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Dual Loopless Nonseparable Near-Triangulations on Projective Plane

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Abstract: In this paper we enumerate the rooted dual loopless nonseparable near-triangular maps on the sphere and the projective plane with the valency of root-face and the number of inner faces as parameters. Explicit expressions of enumerating functions are derived for such maps on the sphere and the projective plane. A parametric expression of the generating function is obtained for the rooted 2-connected triangular maps on the projective plane, from which asymptotics evaluations are derived.

Key words: dual loopless; nonseparable; triangulation.
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1. Introduction

A (rooted) near-triangular map on a surface is a (rooted) map on the surface such that each face except possibly the root face has the valency three. A (rooted) triangular map is a (rooted) near-tiangular map with root face valency also being three. A map M is called separable if its edge set can be partitioned into two disjoint non-null submaps S and T so that there is just one vertex incident with both S and T. The vertex is said to be a separable vertex of M. A rooted nonseparable near-triangular map is a rooted map without any separable vertex. A map is dual loopless if its dual map is loopless. A circuit C on a surface $\sum$ is called *essential* if $\sum - C$ has no connected region homeomorphic to a disc, otherwise it is planar.

Gao[4] has studied the rooted loopless near-triangular maps on the sphere and the projective plane with the valency of root-face and the number of vertices. In this paper we study the rooted dual loopless nonseparable near-triangular maps on the sphere and the projective plane with the valency of root-face and the number of inner faces as parameters.

Let $\mathcal{S}$ and $\mathcal{M}$ be respectively the set of all rooted dual loopless nonseparable near-triangular maps on the sphere and the projective plane. Their enumerating functions are, respectively,

$$f(x,y)=\sum_{M\in\mathcal{S}} x^{m(M)}y^{n(M)};\quad F(x,y)=\sum_{M\in\mathcal{M}} x^{m(M)}y^{n(M)},$$

where $m(M)$ and $n(M)$ be respectively the valency of root-face of M and the number of inner faces of M.

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Given two maps M_1 and M_2 with roots $r_1 = r(M_1)$ and $r_2 = r(M_2)$, respectively, we define $M = M_1 \odot M_2$ to be the map obtained by identifying the vertex $V_{(p\alpha\beta)r_1}$ and the vertex V_{r_2}, the root-edge of M is the same as those of M_1, but the root-face of M is the composition of $f_r(M_1)$ and $f_r(M_2)$, where $f_r(M_i)$ is the root-face of $M_i (i = 1, 2)$. Further, for two sets of maps $\mathcal{M}_1$ and $\mathcal{M}_2$, the set of maps $\mathcal{M}_1 \bigodot \mathcal{M}_2 = \{M_1 \odot M_2 \,|M_i \in \mathcal{M}_i, i = 1, 2\}$ Terms not mentioned here can be found in[5].

For any map M, let $e_r(M)$ be the root-edge of M. Let $M - e_r(M)$ stand for the resultant maps of deleting $e_r(M)$ from M.

2. Maps on the sphere

The set $\mathcal{S}$ is divided into four parts as

$$\mathcal{S} = \mathcal{S}_I + \mathcal{S}_{II} + \mathcal{S}_{III} + \mathcal{S}_{IV}, \tag{1}$$

where $\mathcal{S}_I$ consists of only the triangle;

$$\mathcal{S}_{II} = \{M|M \in \mathcal{S}, M - e_r(M) \in \mathcal{S}\};$$
$$\mathcal{S}_{III} = \{M|M \in \mathcal{S}, M - e_r(M) \in \mathcal{S} \bigodot \mathcal{S}\};$$
$$\mathcal{S}_{IV} = \{M|M \in \mathcal{S}, M \notin \mathcal{S}_I, M \notin \mathcal{S}_{II}, M \notin \mathcal{S}_{III}\}.$$

Let f_i $(i = I, II, III, IV)$ be the enumerating function of $\mathcal{S}_i$, and $f_i (i = 2, 3, \cdots)$ be the coefficient of x^i in f. Then

$$f_I = x^3 y; \quad f_{II} = x^{-1} y(f - x^2 f_2); \quad f_{III} = x^{-1} y f^2; \quad f_{IV} = 2xyf. \tag{2}$$

Applying (1) and (2), we obtain

Theorem 1 *The enumerating function $f = f(x, y)$ satisfies the following functional equation:*

$$y f^2 + (2x^2 y + y - x) f + x^4 y - x^2 y f_2 = 0, \tag{3}$$

where f_2 is the coefficient of x^2 in f.

By means of equation (3), we can get the discriminant:

$$\delta(x, y) = (2x^2 y + y - x)^2 - 4x^2 y^2 (x^2 - f_2).$$

If x is considered as a power series η of y such that $x = \eta$ is a double root of $\delta(x, y)$. Then we may find the following two equation:

$$\delta(x, y)|_{x=\eta} = 0; \quad \frac{\partial \delta(x, y)}{\partial x}|_{x=\eta} = 0. \tag{4}$$

Applying (4), we have

$$y = \eta(1 - 2\eta^2); \quad f_2 = \frac{\eta^2 (1 - 4\eta^2)}{(1 - 2\eta^2)^2}. \tag{5}$$

Applying (5) and Lagrangian inversion, we have

Theorem 2 *The enumerating function $f_2 = f_2(y)$ has the following explicit expression:*

$$f_2(y) = \sum_{n\geq 1} \frac{2^{n+1} \times (3n)!}{n!(2(n+1))!} y^{2n}.$$

3. Maps on the projective plane

The set $\mathcal{M}$ is divided into three parts as

$$\mathcal{M} = \mathcal{M}_I + \mathcal{M}_{II} + \mathcal{M}_{III}, \tag{6}$$

where

$$\mathcal{M}_I = \{M|M \in \mathcal{M}, M - e_r(M) \in \mathcal{M}\};$$
$$\mathcal{M}_{II} = \{M|M \in \mathcal{S}, M - e_r(M) \in \mathcal{M} \odot \mathcal{S}, \quad \text{or} \ \mathrm{M} - \mathrm{e_r(M)} \in \mathcal{S} \odot \mathcal{M}\};$$
$$\mathcal{M}_{III} = \{M|M \in \mathcal{M}, M \notin \mathcal{M}_I, M \notin \mathcal{M}_{II}\}.$$

Let F_i $(i = I, II, III)$ be the enumerating function of $\mathcal{M}_i$, and $F_i (i = 2, 3, \cdots)$ be the coefficient of x^i in F.

For any map M' in $\mathcal{M}_I$, the non-root side of $e_r(M')$ must border a triangle. After removing $e_r(M')$, the triangle is connected to the root face. We can reverse this process by taking a rooted dual loopless nonseparable near-triangulation M and introducing a new root-edge runing to the vertex $V_{(p\alpha\beta)^2 r}$ (Terms not mentioned here can be found in[5]) from the root-vertex V_r two edges backs. If $V_{(p\alpha\beta)^2 r} = V_r$, a loop will be introduced. Letting L_p be the set of those M with $V_{(p\alpha\beta)^2 r} = V_r$, then

$$F_I = x^{-1} y(F - L_p). \tag{7}$$

Lemma 1 *$L_p = x^2 F_2 + L$, where $L = f^2 + f - x^2 f_2 - \frac{x(f_2 - y^2) f}{y f_2} - x^3 y$.*

Proof For any map M in $\mathcal{M}$ with $V_{(p\alpha\beta)^2 r} = V_r$, if the valency of root-face of M is not two, make a cut along a simple closed curve that lies in the root face of M except where it cuts through V_r. Let V' be the vertex split from V_r, then the cut destroy a crosscap, the resulting map lies on the sphere. One may easily check that the process is reversible, except that an edge connecting V_r and V' would become a loop. Since the resulting map may be separable, we have

$$L_p = x^2 F_2 + f^2 + f - x^2 f_2 - f_{\mathcal{J}},$$

where $f_{\mathcal{J}}$ is the enumerating function of $\mathcal{J}$, and $\mathcal{J}$ corresponds to the maps with some edges joining V_r and V'.

Now we analyze $\mathcal{J}$. Let e_1 be the leftmost edge joining V_r and V'. Cutting along e_1, and spliting e_1 into two edges, we get two separate pieces. The left pice is a triangular map with no edges other than e_1 joining V_r and V'. The right piece is a general rooted dual loopless nonseparable near-triangular map. Let $\triangle$ be the generating function for maps like the left piece. Then

$$f_{\mathcal{J}} = \frac{\triangle f}{x^2} + x^3 y. \tag{8}$$

It is easy to see

$$f_2 = y f_3; \quad x^3 f_3 = \triangle f_2 + x^3 y. \tag{9}$$

Applying (8) and (9), we have

$$f_{\mathcal{J}} = \frac{x(f_2 - y^2) f}{y f_2} + x^3 y.$$

It is easy to see

$$F_{II} = 2x^{-1} y f F. \tag{10}$$

An edge is called a double edge if the same face appears on both sides of it. For any map M in $\mathcal{M}$, the valency of inner face of M is three, and M is dual loopless nonseparable. Theorefore, $M - e_r(M)$ has at most two double edges. We can divided $\mathcal{M}_{III}$ into four parts as

$$\mathcal{M}_{III} = \mathcal{M}_{III}^1 + \mathcal{M}_{III}^2 + \mathcal{M}_{III}^3 + \mathcal{M}_{III}^4,$$

where

$\mathcal{M}_{III}^1 = \{M | M \in \mathcal{M}_{III}, M - e_r(M)$ has a double edge and the double edge is not on essential $\}$;

$\mathcal{M}_{III}^2 = \{M | M \in \mathcal{M}_{III}, M - e_r(M)$ has a double edge and the double edge is on essential circuit$\}$;

$\mathcal{M}_{III}^3 = \{M | M \in \mathcal{M}_{III}, M - e_r(M)$ has two double edge and the two double edges are on essential circuit$\}$;

$\mathcal{M}_{III}^4 = \{M | M \in \mathcal{M}_{III}, M - e_r(M)$ has two double edges and the one double edge is on essential circuit and another double edge is not on essential circuit$\}$.

Let $F_{III}^i (i = 1, 2, 3, 4)$ be the enumerating function of $\mathcal{M}_{III}^i$. It is easy to see

$$F_{III}^1 = 2xyF. \tag{11}$$

For any map M in $\mathcal{M}_{III}^2$, if $M - e_r(M)$ is nonseparable, then we delete the double edge of $M - e_r(M)$. The resulting map is a dual loopless near-triangulations on the sphere. Notice that $M - e_r(M)$ may be separable, and $M - e_r(M)$ add a new edge as the root-edge of new map being loopless, we have

$$F_{III}^2 = x^2 \sum_{k \geq 1} [\frac{x^2 \partial (\frac{f}{x})}{\partial x}]^k \times [2x^{-1} y + 2x^{-1} y f] - 2xyf. \tag{12}$$

Similarly, we have

$$F_{III}^3 = x^3 y \sum_{k \geq 1} [\frac{x^2 \partial (\frac{f}{x})}{\partial x}]^k, \tag{13}$$

$$F_{III}^4 = 2x^3y\sum_{k\geq 1}[\frac{x^2\partial(\frac{f}{x})}{\partial x}]^k. \tag{14}$$

Applying (6)–(14), we have

Theorem 3 *The enumerating function $F = F(x,y)$ satisfies the following functional equation*

$$F = x^{-1}y(F - x^2F_2 - L) + 2x^{-1}yfF + 2xyF + \frac{(xf_x - f)(2xy + 2xyf + 3x^3y)}{1 - xf_x + f} - 2xyf. \tag{15}$$

Rewriting (15), we have

$$(2x^2y + y - x + 2yf)F = x^2yF_2 + yL - \frac{x(xf_x - f)(2xy + 2xyf + 3x^3y)}{1 - xf_x + f} + 2x^2yf. \tag{16}$$

Let $A(x,y) = 2x^2y + y - x + 2yf$, then $A(x,y)^2 = \delta(x,y)$.

Therefore, we have

$$A(\eta,y) = 2\eta^2y + y - \eta + 2yf(\eta,y) = 0, \tag{17}$$

$$A_x(x,y)|_{x=\eta} = 4\eta y - 1 + 2yf_x(x,y)|_{x=\eta}, \tag{18}$$

$$2(A_x(x,y)|_{x=\eta})^2 = \delta_{xx}(x,y)|_{x=\eta} = 2(1-2\eta^2)(1-6\eta^2). \tag{19}$$

Let $Y = y^2$, $t = \eta^2$, then $Y = t(1-2t)^2$.

Applying (17)–(19), we have

$$f = \frac{2t^2}{1-2t};\quad A_x(x,y)\big|_{x=t^{\frac{1}{2}}} = -\sqrt{(1-2t)(1-6t)}, \tag{20}$$

$$f_x(x,y)\big|_{x=t^{\frac{1}{2}}} = \frac{-\sqrt{(1-2t)(1-6t)} + 1 - 4t(1-2t)}{2t^{\frac{1}{2}}(1-2t)}. \tag{21}$$

Applying (5), (16), (20), (21) and Lemma 1, we have

$$F_2 = \frac{-\sqrt{(1-2t)(1-6t)} + 1 - 4t}{2t(1-2t)} + \frac{2t^2 - t}{1-4t}. \tag{22}$$

Lemma 2 *Let $t = t(Y)$ be the solution to $Y = t(1-2t)^2$ with $t(0) = 0$. Then $t(Y)$ is analytic at 0, and has a unique singularity at $Y = \frac{2}{27}$. Moreover,*

$$1 - 6t \approx \frac{2}{\sqrt{3}}(1 - \frac{27}{2}Y)^{\frac{1}{2}}.$$

Proof From $Y = t(1-2t)^2$, it is evident that $t(Y)$ is an algebraic function whose singularities are branch points. Let $G(Y,t) = t(1-2t)^2 - Y$. Since $G_t(Y,t) = (1-2t)(1-6t)$ and $G_t(0,0) = 1$, by the Implicit Function Theorem there exists a unique $t(Y)$ which is analytic at 0 and satisfies $t(0) = 0$. To find the branch points, solve $G_t(Y,t) = 0$ and $G(Y,t) = 0$, we obtain $(Y,t) = (0,\frac{1}{2})$ and $(\frac{2}{27},\frac{1}{6})$. Since $t(0) = 0$, $(0,\frac{1}{2})$ is on the wrong Riemann sheet and so $\frac{2}{27}$ is the only singularity of $t(Y)$. As in the proof of Theorem 5 of [1], we have

$$t - \frac{1}{6} \approx -\sqrt{\frac{-2G_Y(\frac{2}{27},\frac{1}{6})(Y - \frac{2}{27})}{G_{tt}(\frac{2}{27},\frac{1}{6})}} = -\frac{1}{3\sqrt{3}}(1 - \frac{27}{2}Y)^{\frac{1}{2}}.$$

Theorem 4 *The number of $2n$ face, rooted 2-connected triangular maps on the projective plane is asymptotic to*

$$-\frac{3^{\frac{5}{4}}}{\Gamma(-\frac{1}{4})}n^{-\frac{5}{4}}(\frac{27}{2})^n.$$

Proof From (22) and Darboux's Theorem (see [1, Theorem 4]), we have

$$F_2 \approx -\frac{3\sqrt{6}}{2}\sqrt{1-6t}.$$

Using Lemma 2 and the fact that $1-6t=0$ at $Y=\frac{2}{27}$, we get

$$F_2 \approx -3^{\frac{5}{4}}(1-\frac{27}{2}Y)^{\frac{1}{4}}.$$

It is easy to see, $F_2 = yF_3$ By Darboux's Theorem, we obtain

$$F_3(y)|_{y^{2n-1}} = F_2(y)|_{y^{2n}} = F_2(Y)|_{Y^n} \approx -\frac{3^{\frac{5}{4}}}{\Gamma(-\frac{1}{4})}n^{-\frac{5}{4}}(\frac{27}{2})^n.$$

Let v and φ be respectively the number of vertices and faces of rooted 2-connected triangular maps on the projective plane. By Euler formula, we obtain

$$\varphi = 2(v-1).$$

Gao[4] has obtained the number of n vertex, rooted 2-connected triangular maps on the projective plane is asymptotic to

$$-\frac{3^{\frac{5}{4}}}{\Gamma(-\frac{1}{4})}n^{-\frac{5}{4}}(\frac{27}{2})^{n-1}.$$

Applying $y^2 = t(1-2t)^2$, (22) and Lagrangian inversion, we obtain

Theorem 5 *The enumerating function $F_2 = F_2(y)$ has the following explicit expression*

$$F_2 = A_{n,i} + B_n + C_{n,i} + D_n + E_n + F_{n,i},$$

where

$$A_{n,i} = -\sum_{n\geq 1}\sum_{n\geq i\geq 0}\frac{3^{n+1-i}\times 2^{-n}(4n+2i)!(2n)!(2n-2i)!}{ni!(2n+i)!(4n)!((n-i)!)^2}y^{2n};$$

$$B_n = \sum_{n\geq 1}\frac{2^{n+1}(3n)!}{n(2n)!n!}y^{2n};$$

$$C_{n,i} = \sum_{n\geq 1}\sum_{n+1\geq i\geq 0}\frac{3^{n+1-i}\times 2^{-2-n}(4n+2i+2)!(2n+1)!(2n-2i+2)!}{ni!(2n+i+1)!(4n+2)!((n+1-i)!)^2}y^{2n};$$

$$D_n = -\sum_{n\geq 1}\frac{2^n(3n+2)!}{n(n+1)!(2n+1)!}y^{2n};\quad E_n = -\sum_{n\geq 1}\frac{2^{n-2}(3n-2)!}{n!(2n-1)!}y^{2n};$$

$$F_{n,i} = -\sum_{n\geq 1}\sum_{n-1\geq i\geq 0}\frac{2^{2n-3-i}(n-i)(2n+i-1)!}{ni!(2n-1)!}y^{2n}.$$

References:

[1] BENDER E A. *Asymptotic methods in enumeration* [J]. SIAM Rev., 1974, **16**: 485–515.
[2] BENDER E A, and CANFIELD E R. *The asymptotic number of rooted maps on a surface* [J]. J. Combin. Theory Ser. A, 1986, **43**: 244–257.
[3] BEBDER E A, WORMALD N C. *The number of loopless planar maps* [J]. Discrete Math., 1985, **54**: 235–237.
[4] GAO Zhi-cheng. *The number of rooted 2-connected triangular maps on the projective plane* [J]. J. Combin. Theory Ser. B, 1991, **53**: 130–142.
[5] LIU Yan-pei. *Enumerative Theory of Maps* [M]. Kluwer, Dordrecht/Boston/London, 1999.
[6] LIU Yan-pei. *Enumerating rooted loopless planar maps* [J]. Acta Math. Appal. Sinica, 1985, **2**: 14–26.
[7] REN Han, LIU Yan-pei. *4-regular maps on the klein bottle* [J]. J. Combin. Theory Ser. B, 2001, **82**: 118–137.
[8] TUTTE W T. *A census of planar triangulations* [J]. Canad. J. Math., 1962, **14**: 21–38.
[9] TUTTE W T. *A census of planar maps* [J]. Canad. J. Math., 1963, **15**: 249–271.

射影平面上的对偶无环不可分近三角剖分地图

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摘要: 本文研究了球面和射影平面上对偶无环不可分近三角剖分带根地图的以根面次和内面数为参数的计数问题, 得到了这类地图在球面和射影平面上的计数函数满足的方程. 还得到了射影平面上 2 连通地图一个参数的显示表达式和渐近估计式.

关键词: 不可分; 三角剖分.