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Littlewood-Paley Operators on Weighted Lipschitz Spaces

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Abstract: Littlewood-Paley operators, the g-function, the area integral and the function g_λ^*, are considered as operators on weighted Lipschitz spaces. It is proved that the image of a weighted Lipschitz function under one of these operators is either equal to infinity almost everywhere or is in weighted Lipschitz spaces.

Key words: Littlewood-Paley operator; weighted Lipschitz spaces; weighted norm inequality.
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1. Introduction

A nonnegative function w defined on R^n is called a weight if it is locally integrable. We denote by $|E|$ the Lebesgue measure of E, and $w(E)=\int_E w(x)\mathrm{d}x$. We use χ_E for the characteristic function of E. If Q is a cube in R^n, dQ stands for the cube concentric with Q and having edge length d times as long. Throughout this paper, the letter C will denote a constant which may change from line to line.

A weight w is said to belong to the Muckenhoutp class A_p, $1<p<\infty$, if there exists a constant C such that

$$\frac{w(Q)}{|Q|}\left(\frac{w^{-\frac{1}{p-1}}(Q)}{|Q|}\right)^{p-1}\leq C$$

for all cube $Q\subset R^n$. The class A_1 is defined by replacing the above inequality by

$$\frac{w(Q)}{|Q|}\leq C\mathrm{ess}\inf_{x\in Q} w(x).$$

We shall say that a weight w satisfies a doubling condition if there exists a constant C such that

$$w(2Q)\leq Cw(Q)$$

for all cube $Q\subset R^n$. Given $p>1$, we shall say that $w\in RH(p)$, if w satisfies a reverse Hölder condition, that is,

$$\left(\frac{1}{|Q|}\int_Q w^{1+\delta}\mathrm{d}x\right)^{\frac{1}{1+\delta}}\leq C_0\frac{1}{|Q|}\int_Q w\mathrm{d}x$$

for all cube $Q\subset R^n$. δ will be called the reverse Hölder constant.

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In [1], E.Harboure, O.Salinas and B.Viviani introduced the weighted Lipschitz spaces.

Let w be a weight and $-1 \leq \beta < 1/n$. We say a locally integrable function f belongs to $\mathcal{L}_w(\beta)$ if there exists a constant C such that

$$\frac{1}{w(Q)|Q|^\beta}\int_Q |f(x)-f_Q|\mathrm{d}x \leq C$$

for all cube $Q \subset R^n$, where $f_Q = (1/|Q|)\int_Q f(x)\mathrm{d}x$. The smallest constant C satisfying this inequality will be denoted by $\|f\|_{\mathcal{L}_w(\beta)}$.

We denote by $\tilde{\mathcal{L}}_w(\beta)$ the space of the locally integrable functions f such that the inequality

$$\frac{1}{\left(w^{\frac{1}{1+\beta}}(Q)\right)^{1+\beta}}\int_Q |f(x)-f_Q|\mathrm{d}x \leq C$$

holds for a fixed constant C and for all cube $Q \subset R^n$. The smallest constant C will be called $\|f\|_{\tilde{\mathcal{L}}_w(\beta)}$.

Let us observe that for $\beta = 0$, both of the spaces $\mathcal{L}_w(\beta)$ and $\tilde{\mathcal{L}}_w(\beta)$ coincide with one of the versions of weighted bounded mean oscilation space, introduced by Muckenhoupt and Wheeden in [2]. Moreover, for the case $w \equiv 1$, the above definitions give the known Lipschitz integral spaces for β in the range $0 < \beta < 1/n$, and the Morrey spaces for $-1 < \beta < 0$.

For $x \in R^n$ and $y > 0$, the Poisson kernel for the upper half-plane, R_+^{n+1}, is $P(x,y) = C_n y/(y^2+|x|^2)^{-(n+1)/2}$. The Poisson integral of f is $u(x,y) = \int_{R^n} f(z)P(x-z,y)\mathrm{d}z$.

Given a point $x \in R^n$, define the cone at x by $\Gamma(x) = \{(z,y) \in R_+^{n+1} : |z-x| < y\}$. The g-function $g(f)$, the Lusin area integral $s(f)$, and the Littlewood-Paley function g_λ^* are defined by

$$g(f)(x) = \left\{\int_0^\infty y|\nabla u(x,y)|^2\mathrm{d}y\right\}^{1/2},$$

$$s(f)(x) = \left\{\iint_{\Gamma(x)} y^{1-n}|\nabla u(z,y)|^2\mathrm{d}z\mathrm{d}y\right\}^{1/2},$$

and

$$g_\lambda^*(f)(x) = \left\{\iint_{R_+^{n+1}} (y/(y+|z-x|))^{\lambda n} y^{1-n}|\nabla u(z,y)|^2\mathrm{d}z\mathrm{d}y\right\}^{1/2},$$

where $\nabla u(x,y) = \left(\frac{\partial u}{\partial x_1}(x,y),\dots,\frac{\partial u}{\partial x_n}(x,y),\frac{\partial u}{\partial y}(x,y)\right)$.

The boundedness of Littlewood-Paley operators in Lipschitz spaces is already sufficiently discussed [3,6,4,5]. Qiu[7] have considered the Littlewood-Paley operators on weighted bounded mean oscilation space.

Let Tf be one of the Littlewood-Paley operators $g(f)$, $s(f)$, g_λ^*, for weighted Lipschitz spaces, then we get the following conclusions.

Theorem 1 *Let $w \in A_1$, $-1 \leq \beta < 1/n$, $\lambda \geq (2+\delta)/(1+\delta) + 2/n$ and $f \in \mathcal{L}_w(\beta)$, then either $Tf(x) = \infty$ a.e., or $Tf(x) < \infty$ a.e. and there is a constant C independent of f and x such that*

$$\|T(f)\|_{\mathcal{L}_w(\beta)} \leq C\|f\|_{\mathcal{L}_w(\beta)}$$

where δ is the weight w's reverse Hölder constant.

Theorem 2 *Let $w \in A_1$, $0 \leq \beta < 1/n$, $\lambda \geq (2+\delta)/(1+\delta) + 2/n$ and $f \in \tilde{\mathcal{L}}_w(\beta)$, then either $Tf(x) = \infty$ a.e., or $Tf(x) < \infty$ a.e. and there is a constant C independent of f and x such that*

$$\|T(f)\|_{\tilde{\mathcal{L}}_w(\beta)} \leq C\|f\|_{\tilde{\mathcal{L}}_w(\beta)},$$

where δ is the weight w's reverse Hölder constant.

2. Some lemmas

The proofs of the theorems are based on the following lemmas.

Lemma 1 *Let $w \in A_1$, $-1 \leq \beta < 1/n$, $\max\{1, 1+\beta\} < p < \infty$, $f \in \mathcal{L}_w(\beta)$, and Q be a cube centered at x_0, having edge length r. There is a constant C depending on n, β so that for $y > 0$*

$$\int_{R^n} \frac{|f(x) - f_Q|}{y^{np} + |x - x_0|^{np}} \mathrm{d}x \leq C y^{n-np}(y^{n\beta} + r^{n\beta}) \max\left\{\frac{w(Q)}{|Q|}, \frac{w(\frac{y}{r}Q)}{|\frac{y}{r}Q|}\right\} \|f\|_{\mathcal{L}_w(\beta)}, \tag{1}$$

and for $d \geq 1$

$$\int_{R^n} \frac{|f(x) - f_Q|}{y^{n+d} + |x - x_0|^{n+d}} \mathrm{d}x \leq C y^{-d}(y^{n\beta} + r^{n\beta}) \max\left\{\frac{w(Q)}{|Q|}, \frac{w(\frac{y}{r}Q)}{|\frac{y}{r}Q|}\right\} \|f\|_{\mathcal{L}_w(\beta)}. \tag{2}$$

Proof Arguing as in [8]. Let Q_k be the cube concentric with Q and having edge length $2^k r$, then we have

$$|f_{Q_k} - f_{Q_{k-1}}| \leq \frac{1}{|Q_{k-1}|} \int_{Q_{k-1}} |f(x) - f_{Q_k}| dx \leq C \frac{w(Q_k)|Q_k|^\beta}{|Q_{k-1}|} \|f\|_{\mathcal{L}_w(\beta)},$$

$$\begin{aligned}
\int_{Q_k} |f(x) - f_Q| \mathrm{d}x &\leq \int_{Q_k} |f(x) - f_{Q_k}| \mathrm{d}x + |f_{Q_k} - f_Q||Q_k| \\
&\leq W(Q_k)|Q_k|^\beta \|f\|_{\mathcal{L}_w(\beta)} + C|Q_k| \sum_{j=1}^{k} \frac{w(Q_j)|Q_j|^\beta}{|Q_j|} \|f\|_{\mathcal{L}_w(\beta)} \\
&\leq C\|f\|_{\mathcal{L}_w(\beta)} \sum_{j=1}^{k} 2^{(k-j)n} w(Q_j)|Q_j|^\beta.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{R^n} \frac{|f(x) - f_Q|}{r^{np} + |x - x_0|^{np}} \mathrm{d}x \\
&= \int_Q \frac{|f(x) - f_Q|}{r^{np} + |x - x_0|^{np}} \mathrm{d}x + \sum_{k=1}^{\infty} \int_{Q_k \setminus Q_{k-1}} \frac{|f(x) - f_Q|}{r^{np} + |x - x_0|^{np}} \mathrm{d}x \quad \text{where } Q_0 = Q \\
&\leq r^{-np} \int_Q |f(x) - f_Q| \mathrm{d}x + \sum_{k=1}^{\infty} (2^{k-1}r)^{-np} \int_{Q_k} |f(x) - f_Q| \mathrm{d}x \\
&\leq C r^{-np} w(Q)|Q|^\beta \|f\|_{\mathcal{L}_w(\beta)} + C\|f\|_{\mathcal{L}_w(\beta)} \sum_{k=1}^{\infty} (2^k r)^{-np} \sum_{j=1}^{k} 2^{(k-j)n} w(Q_j)|Q_j|^\beta.
\end{aligned}$$

Since $w \in A_1$, we have $w(Q_j) \le Cw(Q_j \setminus Q_{j-1})$ and $w \in A_{p-\beta}$ $(p-\beta > 1)$,

$$\int_{R^n} \frac{w(x)}{r^{n(p-\beta)} + |x-x_0|^{n(p-\beta)}} \mathrm{d}x \le C r^{n-n(p-\beta)} \frac{w(Q)}{|Q|}.$$

We have[8]

$$\begin{aligned}
&\sum_{k=1}^{\infty} (2^k r)^{-np} \sum_{j=1}^{k} 2^{(k-j)n} w(Q_j)|Q_j|^{\beta} \\
&\le C \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} 2^{-(k-j)(p-1)n} (2^j r)^{-np} w(Q_j \setminus Q_{j-1})|Q_j|^{\beta} \\
&\le C \sum_{j=1}^{\infty} (2^j r)^{-np} w(Q_j \setminus Q_{j-1})(2^j r)^{n\beta} \\
&\le C \sum_{j=1}^{\infty} \int_{Q_j \setminus Q_{j-1}} \frac{w(x)}{r^{n(p-\beta)} + |x-x_0|^{n(p-\beta)}} \mathrm{d}x \\
&\le C \int_{R^n} \frac{w(x)}{r^{n(p-\beta)} + |x-x_0|^{n(p-\beta)}} \mathrm{d}x \le C r^{n-n(p-\beta)} \frac{w(Q)}{|Q|}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{R^n} \frac{|f(x)-f_Q|}{r^{np} + |x-x_0|^{np}} \mathrm{d}x \le C r^{-np} w(Q)|Q|^{\beta} \|f\|_{\mathcal{L}_w(\beta)} + C r^{n-n(p-\beta)} \frac{w(Q)}{|Q|} \|f\|_{\mathcal{L}_w(\beta)} \\
&\le C r^{n-n(p-\beta)} \frac{w(Q)}{|Q|} \|f\|_{\mathcal{L}_w(\beta)}. \qquad (3)
\end{aligned}$$

To complete the proof of (1), let R be the cube concentric with Q and having edge length y. If $y > r$, let k satisfy $2^k r \le y < 2^{k+1} r$, then $w(R) \le w(2^{k+1}Q) \le Cw(\frac{y}{r}Q)$. Hence, by (3)

$$\begin{aligned}
&\int_{R^n} \frac{|f(x)-f_Q|}{y^{np} + |x-x_0|^{np}} \mathrm{d}x \\
&\le \int_{R^n} \frac{|f(x)-f_R|}{y^{np} + |x-x_0|^{np}} \mathrm{d}x + |f_R - f_Q| \int_{R^n} \frac{1}{y^{np} + |x-x_0|^{np}} \mathrm{d}x \\
&\le C y^{n-n(p-\beta)} \|f\|_{\mathcal{L}_w(\beta)} \frac{w(R)}{|R|} + y^{n-np}|f_R - f_Q| \\
&\le C y^{n-n(p-\beta)} \|f\|_{\mathcal{L}_w(\beta)} \frac{w(\frac{y}{r}Q)}{|\frac{y}{r}Q|} + y^{n-np}|f_R - f_Q|.
\end{aligned}$$

Arguing as in [3], we have

$$|f_R - f_Q| \le C(y^{n\beta} + r^{n\beta}) \frac{w(Q)}{|Q|} \|f\|_{\mathcal{L}_w(\beta)}. \qquad (4)$$

Hence,

$$\int_{R^n} \frac{|f(x)-f_Q|}{y^{np} + |x-x_0|^{np}} \mathrm{d}x \le C y^{n-np}(y^{n\beta} + r^{n\beta}) \max\left\{ \frac{w(Q)}{|Q|}, \frac{w(\frac{y}{r}Q)}{|\frac{y}{r}Q|} \right\} \|f\|_{\mathcal{L}_w(\beta)}.$$

When $y < r$, by exchanging y and r, we shall get the same estimate as above. Thus, we finish the proof of (1).

Taking $p=(n+d)/n$ in (1), (2) follows. This completes the proof of Lemma 1.

Remark 1 Repeating the argument of the proof of Lemma 1, for $f\in\bar{\mathcal{L}}_w(\beta)$, we can prove

Let $w\in A_1$, $0\le\beta<1/n$, $1/(1+\beta)<p<\infty$, $f\in\bar{\mathcal{L}}_w(\beta)$ and Q be a cube centered at x_0, having edge length r. There is a constant C depending on n, β so that for $y>0$

$$\int_{R^n}\frac{|f(x)-f_Q|}{y^{np(1+\beta)}+|x-x_0|^{np(1+\beta)}}\mathrm{d}x$$
$$\le Cy^{n-np(1+\beta)}(y^{n\beta}+r^{n\beta})\|f\|_{\bar{\mathcal{L}}_w(\beta)}\left\{\max\left(\frac{w^{\frac{1}{1+\beta}}(Q)}{|Q|},\frac{w^{\frac{1}{1+\beta}}(\frac{y}{r}Q)}{|\frac{y}{r}Q|}\right)\right\}^{1+\beta},$$

and for $d>0$

$$\int_{R^n}\frac{|f(x)-f_Q|}{y^{n+d}+|x-x_0|^{n+d}}\mathrm{d}x$$
$$\le Cy^{-d}(y^{n\beta}+r^{n\beta})\|f\|_{\bar{\mathcal{L}}_w(\beta)}\left\{\max\left(\frac{w^{\frac{1}{1+\beta}}(Q)}{|Q|},\frac{w^{\frac{1}{1+\beta}}(\frac{y}{r}Q)}{|\frac{y}{r}Q|}\right)\right\}^{1+\beta}.$$

Lemma 2 *Let $w\in A_1$, $f\in\mathcal{L}_w(\beta)$ and Q be a cube. There are constants C_1 and C_2 such that for $t>0$*

$$w(\{x\in Q:\ |f(x)-f_Q||Q|^{-\beta}w(x)^{-1}>t\})\le C_1\exp(-C_2t)w(Q).$$

Proof The proof of Lemma 2 is based on the method of John and Nirenberg[2]. We omit the details.

Lemma 3 *Suppose that $w\in A_1$, $-1\le\beta<1/n$ and $f\in\mathcal{L}_w(\beta)$. Let Q be a cube centered at x_0 with edge length r, and $h(x)=(f(x)-f_Q)\chi_{Q^c}(x)$. If there is an $x'\in dQ$ such that $s(h)(x')<\infty$, where $d=(8\sqrt{n})^{-1}$, then there is a constant C so that for every $x\in \mathrm{d}Q$ $s(h)(x)<\infty$ and*

$$|s(h)(x)-s(h)(x')|\le C\frac{w(Q)}{|Q|}|Q|^{\beta}\|f\|_{\mathcal{L}_w(\beta)}.$$

Proof Arguing as in [4], with minor changes in the proof one can obtain the present lemma. we omit the proof for brevity.

Remark 2 For g-function $g(f)$, we also have the similar results as Lemma 3.

Lemma 4 *Under the hypothesis of Lemma 3, if there is an $x'\in dQ$ such that $g_\lambda^*(h)(x')<\infty$, where $\lambda\ge(2+\delta)/(1+\delta)+2/n$ (δ is w's reverse Hölder constant), there exists a constant C, such that for all $x\in \mathrm{d}Q$ $g_\lambda^*(h)(x)<\infty$ and*

$$|g_\lambda^*(h)(x)-g_\lambda^*(h)(x')|\le C\frac{w(Q)}{|Q|}|Q|^{\beta}\|f\|_{\mathcal{L}_w(\beta)}.$$

Proof Let $J_k=\{(z,y)\in R_+^{n+1}:\ |z|<2^{k-2}r,\ 0<y<2^{k-2}r\}$, $k\ge 0$. For $x\in \mathrm{d}Q$, we have

$$
\begin{aligned}
g_\lambda^*(h)(x)\le&\left\{\iint_{J_0}\left(\frac{y}{y+|z|}\right)^{\lambda n}y^{1-n}|\nabla h(x+z,y)|^2\mathrm{d}z\mathrm{d}y\right\}^{1/2}+\\
&\left\{\iint_{R_+^{n+1}\setminus J_0}\left(\frac{y}{y+|z|}\right)^{\lambda n}y^{1-n}|\nabla h(x+z,y)|^2\mathrm{d}z\mathrm{d}y\right\}^{1/2}\\
=&G^-+G^+.
\end{aligned}
$$

Arguing as in [4] we have

$$
\begin{aligned}
G^-\le& C\left\{\iint_{J_0}\left(\frac{y}{y+|z|}\right)^{\lambda n}y^{1-n}\left(\int_{Q^c}\frac{|f(t)-f_Q|}{(r+|t-x_0|)^{n+1}}\mathrm{d}t\right)^2\mathrm{d}z\mathrm{d}y\right\}^{1/2}\\
\le& C\left\{\int_0^r\int_{|z|<r}\left(\frac{y}{y+|z|}\right)^{\lambda n}y^{1-n}\left(r^{-1+n\beta}\|f\|_{\mathcal{L}_w(\beta)}\frac{w(Q)}{|Q|}\right)^2\mathrm{d}z\mathrm{d}y\right\}^{1/2}\\
\le& C\frac{w(Q)}{|Q|}|Q|^\beta\|f\|_{\mathcal{L}_w(\beta)}.
\end{aligned}
$$

And as for G^+, we have

$$G^+\le g_\lambda^*(h)(x')+\tau,$$

where

$$
\begin{aligned}
\tau=&\left\{\iint_{R_+^{n+1}\setminus J_0}\left(\frac{y}{y+|z|}\right)^{\lambda n}y^{1-n}|\nabla h(x+z,y)-\nabla h(x'+z,y)|^2\mathrm{d}z\mathrm{d}y\right\}^{1/2}\\
\le& C\left\{\sum_{k=1}^\infty(2^kr)^{-\lambda n}\iint_{J_k\setminus J_{k-1}}y^{\lambda n+1-n}\right.\\
&\left.\left(\int_{Q^c}|\nabla p(x+z-t,y)-\nabla p(x'+z-t,y)||f(t)-f_Q|\mathrm{d}t\right)^2\mathrm{d}z\mathrm{d}y\right\}^{1/2}\\
\le& C\left\{\sum_{k=1}^\infty(2^kr)^{-\lambda n}(A_k+B_k)\right\}^{1/2},
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
A_k=&\iint_{J_k\setminus J_{k-1}}y^{\lambda n+1-n}\\
&\left(\int_{Q_{k+1}^c}|\nabla p(x+z-t,y)-\nabla p(x'+z-t,y)||f(t)-f_Q|\mathrm{d}t\right)^2\mathrm{d}z\mathrm{d}y,\\
B_k=&\iint_{J_k\setminus J_{k-1}}y^{\lambda n+1-n}\\
&\left(\int_{Q_{k+1}\setminus Q}|\nabla p(x+z-t,y)-\nabla p(x'+z-t,y)||f(t)-f_Q|\mathrm{d}t\right)^2\mathrm{d}z\mathrm{d}y.
\end{aligned}
$$

By the same reason as in [4], we have

$$\begin{aligned}A_k \le & Cr^2 \iint_{J_k \setminus J_{k-1}} y^{\lambda n+1-n} \left((2^k r)^{-2} [(2^k r)^{n\beta} + r^{n\beta}] \frac{w(Q)}{|Q|} \|f\|_{\mathcal{L}_w(\beta)} \right)^2 \mathrm{d}z\mathrm{d}y \\ \le & Cr^2 (2^k r)^{\lambda n+2n\beta-2} \left(\frac{w(Q)}{|Q|} \|f\|_{\mathcal{L}_w(\beta)} \right)^2. \end{aligned} \tag{7}$$

As for B_k we have

$$B_k \le Cr^2 \left(\int_{Q_{k+1}} |f(z) - f_Q|^s \mathrm{d}z \right)^{2/s}, \tag{8}$$

where $1/s = \lambda/2 - 1/n$, $(2+\delta)/(1+\delta) + 2/n < \lambda < 1 + (2+\delta)/(1+\delta) + 2/n$. We will prove that there is a constant C such that

$$\left(\int_{Q_{k+1}} |f(z) - f_Q|^s \mathrm{d}z \right)^{2/s} \le C(2^{k+1} r)^{\lambda n+2n\beta-2} \left(\|f\|_{\mathcal{L}_w(\beta)} \frac{w(Q)}{|Q|} \right)^2. \tag{9}$$

Note

$$\left(\int_{Q_{k+1}} |f(z) - f_Q|^s \mathrm{d}z \right)^{2/s} \le C \left(\int_{Q_{k+1}} |f(z) - f_{Q_{k+1}}|^s \mathrm{d}z \right)^{2/s} + C|f_{Q_{k+1}} - f_Q|^2 |Q_{k+1}|^{2/s}. \tag{10}$$

For the first term in the above inequality, by Hölder inequality, we have

$$\begin{aligned}&\left(\int_{Q_{k+1}} |f(z) - f_{Q_{k+1}}|^s \mathrm{d}z \right)^{2/s} \\ &\left(\int_{Q_{k+1}} \left(|f - f_{Q_{k+1}}| w^{-1} \right)^s w^{s-1} w \mathrm{d}z \right)^{2/s} \\ \le & \left(\int_{Q_{k+1}} \left(|f - f_{Q_{k+1}}| w^{-1} \right)^2 w \mathrm{d}z \right) \left(\int_{Q_{k+1}} w^{\frac{s}{2-s}} \mathrm{d}z \right)^{(2-s)/s} \quad (s<2). \end{aligned} \tag{11}$$

For the first integral in the above inequality, we have

$$\begin{aligned}&\int_{Q_{k+1}} \left(|f - f_{Q_{k+1}}| w^{-1} \right)^2 w \mathrm{d}z \\ = & |Q_{k+1}|^{2\beta} \int_{Q_{k+1}} \left(|f - f_{Q_{k+1}}| |Q_{k+1}|^{-\beta} w^{-1} \right)^2 w \mathrm{d}z \\ = & |Q_{k+1}|^{2\beta} \int_0^\infty t w(\{x \in Q_{k+1} : |f - f_{Q_{k+1}}| |Q_{k+1}|^{-\beta} w^{-1} > t\}) \mathrm{d}t \\ \le & C|Q_{k+1}|^{2\beta} \int_0^\infty t \exp(-t/\|f\|_{\mathcal{L}_w(\beta)}) \mathrm{d}t \, w(Q_{k+1}) \quad \text{(by Lemma 2)} \\ \le & C|Q_{k+1}|^{2\beta} \|f\|^2_{\mathcal{L}_w(\beta)} w(Q_{k+1}). \end{aligned} \tag{12}$$

For the second integral in (11), since $1/s = \lambda/2 - 1/n$, and $(2+\delta)/(1+\delta) + 2/n \le \lambda$, it follows that $s/(2-s) \leq 1+\delta$. By Hölder inequality, we have

$$\begin{aligned}\left(\int_{Q_{k+1}} w^{\frac{s}{2-s}} \mathrm{d}z\right)^{(2-s)/s} &\leq \left(\frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} w^{1+\delta}\mathrm{d}z\right)^{1/1+\delta} |Q_{k+1}|^{(2-s)/s} \\ &\leq C\frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} w dz |Q_{k+1}|^{(2/s)-1} \\ &\leq C\frac{w(Q_{k+1})}{|Q_{k+1}|^2}|Q_{k+1}|^{\lambda-2/n}. \end{aligned} \tag{13}$$

Therefore, by (11), (12), and (13), we obtain

$$\begin{aligned}\left(\int_{Q_{k+1}} |f-f_{Q_{k+1}}|^s \mathrm{d}z\right)^{2/s} &\leq C|Q_{k+1}|^{\lambda+2\beta-2/n}\left(\|f\|_{\mathcal{L}_w(\beta)}\frac{w(Q_{k+1})}{|Q_{k+1}|}\right)^2 \\ &\leq C(2^{k+1}r)^{\lambda n+2n\beta-2}\left(\|f\|_{\mathcal{L}_w(\beta)}\frac{w(Q)}{|Q|}\right)^2. \end{aligned}$$

As for the second term on the right in (10), by (4), we have

$$|f_{Q_{k+1}}-f_Q|^2|Q_{k+1}|^{2/s} \leq C\left((2^{k+1}r)^{n\beta}+r^{n\beta}\right)^2\left(\|f\|_{\mathcal{L}_w(\beta)}\frac{w(Q)}{|Q|}\right)^2|Q_{k+1}|^{\lambda-2/n}. \tag{15}$$

Thus, by (15), (14), and (10), it follows that

$$\left(\int_{Q_{k+1}} |f-f_Q|^s \mathrm{d}z\right)^{2/s} \leq C(2^{k+1}r)^{\lambda n+2n\beta-2}\left(\|f\|_{\mathcal{L}_w(\beta)}\frac{w(Q)}{|Q|}\right)^2,$$

so (9) holds.

Thus, by (9) and (8) we have

$$B_k \leq Cr^2(2^k r)^{\lambda n+2n\beta-2}\left(\|f\|_{\mathcal{L}_w(\beta)}\frac{w(Q)}{|Q|}\right)^2. \tag{16}$$

Thus, by (6), (7) and (16), we get

$$\begin{aligned}\tau &\leq Cr\|f\|_{\mathcal{L}_w(\beta)}\frac{w(Q)}{|Q|}\left\{\sum_{k=1}^{\infty}(2^k r)^{-\lambda n}(2^k r)^{\lambda n+2n\beta-2}\right\}^{1/2} \\ &\leq C\frac{w(Q)}{|Q|}|Q|^{\beta}\|f\|_{\mathcal{L}_w(\beta)}, \end{aligned}$$

which completes the proof of Lemma 4.

Remark 3 For $f \in \tilde{\mathcal{L}}_w(\beta)$, we also have the similar results as lemma 3 and Lemma 4.

3.The proofs of the theorems

We only prove Theorem 1. Theorem 2 can be dealt with quite similarly.

Let Tf be one of the Littlewood-Paley operators. $|E| = |\{x \in R^n : Tf(x) < \infty\}| > 0$. Let x_0 be a density point of E, and Q' be a cube with center x_0. Set $Q = (1/d)Q'$(then $Q' = \mathrm{d}Q$). We write f as

$$\begin{aligned} f(x) &= f_Q + [f(x) - f_Q]\chi_Q(x) + [f(x) - f_Q]\chi_{Q^c}(x) \\ &\doteq f_Q + g(x) + h(x). \end{aligned}$$

Since $Tf_Q \equiv 0$, we have

$$Tf(x) \le Tg(x) + Th(x) \tag{17}$$

and

$$Th(x) \le Tf(x) + Tg(x). \tag{18}$$

For $w \in A_1$, we have $w \in A_2$ and also $w^{-1} \in A_2$. It follows that

$$\begin{aligned} \|T(g)\|^2_{L^2(w^{-1}\mathrm{d}x)} &\le C\|g\|^2_{L^2(w^{-1}\mathrm{d}x)} = C\int_Q |f - f_Q|^2 w^{-1}\mathrm{d}x \\ &= C|Q|^{2\beta}\int_Q \left(|f - f_Q||Q|^{-\beta}w^{-1}\right)^2 w(x)\mathrm{d}x \\ &= C|Q|^{2\beta}\int_0^\infty tw(\{x \in Q : \ |f - f_Q||Q|^{-\beta}w^{-1} > t\})\mathrm{d}t \\ &\le C|Q|^{2\beta}\int_0^\infty t\exp(-ct/\|f\|_{\mathcal{L}_w(\beta)})w(Q)\mathrm{d}t \quad \text{(By Lemma 2)} \\ &\le C|Q|^{2\beta}\|f\|^2_{\mathcal{L}_w(\beta)}w(Q). \end{aligned} \tag{19}$$

Thus $T(g)(x) < \infty$ *a.e.* on R^n. Taking $x' \in Q' \cap E \subset dQ$ such that $T(f)(x') < \infty$ and $T(g)(x') < \infty$. By (18), we have $T(h)(x') < \infty$. Using lemmas 3 and 4, it follows that $T(h)(x) < \infty$ for all $x \in \mathrm{d}Q = Q'$.

By (17), $T(f)(x)$ is finite for almost every $x \in Q'$, and, consequently, for almost every $x \in R^n$.

Let Q' be any cube and $Q = (1/d)Q'$. By (19), we have

$$\int_Q |T(g)(x)|dx \le C(w(Q))^{1/2}\|T(g)\|_{L^2(w^{-1}dx)} \le C|Q|^\beta\|f\|_{\mathcal{L}_w(\beta)}w(Q).$$

Choose an $x' \in \mathrm{d}Q$ such that $T(h)(x') < \infty$. Then, it follows from Lemmas 3 and 4 that

$$\begin{aligned} \int_{Q'} |T(f)(x) - T(h)(x')|\mathrm{d}x &\le \int_Q |T(g)(x)|\mathrm{d}x + \int_{Q'} |T(h)(x) - T(h)(x')|\mathrm{d}x \\ &\le C\|f\|_{\mathcal{L}_w(\beta)}|Q|^\beta w(Q) \le C\|f\|_{\mathcal{L}_w(\beta)}|Q'|^\beta w(Q'). \end{aligned}$$

This completes the proof of the Theorem 1.

Remark 4 For Marcinkiewicz integral $\mu(f)$[6], we also have the similar results as Theorems 1 and 2.

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加权 Lipschitz 空间上的 Littlewood-Paley 算子

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摘要: 本文研究了加权 Lipschitz 空间上的 Littlewood-Paley 算子.，证明了一个加权 Lipschitz 函数在 Littlewood-Paley 算子下的象或者几乎处处等于无穷或者仍是一个加权 Lipschitz 函数.

关键词: Littlewood-Paley 算子；加权 Lipschitz 空间；加权模不等式.