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Convolution of Absolute Value Sum of Coefficients on Chebyshev Polynomials of the Second Kind

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Abstract: The main purpose of this paper is in using the generating function of generalized Fibonacci polynomials and its partial derivative to study the convolution evaluation of the second-kind Chebyshev polynomials, and give an interesting formula.

Key words: generalized Fibonacci polynomials; generating function; identity; Chebyshev polynomials.
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1. Introduction and results

As usual, Chebyshev polynomials of the second kind, $U(x) = \{U_n(x)\}$, $n = 0, 1, 2, \ldots$, are defined by the second-order linear recurrence sequence

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x), \tag{1.1}$$

for $n \geq 0$, $U_0(x) = 1$, $U_1(x) = 2x$.

It is well known that Chebyshev polynomials of the second kind play a very important role in the study of the orthogonality of functions[1]. The various properties of $U_n(x)$ were investigated by many people, but there are a few to study their arithmetic properties. The main purpose of this paper is to study how to calculate the summation involving the Chebyshev polynomials of the second kind:

$$\sum_{l_1+l_2+\cdots+l_k=n} N(U_{l_1}(x)) \cdot N(U_{l_2}(x)) \cdot \cdots \cdot N(U_{l_k}(x)), \tag{1.2}$$

where the summation is over all the $k-$dimension nonnegative integer coordinates $(l_1, l_2, \cdots, l_k)$ such that $l_1 + l_2 + \cdots + l_k = n$, k is any positive integer, and $N(U_m(x))$ is the sum of the absolute value of coefficients in Chebyshev polynomials of the second kind $U_m(x)$.

Regarding (1.2), it seems that they have not been studied yet; at least, we have not seen such a mean value before. The problem is interesting because it can help us to find some relationships between Chebyshev polynomials and generalized Fibonacci polynomials. In this paper, we use the generating function of generalized Fibonacci polynomials $F(\alpha; x)$ and its partial derivative to study the evaluation of (1.2), and give an interesting identity for any fixed integers k and n.

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That is, we shall prove the following theorem.

Theorem 1 *Let $U(x)=\{U_n(x)\}$ be defined by Eq (1.1). Then, for any positive integers k and nonnegative integer n, we have the calculating formula:*

$$\sum_{l_1+l_2+\cdots+l_k=n} N(U_{l_1}(x))\cdot N(U_{l_2}(x))\cdot\cdots\cdot N(U_{l_k}(x))$$

$$=\sum_{m=0}^{[\frac{n}{2}]}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}2^{n-2m},$$

where the summation is over all $k-$dimension nonnegative integer coordinates $(l_1,l_2,\cdots,l_k)$ such that $l_1+l_2+\cdots+l_k=n$, $N(U_m(x))$ is the same as that in (1.2).

2. Preliminary and several lemmas

In order to prove Theorem 1, we introduce the generalized Fibonacci polynomials, and the give several lemmas.

The generalized Fibonacci polynomials $F(\alpha;x)=\{F_n(\alpha;x)\}$, $n=0,1,2,\cdots$, are defined by the second-order linear recurrence sequence

$$F_{n+2}(\alpha;x)=xF_{n+1}(\alpha;x)+\alpha F_n(\alpha;x), \tag{2.1}$$

for $n\geq 0$, $F_0(\alpha;x)=0$, $F_1(\alpha;x)=1$, α is any real number.

The characteristic equation of Eq (2.1) is $\lambda^2-x\lambda-\alpha=0$. Its characteristic roots are

$$\omega=\frac{x+\sqrt{x^2+4\alpha}}{2} \quad\text{and}\quad \theta=\frac{x-\sqrt{x^2+4\alpha}}{2},$$

then the terms of the sequence $F(\alpha;x)$[2] can be expressed as

$$F_n(\alpha;x)=\frac{1}{\omega-\theta}\{\omega^n-\theta^n\},\quad n=0,1,2,\cdots.$$

If $\alpha=1$, then the sequence $F(\alpha;x)$ is called the Fibonacci polynomials, and we shall denote it by $F(x)=F(1;x)=\{F_n(x)\}$.

If $\alpha=x=1$, then the sequence $F(\alpha;x)$ is called the Fibonacci sequences, and we shall denote it by $F=F(1;1)=\{F_n\}$.

Regarding generalized Fibonacci polynomials $F(\alpha;x)$, we have the following lemmas.

Lemma 1 *Let $F(\alpha;x)=\{F_n(\alpha;x)\}$ be defined by Eq (2.1). Then we have the following identity:*

$$F_{n+1}(\alpha;x)=\sum_{m=0}^{[\frac{n}{2}]}\binom{n-m}{m}x^{n-2m}\cdot\alpha^m. \tag{2.2}$$

Proof In fact, from the definition of $F_n(\alpha;x)$, we know that (2.2) is true for $n=0$ and $n=1$.

Assume that (2.2) is true for all integers $0 \le n \le k$. Then, for $n = k+1$, we immediately obtain

$$\begin{aligned}
F_{k+2}(\alpha;x) &= xF_{k+1}(\alpha;x) + \alpha F_k(\alpha;x) \\
&= x\sum_{m=0}^{[\frac{k}{2}]}\binom{k-m}{m}x^{k-2m}\cdot\alpha^m + \alpha\sum_{m=0}^{[\frac{k-1}{2}]}\binom{k-1-m}{m}x^{k-1-2m}\cdot\alpha^m \\
&= \sum_{m=0}^{[\frac{k+1}{2}]}\binom{k-m}{m}x^{k+1-2m}\cdot\alpha^m + \sum_{m=0}^{[\frac{k-1}{2}]}\binom{k-1-m}{m}x^{k-1-2m}\cdot\alpha^{m+1} \\
&= x^{k+1} + \sum_{m=0}^{[\frac{k-1}{2}]}\binom{k-1-m}{m+1}x^{k-1-2m}\cdot\alpha^{m+1} + \sum_{m=0}^{[\frac{k-1}{2}]}\binom{k-1-m}{m}x^{k-1-2m}\cdot\alpha^{m+1} \\
&= x^{k+1} + \sum_{m=0}^{[\frac{k-1}{2}]}\binom{k-m}{m+1}x^{k-1-2m}\cdot\alpha^{m+1} \\
&= \sum_{m=0}^{[\frac{k+1}{2}]}\binom{k+1-m}{m}x^{k+1-2m}\cdot\alpha^m.
\end{aligned}$$

where we have used the fact that $\binom{k-m}{m} = 0$ if $m > \frac{k}{2}$. So by induction we know that (2.2) is true for all nonnegative integer n. This completes the proof of Lemma 1.

Lemma 2 *Let $F(\alpha;x) = \{F_n(\alpha;x)\}$ be defined by Eq (2.1), $F_n^{(k)}(\alpha;x)$ denote the k^{th} derivative of $F_n(\alpha;x)$. Then we have the following identity:*

$$\frac{1}{(1-xt-\alpha t^2)^k} = \frac{1}{(k-1)!}\sum_{n=0}^{+\infty}F_{n+k}^{(k-1)}(\alpha;x)\cdot t^n. \tag{2.3}$$

Proof First, note that

$$F_n(\alpha;x) = \frac{1}{\sqrt{x^2+4\alpha}}\left[(\frac{x+\sqrt{x^2+4\alpha}}{2})^n - (\frac{x-\sqrt{x^2+4\alpha}}{2})^n\right],$$

so we can easily deduce that the generating function of $F(\alpha;x)$ is

$$G(\alpha;t,x) = \sum_{n=0}^{+\infty}F_{n+1}(\alpha;x)t^n = \frac{1}{(1-\omega t)(1-\theta t)} = \frac{1}{1-xt-\alpha t^2}. \tag{2.4}$$

Let $\frac{\partial^k G(\alpha;t,x)}{\partial x^k}$ denote the k^{th} partial derivative of $G(\alpha;t,x)$ with respect to x. Then we have

$$\frac{\partial G(\alpha;t,x)}{\partial x} = \frac{t}{(1-xt-\alpha t^2)^2} = \sum_{n=0}^{+\infty}F_{n+1}^{(1)}(\alpha;x)\cdot t^n$$

$$\frac{\partial^2 G(\alpha;t,x)}{\partial x^2} = \frac{2!\cdot t^2}{(1-xt-\alpha t^2)^3} = \sum_{n=0}^{+\infty}F_{n+1}^{(2)}(\alpha;x)\cdot t^n \tag{2.5}$$

$$\cdots$$

$$\frac{\partial^{k-1}G(\alpha;t,x)}{\partial x^{k-1}}=\frac{(k-1)!\cdot t^{k-1}}{(1-xt-\alpha t^2)^k}=\sum_{n=0}^{+\infty}F_{n+1}^{(k-1)}(\alpha;x)\cdot t^n$$
$$=\sum_{n=0}^{+\infty}F_{n+k}^{(k-1)}(\alpha;x)\cdot t^{n+k-1},$$

where we have used the fact that $F_{n+1}(\alpha;x)$ is a polynomial of degree n. Therefore it follows that

$$\frac{1}{(1-xt-\alpha t^2)^k}=\frac{1}{(k-1)!\cdot t^{k-1}}\sum_{n=0}^{+\infty}F_{n+k}^{(k-1)}(\alpha;x)\cdot t^{n+k-1}=\frac{1}{(k-1)!}\sum_{n=0}^{+\infty}F_{n+k}^{(k-1)}(\alpha;x)\cdot t^n.$$

This completes the proof of Lemma 2.

Lemma 3 *Let $F(\alpha;x)=\{F_n(\alpha;x)\}$ be defined by Eq (2.1), $F_n^{(k)}(\alpha;x)$ denote the k^{th} derivative of $F_n(\alpha;x)$. Then we have the following identity:*

$$\sum_{l_1+l_2+\cdots+l_k=n}F_{l_1+1}(\alpha;x)\cdot F_{l_2+1}(\alpha;x)\cdot\cdots\cdot F_{l_k+1}(\alpha;x)=\frac{1}{(k-1)!}F_{n+k}^{(k-1)}(\alpha;x).$$

Proof For any two absolutely convergent power series $\sum_{n=0}^{+\infty}a_nx^n$ and $\sum_{n=0}^{+\infty}b_nx^n$, note that

$$\left(\sum_{n=0}^{+\infty}a_nx^n\right)\cdot\left(\sum_{n=0}^{+\infty}b_nx^n\right)=\sum_{n=0}^{+\infty}\left(\sum_{u+v=n}a_ub_v\right)x^n. \tag{2.6}$$

So from (2.6), (2.4) and (2.3), we obtain

$$\sum_{n=0}^{\infty}\left(\sum_{l_1+l_2+\cdots+l_k=n}F_{l_1+1}(\alpha;x)\cdot F_{l_2+1}(\alpha;x)\cdot\cdots\cdot F_{l_k+1}(\alpha;x)\right)\cdot t^n$$
$$=\left(\sum_{n=0}^{\infty}F_{n+1}(\alpha;x)\cdot t^n\right)^k=\frac{1}{(1-xt-\alpha t^2)^k}=\frac{1}{(k-1)!}\sum_{n=0}^{\infty}F_{n+k}^{(k-1)}(\alpha;x)\cdot t^n.$$

Equating the coefficients of t^n on both sides of the expression above, we obtain the identity

$$\sum_{l_1+l_2+\cdots+l_k=n}F_{l_1+1}(\alpha;x)\cdot F_{l_2+1}(\alpha;x)\cdot\cdots\cdot F_{l_k+1}(\alpha;x)=\frac{1}{(k-1)!}F_{n+k}^{(k-1)}(\alpha;x).$$

The proof of this Lemma 3 ends.

Lemma 4 *Let $F(\alpha;x)=\{F_n(\alpha;x)\}$ be defined by Eq (2.1). Then, for any positive integers k and n, any real α, we have the calculating formula*

$$\sum_{l_1+l_2+\cdots+l_k=n}F_{l_1+1}(\alpha;x)\cdot F_{l_2+1}(\alpha;x)\cdot\cdots\cdot F_{l_k+1}(\alpha;x)$$
$$=\sum_{m=0}^{[\frac{n}{2}]}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}x^{n-2m}\cdot\alpha^m,$$

where $\binom{n}{m} = \frac{n!}{m!\cdot(n-m)!}$, *and* $[z]$ *denotes the greatest integer not exceeding* z.

Proof From (2.2) we can deduce that the $(k-1)^{th}$ derivative of $F_{n+k}(\alpha;x)$ is

$$F_{n+k}^{(k-1)}(\alpha;x) = \left(\sum_{m=0}^{[\frac{n+k-1}{2}]} \binom{n+k-1-m}{m} x^{n+k-1-2m}\cdot\alpha^m\right)^{(k-1)}$$

$$= \sum_{m=0}^{[\frac{n}{2}]} \frac{(n+k-1-m)!}{m!\cdot(n-2m)!} x^{n-2m}\cdot\alpha^m. \tag{2.7}$$

Combining (2.7) with Lemma 3, we obtain the identity

$$\sum_{l_1+l_2+\cdots+l_k=n} F_{l_1+1}(\alpha;x)\cdot F_{l_2+1}(\alpha;x)\cdot\cdots\cdot F_{l_k+1}(\alpha;x)$$

$$= \frac{1}{(k-1)!}F_{n+k}^{(k-1)}(\alpha;x) = \frac{1}{(k-1)!}\sum_{m=0}^{[\frac{n}{2}]} \frac{(n+k-1-m)!}{m!\cdot(n-2m)!} x^{n-2m}\cdot\alpha^m$$

$$= \sum_{m=0}^{[\frac{n}{2}]} \binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1} x^{n-2m}\cdot\alpha^m.$$

This completes the proof of Lemma 4.

3. Proof of Theorem 1

At last we give the proof of Theorem 1.

Proof of Theorem 1 Since

$$U_n(x) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k \frac{(n-k)!}{k!(n-2k)!}(2x)^{n-2k},$$

then

$$N(U_n(x)) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(n-k)!}{k!(n-2k)!}2^{n-2k} = \sum_{k=0}^{[\frac{n}{2}]}\binom{n-k}{k}2^{n-2k}.$$

Taking $\alpha=1$, $x=2$ in the Eq (2.2), we obtain

$$F_{n+1}(1;2) = \sum_{m=0}^{[\frac{n}{2}]}\binom{n-m}{m}2^{n-2m},$$

therefore, the relationship between $N(U_n(x))$ and generalized Fibonacci polynomials $F_{n+1}(\alpha;x)$ is

$$N(U_n(x)) = F_{n+1}(1;2).$$

Hence (1.2) becomes to be of the form

$$\sum_{l_1+l_2+\cdots+l_k=n} F_{l_1+1}(1;2)\cdot F_{l_2+1}(1;2)\cdot\cdots\cdot F_{l_k+1}(1;2)$$

Taking $\alpha = 1$, $x = 2$ in the Lemma 4, we obtain

$$\sum_{l_1+l_2+\cdots+l_k=n} F_{l_1+1}(1;2)\cdot F_{l_2+1}(1;2)\cdot\cdots\cdot F_{l_k+1}(1;2)$$

$$=\sum_{m=0}^{[\frac{n}{2}]}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}2^{n-2m}.$$

So

$$\sum_{l_1+l_2+\cdots+l_k=n} N(U_{l_1}(x))\cdot N(U_{l_2}(x))\cdot\cdots\cdot N(U_{l_k}(x))$$

$$=\sum_{m=0}^{[\frac{n}{2}]}\binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}2^{n-2m}.$$

This completes the proof of the Theorem.

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第二类 Chebyshev 多项式系数的绝对值和的卷积

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摘要: 本文主要的目的是利用广义 Fibonacci 多项式的生成函数及其偏导数来研究第二类 Chebyshev 多项式卷积的计算, 并给出一个有趣的计算公式.

关键词: 广义 Fibonacci 多项式; 生成函数; 恒等式; Chebyshev 多项式.