective envelopes.

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1. Introduction

Throughout this paper, R will be an associative ring with identity, a module will mean a right R-module, and the symbol $\mathcal{L}$ will denote a class of modules closed under isomorphisms. We freely use the terminology and notationies of Anderson and Fuller[1].

The concepts of envelopes and covers of modules have been studied by serval authors (e.g., [2–7], [12]). Recently, we studied $\mathcal{L}$-injective modules and $\mathcal{L}$-projective modules in [11] and obtained some properties which are similar to those of injective modules and projective modules, and then we used these two concepts to introduce and investigate n-$\mathcal{L}$-injective envelopes and n-$\mathcal{L}$-projective covers in [12], and provided some characterizations of their existences. In particular, taking $\mathcal{L}$ a special class of right R-modules we can obtain the characterizations of different known envelopes and covers (e.g., cotorsion envelopes, flat covers).

As we have known, if two modules are isomorphic, then their $\mathcal{L}$-injective envelopes are isomorphic. However, it is easy to see that the converse is false. Motiviated by these, in this paper we mainly study the properties of a homomorphism $\varphi : M_1 \to M_2$ (it is called below an $\mathcal{L}$-injective enveloping homomorphism) such that there exists an isomorphism $g : E_1 \to E_2$ with $g\varphi_1 = \varphi_2 f$, where $\varphi_i : M_i \to E_i$ is an $\mathcal{L}$-injective envelope($i = 1, 2$). We give some relations between $\mathcal{L}$-injective enveloping homomorphisms and $\mathcal{L}$-injective envelopes, and obtain some necessary and equivalent conditions for a homomorphism to be an $\mathcal{L}$-injective enveloping homomorphism.

2. Preliminaries

In this section we recall some notions and facts which we will use in the later sections.

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As a generalization of injective modules, for a given class of right R-modules, denoted by $\mathcal{L}$, a right R-module M is called an *$\mathcal{L}$-injective module*[11], if every diagram with exact row and $L \in \mathcal{L}$

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "\eta"] \arrow[d, "f"'] & K \arrow[r] \arrow[dl, dotted, "\overline{f}"] & L \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

can be completed along the dotted arrow to commute. Obviously, every injective module is $\mathcal{L}$-injective. Dually[10] also studied $\mathcal{L}$-projective modules, and obtained some interesting properties similar to those of the injective and the projective modules.

In the following part, the class of all $\mathcal{L}$-injective (resp. $\mathcal{L}$-projective) right R-modules will be denoted by $I(\mathcal{L})$ (resp. $P(\mathcal{L})$). The notations $P(I(\mathcal{L}))$ and $I(P(\mathcal{I}))$ are defined similarly. From [11] we have the facts that $I(\mathcal{L}) = I(P(I(\mathcal{L}))$ and $P(\mathcal{L}) = P(I(P(\mathcal{L})))$.

Let $M \in$ Mod-R. A homomorphism $\varphi : M \to E$ is called *an $\mathcal{L}$-injective envelope* of M[12], if E is $\mathcal{L}$-injective and the following conditions hold:

(1) For each homomorphism $\psi : M \to F$ with F $\mathcal{L}$-injective there exists a homomorphism $g : E \to F$ such that $\psi = g\varphi$

$$\begin{tikzcd} M \arrow[d, "\varphi"'] \arrow[dr, "\psi"] & \\ E \arrow[r, "g"] & F \end{tikzcd}$$

(2) If g is an endomorphism of E such that $\varphi = g\varphi$

$$\begin{tikzcd} M \arrow[d, "\varphi"'] \arrow[dr, "\varphi"] & \\ E \arrow[r, "g"] & E \end{tikzcd}$$

then g must be an automorphism.

If (1) holds (maybe not (2)), we call it *an $\mathcal{L}$-injective preenvelope* of M. From this definition we easily see that φ must be a monomophism, and it was showed that if φ is an $\mathcal{L}$-injective envelope, then coker$\varphi \in P(I(\mathcal{L}))$. Dually, we define $\mathcal{L}$-projective (pre)covers of a module.

Taking $\mathcal{L}$ a special class of right R-modules we can obtain some known envelopes and covers. For instance, the flat covers coincide with the $\mathcal{L}$-projective covers when taking $\mathcal{L}$ the class of all right cotorsion modules. On the other hand, we see from [8] and [11] that the $\mathcal{L}$-injective and the $\mathcal{L}$-projective modules satisfy many properties similar to that of injective and projective modules, which are useful in the following considerations of relative envelopes and covers, and related topics.

3. Enveloping homomorphisms

As applications, in this section, we assume that for each right R-module its $\mathcal{L}$-injective envelopes always exist, and study some properties of the morphisms of right R-modules whose envelopes are isomorphic.

Definition 3.1 *If $f: M_1 \to M_2$ is a homomorphism and $\varphi_1: M_1 \to E_1, \varphi_2: M_2 \to E_2$ are $\mathcal{L}$-injective prenvelopes, thus φ_1 is monic and coker$\varphi_1 \in P(I(\mathcal{L}))$ by [12, Theorem 3.3], then the diagram*

$$\begin{array}{ccc} M_1 & \xrightarrow{\varphi_1} & E_1 \\ f\downarrow & & \downarrow g \\ M_2 & \xrightarrow{\varphi_2} & E_2 \end{array}$$

can be completed to a commutative diagram by the definition of $\mathcal{L}$-injectivity. In this situation, g is called an extending of f (relative to the two preenvelopes).

The following mainly concerns with such extendings when φ_1 and φ_2 are envelopes. it is easy to see that if $f: M_1 \to M_2$ is an isomorphism, so is any extending by [12, Proposition 3.2]. However, there are examples with g is an isomorphism where f is not.

We now aim to study the homomorphism f such that an extending g is an isomorphism.

Proposition 3.2 *Assume that $\varphi_i: M_i \to E_i$ is an $\mathcal{L}$-injective envelope $(i = 1, 2)$, $f: M_1 \to M_2$ is a homomorphism. Then f has some extending which is an isomorphism if and only every extending of f is an isomorphism.*

Proof Suppose that the extending $g: E_1 \to E_2$ of f is an isomorphism. Let $h: E_1 \to E_2$ be an arbitrary extending of f, then $h\varphi_1 = \varphi_2 f = g\varphi_1$, hence $g^{-1}h\varphi_1 = \varphi_1$. Since φ_1 is an $\mathcal{L}$-injective envelope, $g^{-1}h$ must be an isomorphism, so is h. □

Thus we have the following definition.

Definition 3.3 *A homomorphism $f: M_1 \to M_2$ is said to be $\mathcal{L}$-injective enveloping, if M_1 and M_2 have $\mathcal{L}$-injective envelopes $\varphi_1: M_1 \to E_1$, $\varphi_2: M_2 \to E_2$ and every extending $g: E_1 \to E_2$ is an isomorphism.*

The homomorphism is said to be *$\mathcal{L}$-projective covering*, if the dual situation holds.

Proposition 3.4 *If $f: M_1 \to M_2$ is an $\mathcal{L}$-injective enveloping homomorphism, then f is monic.*

Proof Suppose that $\varphi_1: M_1 \to E_1, \varphi_2: M_2 \to E_2$ are $\mathcal{L}$-injective envelope, $g: E_1 \to E_2$ is extending of f, then $g\varphi_1 = \varphi_2 f$ is monic, so is f. □

We first give some relations between $\mathcal{L}$-injectvie enveloping homomorphisms and $\mathcal{L}$-injective envelopes.

Proposition 3.5 *If $\varphi_2: M_2 \to E_2$ is an $\mathcal{L}$-injective envelope and $f: M_1 \to M_2$ is a homomorphism, then f is $\mathcal{L}$-injective enveloping if and only if $\varphi_2 f: M_1 \to E_2$ is an $\mathcal{L}$-injective envelope.*

Proof $(\Rightarrow)$. Suppose that $\varphi_1: M_1 \to E_1$ is an $\mathcal{L}$-injective envelope. It suffices to show that $E_1 \cong E_2$. There exists a homomorphism $g: E_1 \to E_2$ such that g is an extending of f, thus g is an isomorphism since f is $\mathcal{L}$-injective enveloping.

($\Leftarrow$). By hypothesis and [12, Proposition 3.2] $g: E_1 \to E_2$ is an isomorphism such that $\varphi_2 f = g\varphi_1$, that is, g is an extending, thus f is $\mathcal{L}$-injective enveloping. $\Box$

Proposition 3.6 *If $\varphi_1 : M_1 \to E_1$ is an $\mathcal{L}$-injective envelope and $f : M_1 \to M_2$ is a homomorphism, then f is $\mathcal{L}$-injective enveloping if and only if there is a homomorphism $h : M_2 \to E_1$ with $hf = \varphi_1$ such that h is an $\mathcal{L}$-injective envelope.*

Proof ($\Rightarrow$). Let $\varphi_2 : M_2 \to E_2$ be an $\mathcal{L}$-injective envelope, g an extending of f. Then g is an isomorphism, thus let $h = g^{-1}\varphi_2$, we have $h : M_2 \to E_1$ such that $hf = \varphi_1$ and h is an $\mathcal{L}$-injective envelope since $g : E_1 \to E_2$ is an isomorphism.

($\Leftarrow$). Since h is an $\mathcal{L}$-injective envelope, we have $g : E_1 \to E_2$ is an isomorphism. It is easy to see that g is an extending of f, so f is $\mathcal{L}$-injective enveloping. $\Box$

Proposition 3.7 *E is an $\mathcal{L}$-injective if and only if every $\mathcal{L}$-injective enveloping homomorphism $E \to M$ is an isomorphism.*

Proof ($\Rightarrow$). Let E be an $\mathcal{L}$-injective. Then $id_E : E \to E$ is an $\mathcal{L}$-injective envelope. Let $f : E \to M$ be an $\mathcal{L}$-injective enveloping homomorphism, $\varphi : M \to F$ an $\mathcal{L}$-injective envelope and $g : F \to E$ an extending of f (so an isomorphism). Since φ is injective and $\varphi f = g(id_E)$ is an isomorphism, thus φ is surjective and hence φ is an isomorphism. Therefore f is an isomorphism.

($\Leftarrow$). Suppose that $\varphi : E \to F$ is an $\mathcal{L}$-injective envelope. Note that $id_F\varphi : E \to F$ is an $\mathcal{L}$-injective envelope, thus φ is $\mathcal{L}$-injective enveloping by Proposition 3.6. So φ is an isomorphism, that is, E is $\mathcal{L}$-injective. $\Box$

Next, we provide some necessary conditions for a homomorphism to be $\mathcal{L}$-injective enveloping.

Proposition 3.8 *Let $f : M_1 \to M_2$ be an $\mathcal{L}$-injective enveloping morphism. Then* $\mathrm{Im}f$ *can not be contained in a proper direct summand of M_2.*

Proof Suppose that there exists a decomposition $M_2 = X \oplus Y$ such that $\mathrm{Im}f \subseteq X$ and $Y \neq 0$. Let $\varphi_1 : M_1 \to E_1, \varphi_2 : M_2 \to E_2$ be $\mathcal{L}$-injective envelopes. Thus there exists a decomposition $E_2 = E_X \oplus E_Y$ such that $\varphi_X : X \to E_X, \varphi_Y : Y \to E_Y$ are $\mathcal{L}$-injective envelopes, hence there exists $k : E_1 \to E_X$ such that $\varphi_X f = k\varphi_1$. Let $g : E_1 \to E_2 = E_X \oplus E_Y, via\ e_1\mathcal{L}$ ongmapsto $(k(e_1), 0)$. Obviously, g is not surjective and $g\varphi_1 = \varphi_2 f$. A contradiction. $\Box$

Recall that $\mathcal{L}$ is said to be ker-closed, if whenever $0 \to F' \to F \to F'' \to 0$ is exact with $F, F'' \in \mathcal{L}$, F' is also in $\mathcal{L}$. $\mathcal{L}$ is said to be coker-closed if it satisfies the dual properties. Using the long exact sequence of Ext, it is not hard to argue that if $\mathcal{L}$ is ker-closed, thus $I(\mathcal{L})$ is coker-closed.

Proposition 3.9 *Assume that* $\mathrm{P(I(}\mathcal{L}))$ *is ker-closed. If a morphism $f : M_1 \to M_2$ is $\mathcal{L}$-injective enveloping, then* $\mathrm{coker}f \in \mathrm{P(I(}\mathcal{L}))$.

Proof Suppose that $\varphi_1 : M_1 \to E_1, \varphi_2 : M_2 \to E_2$ are $\mathcal{L}$-injective envelope, $g : E_1 \to E_2$ is extending of f, then $g\varphi_1 = \varphi_2 f$. Hence the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{f} & M_2 & \longrightarrow & \mathrm{coker}f & \longrightarrow & 0 \\ & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow & & \\ 0 & \longrightarrow & E_1 & \xrightarrow{g} & E_2 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}.$$

By the Snake Lemma we get an exact sequence

$$0 \to \mathrm{coker}f \to \mathrm{coker}\varphi_2 \to \mathrm{coker}\varphi_1 \to 0.$$

By [12, Theorem 3.3] $\mathrm{coker}\varphi_1, \mathrm{coker}\varphi_2 \in \mathrm{P}(\mathrm{I}(\mathcal{L}))$. Since $\mathrm{P}(\mathrm{I}(\mathcal{L}))$ is ker-closed, $\mathrm{coker}f \in \mathrm{P}(\mathrm{I}(\mathcal{L}))$.

This result says that if $\mathrm{P}(\mathrm{I}(\mathcal{L}))$ is ker-closed such that M_1, M_2 have $\mathcal{L}$-injective envelopes and a morphism $f: M_1 \to M_2$ is $\mathcal{L}$-injective enveloping, then $f \in \mathrm{Ext}^1(\mathrm{P}(\mathrm{I}(\mathcal{L})), M_1)$. The same type of results holds for covering morphisms. Following the same type of argument we get

Proposition 3.10 *Assume that* $\mathrm{I}(\mathrm{P}(\mathcal{L}))$ *is coker-closed such that* M_1, M_2 *both have* $\mathcal{L}$*-projective covers. If a morphism* $f: M_1 \to M_2$ *is* $\mathcal{L}$*-projective covering, then* f *is surjective and* $\ker f \in \mathrm{I}(\mathrm{P}(\mathcal{L}))$.

We now provide an equivalent characterization of $\mathcal{L}$-injective enveloping homomorphisms. Recall that $\mathcal{L}$ is *homomorphically closed*[8], if A $\to$ B $\to$ 0 with A $\in \mathcal{L}$, then B $\in \mathcal{L}$; $\mathcal{L}$ is said to be *hereditary*, if $0 \to B \to A$ with $A \in \mathcal{L}$, then $B \in \mathcal{L}$.

Theorem 3.11 *Assume that* $\mathcal{L}$ *is homomorphically closed,* $f: M_1 \to M_2$ *is a homomorphism with* $M_2/f(M_1) \in \mathcal{L}$*, and* $\varphi_1: M_1 \to E_1$ *is an* $\mathcal{L}$*-injective envelope with* $E_1/\varphi_1(M_1) \in \mathcal{L}$*. Then* f *is* $\mathcal{L}$*-injective enveloping if and only if* f *is monic and* $\mathrm{Im}f$ *is essential in* M_2*.*

Proof ($\Rightarrow$). If $g: E_1 \to E_2$ is an extending of f (and so an isomorphism), then $g\varphi_1 = \varphi_2 f$ is injective, so f is injective. Now if $\mathrm{Im}f \cap L = 0$ for some $L \subseteq M_2$, thus $k: M_1 \to M_2/L$, via $m_1 \longmapsto f(m_1)$ is injective and we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{f} & M_2 & \longrightarrow & M_2/f(M_1) & \longrightarrow & 0 \\ & & \downarrow 1_M & & \downarrow \pi & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \xrightarrow{k} & M_2/L & \longrightarrow & C & \longrightarrow & 0 \end{array}.$$

So $C \in \mathcal{L}$, since $M_2/f(M_1) \in \mathcal{L}$ and $\mathcal{L}$ is homomorphically closed. Now because E_1 is $\mathcal{L}$-injective, there is an $h: M_2/L \to E_1$ such that $hk = \varphi_1$, that is, $h\pi f = \varphi_1$. By Proposition 3.6, $h\pi: M_2 \to E_1$ is an $\mathcal{L}$-injective envelope and so injective, hence $L = 0$, that is, $\mathrm{Im}f$ is essential in M_2.

($\Leftarrow$). If $\varphi_1: M_1 \to E_1$ is an $\mathcal{L}$-injective envelope, there exists an $h: M_2 \to E_1$ such that $\varphi_1 = hf$ since f is injective. Then h is injective since $\mathrm{Im}f$ is essential in M_2 and φ_1 is injective. Also $\mathrm{Im}\varphi_1 \subseteq \mathrm{Im}h$ and $\mathrm{Im}\varphi_1$ is essential in E_1 by [12, Theorem 3.8]. So $\mathrm{Im}h$ is essential in E_1. Note that $E_1/h(M_2)$ is a homomorphism image of $M_1/\mathrm{Im}\varphi_1 \in \mathcal{L}$, so $E_1/h(M_2) \in \mathcal{L}$. By [12, Theorem 3.8], h is an $\mathcal{L}$-injective envelope, so f is $\mathcal{L}$-injective enveloping by Proposition 3.6. □

Corollary 3.12 *A homomorphism* $f: M_1 \to M_2$ *is injective enveloping if and only if* f *is monic and* $\mathrm{Im}f$ *is essential in* M_2 *(i.e.,* M_1 *is an essential submodule of* M_2 *under isomorphisms).*

Dually we have the following results.

Theorem 3.13 *Assume that $\mathcal{L}$ is hereditary, $f : M_1 \to M_2$ is a homomorphism with* $\ker f \in \mathcal{L}$ *and $\psi_2 : F_2 \to M_2$ is an $\mathcal{L}$-projective cover with* $\ker\psi_2 \in \mathcal{L}$. *Then f is $\mathcal{L}$-projective covering if and only if f is surjective and* $\ker f$ *is superfluous in M_1.*

For two rings R and S, a bimodule ${}_S\mathrm{U}_R$ is said to *define a Morita duality*, if ${}_S\mathrm{U}_R$ is a faithfully balanced bimodule such that ${}_S\mathrm{U}$ and U_R are injective cogenerators. A presentation of Morita duality can be found in [1, §23, §24] and [9]. If M is a right R-module (left S-module), we let $M^* = {}_S\mathrm{Hom}_R(M,U)(= \mathrm{Hom}_S(M,U)_R)$, $\mathcal{L}^* = \{L^* | L \in \mathcal{L}\}$ and M is said to be *U-reflexive* if the evalution homomorphism $e_M : M \to M^{**}$ is an isomorphism. According to [1] let $R_R[U]$ and $R_S[U]$ denote the class of all U-reflexive right R-modules and that of all U-reflexive left S-modules, respectively. It is showed in [12] that if $\mathcal{L} \subseteq R_R[U]$ then the $\mathcal{L}$-injective (resp. the $\mathcal{L}$-projective) envelopes (resp. covers) and the $\mathcal{L}^*$-projective (resp. the $\mathcal{L}^*$-injective) covers (resp. envelopes) are dual to each other under Morita duality.

Theorem 3.14 *Let ${}_SU_R$ define a Morita duality, $\mathcal{L} \subseteq \mathrm{R}_R[U]$, $M_1, M_2 \in \mathrm{R}_R[U]$, and $\varphi_1 : M_1 \to E_1$ be an $\mathcal{L}$-injective envelope with* $\mathrm{coker}\varphi_1 \in \mathcal{L}$. *If $\mathcal{L}$ is homomorphically closed, then $f : M_1 \to M_2$ is $\mathcal{L}$-injective enveloping with* $\mathrm{coker} f \in \mathcal{L}$ *if and only if $f^* : M_2^* \to M_1^*$ in Mod-S is $\mathcal{L}^*$-projective covering with* $\ker f^* \in \mathcal{L}^*$.

Proof ($\Rightarrow$). By Proposition 3.6 there exists $\varphi_2 : M_2 \to E_2$ which is an $\mathcal{L}$-injective envelope with $\mathrm{coker}\varphi_2 \in \mathcal{L}$. By [12, Proposition 3.10] $\varphi_i^* : E_i^* \to M_i^*$ is $\mathcal{L}^*$-projective cover with $\ker\varphi_i^* \in \mathcal{L}^*$. On the other hand, by Theorem 3.11 and [1] f^* is surjective and $\ker f^*$ is superfluous in M_2^*, hence $f^* : M_2^* \to M_1^*$ is $\mathcal{L}^*$-projective covering with $\ker f^* \in \mathcal{L}^*$.

($\Leftarrow$). Dually. □

Take $\mathcal{L} = \mathrm{R}_R[U]$, since we assume ${}_SU_R$ define a Morita duality, $R/I \in R_R[U]$ for every right ideal I of R. Hence by the Baer criterion, every $R_R[U]$-injective R-module is injective. Similarly every ${}_SR[U]$-injective left S-module is injective. We immediately have

Corollary 3.15 *Let ${}_SU_R$ define a Morita duality. $\varphi_1 : M_1 \to E_1$ is an injective envelope with $M_1, E_1 \in {}_R\mathrm{R}[U]$. Then $f : M_1 \to M_2$ is injective enveloping if and only if $f^* : M_2^* \to M_1^*$ in Mod-S is ${}_S\mathrm{R}[U]$-projective covering.*

Corollary 3.16 *Let ${}_SU_R$ define a Morita duality. $\psi_2 : F_2 \to M_2$ is a ${}_RR[U]$-projective cover with $M_2, F_2 \in {}_R\mathrm{R}[U]$. Then $f : M_1 \to M_2$ is ${}_RR[U]$-projective covering if and only if f is epic and* $\ker f$ *is superfluous in M_1.*

Let R be a commutative ring with a Morita duality. By [10, Theorem 4.8], there is an R-bimodule ${}_RU_R$ which defines a Morita duality, i.e., a self-duality. It follows [11, Corollary 10] that a U-reflexive R-module is $\mathrm{R}_R[U]$-projective if and only if it is flat. Hence we have

Corollary 3.17 *Assume that R is a commutative ring, and ${}_RU_R$ define a Morita duality, $\varphi_1 : M_1 \to E_1$ is an injective envelope with $M_1, E_1 \in {}_R\mathrm{R}[U]$. Then $f : M_1 \to M_2$ is injective* *enveloping if and only if $f^* : M_2^* \to M_1^*$ is flat covering.*

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$\mathcal{L}$- 内射包络的一些应用

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摘要: 作为 $\mathcal{L}$- 内射包络的应用，本文主要研究具有同构的 $\mathcal{L}$- 内射包络的模之间的同态的若干性质.

关键词: $\mathcal{L}$- 内射包络；同态.