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Lie Ideals in AF C^*-Algebras

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Abstract: We study Lie ideals in unital AF C^*-algebras. It is shown that if a linear manifold L in an AF C^*-algebra A is a closed Lie ideal in A, then there exists a closed associative ideal I and a closed subalgebra E_I of the canonical masa D of A such that $\overline{[A,I]} \subset L \subset I + E_I$, and that every closed subspace in this form is a closed Lie ideal in A.

Key words: AF C^*-algebra; Lie ideal; canonical masa.
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1. Introduction

Let A be an associative complex algebra, then under the Lie multiplication $[x,y]=xy-yx$, A becomes a Lie algebra. A Lie ideal in A is a linear manifold L in A for which $[a,k]\in L$ for every $a\in A$ and $k\in L$. In many instances, there is a closed connection between the Lie ideal structure and the (associative) ideal structure of A. This connection has been investigated for prime rings in [1], in [2] for $B(H)$, the set of bounded operators on a Hilbert space H, in [3] for certain von Nuemann algebras, in [4,5] for nest algebras, TUHF algebras and TAF algebras, and in [6] for UHF algebras. It is interesting to determine the closed Lie ideals in AF C^*-algebras. In [6], Marcoux had pointed out this problem. In this paper, we give it a groupoid description. It is shown that if a linear manifold L in an AF C^*-algebra A is a closed Lie ideal in A, then there exist a closed associative ideal I and a closed subalgebra E_I of the canonical masa D of A such that $\overline{[A,I]}\subset L\subset I+E_I$, and that every closed subspace in this form is a closed Lie ideal in A.

2. Closed Lie ideals in AF C^*-algebras

Let $A=\overline{\cup_{n=1}^{\infty}A_n}$ be a unital AF (approximately finite) C^*-algebra, where $A_n\subset A_{n+1}$ for all $n\geq 1$ and each A_n is a finite dimensional C^*-algebra[7−9].

A canonical masa D of A is a maximal abelian self-adjoint subalgebra of A such that $D_n=A_n\cap D$ is a masa of A_n for all n, and $D=\overline{\cup_{n=1}^{\infty}D_n}$.

In addition to the presentation for A described above, we shall make use of coordinization for A. For the convenience of the reader, we provide a brief sketch of the most relerant aspects of coordinization.

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Since A is an AF C^*-algebra, it is a groupoid C^*-algebra. The groupoid can be realized as a topological equivalence relation on the maximal ideals space X of D. So $D \cong C(X)$. It is possible to pick a system of matrix units $\{e_{ij}^n : i,j,n\}$ for A such that, for each n, $\{e_{ij}^n : i,j\}$ are the matrix units for A_n which generate A_n, and each e_{ij}^n can be written as a sum of $\{e_{ij}^{n+1} : i,j\}$. The matrix units are all normalizing partial isometries for D. (A partial isometry v is normalizing for D if $v^*Dv \subset D$ and $vDv^* \subset D$). The action of a normalizing partial isometry on D induces a partial homeomorphism on X and the equivalence relation G is exactly the union of the graphs of all the partial homeomorphisms induced by normalizing partial isometries. The multiplication on G is defined for these pairs of elements (x,y) and (w,z) for which $w = y$; the product for a composable pair is given by $(x,y)(y,z) = (x,z)$. Inversion is given by $(x,y)^{-1} = (y,x)$. The topology on G is the one obtained by declaring that each such graph is an open subset of G. It turns out that these are also closed and in fact, compact. This description makes it clear that the groupoid is independent of the presentation, but a very hardy fact is that G is the union of the graphs of the matrix units in the presentations.

The elements of A can be identified with elements of $C_0(G)$ (but not all elements of $C_0(G)$ correspond to elements of A). We will need the formula for multiplication: if f and g are elements of A viewed as functions in $C_0(G)$, then

$$fg(x,y) = \sum_u f(x,u)g(u,y),$$

where u varies over the equivalence class of x (which is the same as the equivalence class of y). Note, in particular, that if $g \in D$, then the support of g is in $\{(x,x) : x \in X\}$ and $fg(x,y) = f(x,y)g(y,y)$ and $gf(x,y) = g(x,x)f(x,y)$.

We will use frequently the spectral theorem for bimodules[10].

Theorem 2.1[10] *If $B \subset A$ is any closed bimodule over D, then the spectral of B,* $\mathrm{spec}(B) = \{(x,y) \in G : f(x,y) \neq 0$ *for some* $f \in B\}$ *has the property that* $B = \{f \in A : \mathrm{supp}(f) \subset \mathrm{spec}(B)\}$.

With this terminology, $\mathrm{spec}(D) = \{(x,x) : x \in X\}$. It is customary to identity $\mathrm{spec}(D)$ with X by writing x in place of (x,x).

If I is a closed ideal of A, then I is a bimodule over D. The spectral of I, $\mathrm{spec}(I)$ is an ideal subset of G (i.e. $G \cdot \mathrm{spec}(I) \cdot G \subset \mathrm{spec}(I)$). Consequently, if $(x,y) \in G \setminus \mathrm{spec}(I)$, then $(x,x),(y,y) \in G \setminus \mathrm{spec}(I)$. Let A/I denote the quotient C^*-algebra of A with respect to I, $\pi : A \longrightarrow A/I$ be the canonical quotient map. When $I \neq A$, denote $E_I = \{f \in D : f(x,x) = f(y,y)$ whenever $(x,y) \in G \setminus \mathrm{spec}(I)\}$; when $I = A$, denote $E_I = 0$.

Proposition 2.2 *E_I is a closed subalgebra of D, and $E_I + I = \pi^{-1}(Z(A/I))$, where $Z(A/I)$ is the center of A/I.*

Proof Clearly, E_I is a subalgebra of D. Let $\{f_n\}$ be a sequence in E_I such that $\lim f_n = f$. Since D is closed, $f \in D$. Because Gelfand transform is an $*$-isomorphism, $f_n(x)$ converges to $f(x)$ uniformly on X. Since $f_n(x,x) = f_n(y,y)$ whenever $(x,y) \in G \setminus \mathrm{spec}(I)$, we have

$f(x,x) = f(y,y)$. Thus $f \in E_I$, and E_I is closed.

Fix $f \in E_I$, $g \in A$. If $(x,y) \in G \setminus \mathrm{spec}(I)$, then $f(x,x) = f(y,y)$, and $(fg-gf)(x,y) = f(x,x)g(x,y) - g(x,y)f(y,y) = 0$. By Theorem 2.1, $fg - gf \in I$. Consequently, $\pi(f)\pi(g) = \pi(g)\pi(f)$. This implies that $\pi(f) \in Z(A/I)$. Thus $E_I + I \subset \pi^{-1}(Z(A/I))$.

On the other hand, fix $f \in \pi^{-1}(Z(A/I))$, so $\pi(f) \in Z(A/I)$. Since $\pi(D)$ is a masa of A/I, we have $\pi(f) \in \pi(D)$. Thus there exists $f_1 \in D$ such that $f - f_1 \in I$. Let $(x,y) \in G\setminus\mathrm{spec}(I)$, by Theorem 2.1. Then there exists $h \in A \setminus I$ such that $h(x,y) \neq 0$. Since $\pi(f_1)\pi(h) = \pi(h)\pi(f_1)$, we have $f_1h - hf_1 \in I$. Thus $(f_1h - hf_1)(x,y) = 0$, i.e. $f_1(x,x)h(x,y) = h(x,y)f_1(y,y)$. Since $h(x,y) \neq 0$, we obtain $f_1(x,x) = f_1(y,y)$. Thus $f_1 \in E_I$. So $f = f_1 + g \in E_I + I$. Thus $E_I + I = \pi^{-1}(Z(A/I))$.

Lemma 2.3[3] *If A is a C^*-algebra and I is a closed ideal in A, then $\overline{[A,I]} = I \cap \overline{[A,A]}$.*

Proposition 2.4 $[A, I + E_I] \subset \overline{[A,I]}$.

Proof By Lemma 2.3, we only prove $[A,E_I] \subset I$. In fact, if $f \in A$, $g \in E_I$, and $(x,y) \in G\setminus\mathrm{spec}(I)$, then $(fg-gf)(x,y) = f(x,x)g(x,y)-g(x,y)f(y,y) = 0$. By Theorem 2.1, $fg-gf \in I$.

We will use a classical result of Herstein[1] as follows.

Theorem 2.5[1] *Let A be a ring with no non-zero locally nilpotent ideals. Suppose that in A, $2x = 0$ implies that $x = 0$. Suppose further that U is a Lie ideal of A and also an associative subring of A. Then, either U contains a non-zero ideal of A or U is contained in the center of A.*

The kind of rings stated in Theorem 2.5 includes all C^*-algebras.

Theorem 2.6 *Let A be a unital AF C^*-algebra A. If L is a closed Lie ideal in A, then there exist a closed associative ideal I and a closed subalgebra E_I of the canonical masa D of A such that $\overline{[A,I]} \subset L \subset I + E_I$. On the other hand, every closed subspace in this form is a closed Lie ideal in A.*

Proof Let $U(L) = \{x \in A : [x,A] \subset L\}$. Then $U(L)$ is a Lie ideal and subring of A, containing L, and is closed since L is closed. Hence either $U(L) \subset Z(A)$ or there exists a non-zero ideal $I \subset U(L)$. If $U(L) \subset Z(A)$, then $L \subset Z(A)$ and $I = \{0\}$ in the theorem. Otherwise, let I be a maximal non-zero ideal of A in $U(L)$. Since $U(L)$ is closed, so is I. We claim that $I \subset U(L) \subset E_I + I$. In fact, if $U(L)/I \neq 0$ in A/I, then $U(L)/I$ is a Lie ideal and subring in A/I, and is either contained in the center of A/I, or contains a non-zero ideal K of A/I. In the first case, by Proposition 2.3, $I \subset U(L) \subset \pi^{-1}(Z(A/I)) = E_I + I$. Consequently, $[A,I] \subset [A,U(L)] \subset L \subset E_I + I$. Since L is closed, we have $\overline{[A,I]} \subset L \subset E_I + I$. In the second case, let $J = \pi^{-1}(K)$, note that $I \neq J$ and $I \subset J \subset U(L) + I = U(L)$, which is impossible since J is an ideal of A contained in $U(L)$ and I is maximal in this kind of ideals of A.

If L is a closed subspace of A, and there exists a closed ideal I of A such that $\overline{[A,I]} \subset L \subset E_I + I$, then, by Proposition 2.4, $[A,L] \subset [A, E_I + I] = \overline{[A,I]} \subset L$. This implies L is a closed Lie ideal of A.

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AF C^*- 代数中的 Lie 理想

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摘要: 本文描述了 AF C^*- 代数中闭 Lie 理想，证明了如果 AF C^*- 代数 A 中的线性流形 L 是 A 的闭 Lie 理想，则存在 A 的闭结合理想 I 和 A 的典型 masa D 中的闭子代数 E_I 使得 $\overline{[A,I]} \subset L \subset I + E_I$，并且 A 中每一个这种形式的闭子空间都是 A 的闭 Lie 理想.

关键词: AF C^*- 代数； Lie 理想； 典型 masa.