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Some Results about Derivations of Prime Rings

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Abstract: Let R be a 2-torsion free noncommutative prime ring and d be a derivation of R. If $[x^d, x]x^d = 0$ for all $x \in R$, then $d = 0$. Furthermore, if $[[x^d, x], x^d] = 0$ for all $x \in R$, then $d = 0$.

Key words: derivation; prime ring; semiprime ring.
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1. Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We often use basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = 0$ implies $a = 0$. An additive map $d : R \longrightarrow R$ is called a derivation if $(xy)^d = x^d y + xy^d$ holds for all $x, y \in R$. A derivation d is inner if there exists $a \in R$ such that $x^d = [a, x]$ for all $x \in R$.

The theory of commuting and centralizing maps on (semi-)prime rings was motivated by the results of Posner[1] and was developed by Vukman[2] and Breasar[3−5]. Posner's second theorem states that if there exists a nonzero centralizing derivation on a prime ring R, then R is commutative. Many people have extended this result in various ways and obtained many powerful results. In the representative works, the work of Vukman[2] and Bresar[3−5] should be mentioned at least. Vukman[2] proved that if d is a derivation of a 2-torsion free prime ring such that $[[x^d, x], x] = 0$ for all $x \in R$, then $d = 0$ or R is commutative. Bresar[4] generalized this result by showing that the same conclusion holds for each additive map. Moreover, Bresar[5] described all commuting traces of biadditive maps on certain prime rings. I.N.Herstein[6] proved that if there exists a nonzero derivation d on a prime ring R such that the map $x \longrightarrow (x^d)^2$ is commuting on R, then R may be noncommutative. That is, the following relation

$$[x^d, x]x^d + x^d[x^d, x] = 0, \quad x \in R$$

does not imply that $d = 0$. There arises the question of whether we can obtain some similar results when the Jordan Version of the above relation holds on a noncommutative prime ring R. This leads to our work, which can be considered as an extension of Posner's second theorem.

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2. Main results

Theorem 1 *Let R be a 2-torsion free noncommutative prime ring and d be a derivation of R. If $[x^d,x]x^d=0$ for all $x\in R$, then $d=0$.*

Proof We define a map $F(\cdot,\cdot):R\times R\longrightarrow R$ by the relation

$$F(x,y)=[x^d,y]+[y^d,x],\quad x,y\in R.$$

Obviously, $F(x,y)=F(y,x)$ for all $x,y\in R$ and $F(\cdot,\cdot)$ is additive in two variables. Moreover, a simple calculation shows that the relation

$$F(xy,z)=F(x,z)y+xF(y,z)+x^d[y,z]+[x,z]y^d$$

holds for all $x,y,z\in R$. Let us write $f(x)$ for $F(x,x)$ briefly. Then

$$f(x)=2[x^d,x],\quad x\in R.$$

It is easy to see that $f(x+y)=f(x)+f(y)+2F(x,y)$ for all $x,y\in R$. Now the assumption of the theorem can be written as follows:

$$f(x)x^d=0,\quad x\in R. \tag{1}$$

The linearization of (1) gives

$$\begin{aligned}0&=f(x+y)(x+y)^d=(f(x)+f(y)+2F(x,y))(x^d+y^d)\\&=f(x)x^d+f(x)y^d+f(y)x^d+f(y)y^d+2F(x,y)x^d+2F(x,y)y^d,\end{aligned}$$

which reduces to

$$f(x)y^d+f(y)x^d+2F(x,y)x^d+2F(x,y)y^d=0,\quad x,y\in R. \tag{2}$$

Replacing x by $-x$ in (2), we obtain

$$f(x)y^d-f(y)x^d+2F(x,y)x^d-2F(x,y)y^d=0\quad x,y\in R, \tag{3}$$

since $f(x)=f(-x)$. Combining (2) with (3), we have

$$f(x)y^d+2F(x,y)x^d=0, \tag{4}$$

since R is 2-torsion free.

Let y be yz in (4), we obtain $f(x)(yz)^d+2F(x,yz)x^d=0$. Expanding it, we have

$$f(x)y^dz+f(x)yz^d+2F(x,y)zx^d+2yF(x,z)x^d+2y^d[z,x]x^d+2[y,x]z^dx^d=0.$$

It follows from (4) that $f(x)y^d=-2F(x,y)x^d$ and $2F(x,z)x^d=-f(x)z^d$. Hence, we get

$$2F(x,y)[z,x^d]+[f(x),y]z^d+2y^d[z,x]x^d+2[y,x]z^dx^d=0. \tag{5}$$

Replacing z by x^d in (5), we obtain

$$[f(x),y]x^{d^2}+2[y,x]x^{d^2}x^d=0,\quad x,y\in R. \tag{6}$$

Substituting yz for y in (6), we get

$$\begin{aligned}0&=([f(x),y]z+y[f(x),z])x^{d^2}+2y[z,x]x^{d^2}x^d+2[y,x]zx^{d^2}x^d\\&=[f(x),y]zx^{d^2}+2[y,x]zx^{d^2}x^d+y([f(x),z]x^{d^2}+2[z,x]x^{d^2}x^d).\end{aligned}$$

Combining (5) with (6), we obtain

$$[f(x),y]zx^{d^2}+2[y,x]zx^{d^2}x^d=0,\quad x,y,z\in R. \tag{7}$$

Replacing y by x in (7), we have

$$[f(x),x]zx^{d^2}=0,\quad x,z\in R. \tag{8}$$

Substituting x^d for y in (7) and using $f(x)x^d=0$, we get

$$f(x)zx^{d^2}x^d-x^df(x)zx^{d^2}=0. \tag{9}$$

Now we are ready to prove that

$$x^{d^2}x^d=0,\qquad x\in R. \tag{10}$$

Suppose on the contrary that $r^{d^2}r^d\neq 0$ for some $r\in R$. Obviously, we have $r^{d^2}\neq 0$. Since R is a prime ring, $[f(r),r]=0$ by (8).
Replacing y by x in (5), we obtain

$$2f(x)[z,x^d]+[f(x),x]z^d+2x^d[z,x]x^d=0,\quad x\in R.$$

Particularly $2f(r)[z,r^d]+2r^d[z,r]r^d=0$, which reduces to

$$f(r)zr^d+r^d[z,r]r^d=0,\quad z\in R. \tag{11}$$

Replacing z by r^dx in (11), we get

$$f(r)r^dxr^d+r^d[r^dx,r]r^d=0.$$

By $f(r)r^d=0$, we have

$$0=r^d[r^d,r]xr^d+(r^d)^2[x,r]r^d=2(r^d)^2[x,r]r^d+r^df(r)xr^d,\quad x\in R. \tag{12}$$

On the other hand left multiplication by r^d of (11) and putting z for x , we obtain

$$(r^d)^2[x,r]r^d+r^df(r)xr^d=0,\quad x\in R. \tag{13}$$

Combining (12) with (13), we have

$$r^df(r)xr^d=0,\quad x\in R. \tag{14}$$

Since R is a prime ring , $r^d f(r)=0$. Substituting r for x in (9), we get

$$f(r)zr^{d^2}r^d=0, \quad z\in R. \tag{15}$$

Since $r^{d^2}r^d\neq 0$ and R is a prime ring , $f(r)=0$. Replacing x by r in (7), we obtain $[y,r]zr^{d^2}r^d=0, y,z\in R$, which reduces to $[y,r]=0, y\in R$. We therefore prove that $x^{d^2}x^d=0$ in the case $x\notin Z$. It remains to prove that $x^{d^2}x^d=0$ also holds in the case of $x\in Z$. Let $x\in Z$ and $y\notin Z$. We have $x+y\notin Z$. We note that $(x+y)^{d^2}(x+y)^d=0$ and $y^{d^2}y^d=0$. Then

$$x^{d^2}x^d+y^{d^2}x^d+x^{d^2}y^d=0. \tag{16}$$

Replacing x by $-x$ in (16), we have

$$x^{d^2}x^d-y^{d^2}x^d-x^{d^2}y^d=0. \tag{17}$$

It follows from (16) and (17) that $x^{d^2}x^d=0$, which completes the proof of (10). The linearization of (10) leads to

$$x^{d^2}y^d+y^{d^2}x^d=0, \quad x,y\in R. \tag{18}$$

Substituting yz for y in (18), we get

$$\begin{aligned}0&=x^{d^2}(yz)^d+(yz)^{d^2}x^d\\&=x^{d^2}y^dz+x^{d^2}yz^d+y^{d^2}zx^d+2y^dz^dx^d+yz^{d^2}x^d.\end{aligned}$$

which reduces to

$$\begin{aligned}0&=-y^{d^2}x^dz+x^{d^2}yz^d+y^{d^2}zx^d+2y^dz^dx^d-yx^{d^2}z^d\\&=y^{d^2}[z,x^d]+[x^{d^2},y]z^d+2y^dz^dx^d.\end{aligned} \tag{19}$$

Putting x^d for z, we obtain $[x^{d^2},y]x^{d^2}+2y^dx^{d^2}x^d=0$. Since $x^{d^2}x^d=0$, we get

$$[x^{d^2},y]x^{d^2}=0, \quad x,y\in R. \tag{20}$$

For a fixed $x\in R$, the map $y\longrightarrow[x^{d^2},y]$ is an inner derivation of R. Then (20) and [1, Lemma 1] imply that $[x^{d^2},y]=0$ or $x^{d^2}=0$ for all $y\in R$. Therefore, $x^{d^2}\in Z$ for all $x\in R$.

Left multiplication by y of (10) and since $x^{d^2}\in Z$, we obtain $yx^{d^2}x^d=x^{d^2}yx^d=0$ for all $y\in R$. Since R is a prime ring , $x^{d^2}=0$ for all $x\in R$. Applying [3, Theorem 1] yields that $x^d=0$ for all $x\in R$. The proof of Theorem 1 is completed.

By Theorem 1, we can prove the following result, which can be viewed as an extension of Posner's second theorem.

Theorem 2 *Let R be a 2-torsion free noncommutative prime ring and d be a derivation of R. If $[[x^d,x],x^d]=0$ for all $x\in R$, then $d=0$.*

Proof We define a map $F(\cdot,\cdot):R\times R\longrightarrow R$ by the relation

$$F(x,y)=[x^d,y]+[y^d,x], \quad x,y\in R.$$

By Theorem 1, we know that

$$F(xy,z)=F(z,xy)=F(x,z)y+xF(y,z)+x^d[y,z]+[x,z]y^d$$

holds for all $x,y,z\in R$. Let us write $f(x)$ for $F(x,x)$. Then $f(x)=F(x,x)=2[x^d,x]$ for all $x\in R$. Thus

$$f(x+y)=f(x)+f(y)+2F(x,y). \tag{21}$$

Now the assumption of the theorem can be written as follows

$$[f(x),x^d]=0,\quad x\in R. \tag{22}$$

Obviously,

$$\begin{aligned}0&=[[f(x),x^d],x]=[f(x)x^d-x^df(x),x]\\&=f(x)x^dx-x^df(x)x-xf(x)x^d+xx^df(x),\\0&=[[x^d,x],f(x)]=[x^dx-xx^d,f(x)]\\&=x^dxf(x)-xx^df(x)-f(x)x^dx+f(x)xx^d.\end{aligned}$$

Combining the above two equations, we obtain

$$0=x^dxf(x)-x^df(x)x+f(x)xx^d-xf(x)x^d=[[f(x),x],x^d].$$

So

$$[[f(x),x],x^d]=[[f(x),x^d],x]=0. \tag{23}$$

The linearization of (22) gives

$$\begin{aligned}0&=[f(x+y),(x+y)^d]\\&=[f(x),y^d]+[f(y),x^d]+2[F(x,y),x^d]+2[F(x,y),y^d].\end{aligned}$$

Replacing x by $-x$, we have

$$0=[f(x),y^d]-[f(y),x^d]+2[F(x,y),x^d]-2[F(x,y),y^d].$$

Combining the above two equations, we get $2[f(x),y^d]+4[F(x,y),x^d]=0$. Since R is 2-torsion free, we obtain

$$[f(x),y^d]+2[F(x,y),x^d]=0,\quad x,y\in R. \tag{24}$$

Substituting xy for y, we have

$$\begin{aligned}0=&[f(x),(xy)^d]+2[F(x,xy),x^d]\\=&[f(x),x^dy+xy^d]+2[f(x)y+xF(x,y)+x^d[y,x],x^d]\\=&[f(x),x^d]y+x^d[f(x),y]+[f(x),x]y^d+x[f(x),y^d]+\\&2[f(x),x^d]y+2f(x)[y,x^d]+2[x,x^d]F(x,y)+\\&2x[F(x,y),x^d]+2x^d[[y,x],x^d].\end{aligned}$$

Using (24) and $[f(x), x^d] = 0$, we obtain

$$0 = x^d[f(x), y] + [f(x), x]y^d + 2f(x)[y, x^d] - f(x)F(x, y) + 2x^d[[y, x], x^d]. \tag{25}$$

Putting yx for y in (25), then by (25), we get

$$0 = x^d y[f(x), x] + [f(x), x]yx^d - 2f(x)yf(x) - f(x)[y, x]x^d - x^d[y, x]f(x). \tag{26}$$

Replacing y by $x^d y$, we have

$$\begin{aligned} 0 =& x^d x^d y[f(x), x] + [f(x), x]x^d yx^d - 2f(x)x^d yf(x) - \\ & f(x)[x^d y, x]x^d - x^d[x^d y, x]f(x) \\ =& (x^d)^2 y[f(x), x] + [f(x), x]x^d yx^d - 2f(x)x^d yf(x) - \\ & f(x)[x^d, x]yx^d - f(x)x^d[y, x]x^d - (x^d)^2[y, x]f(x) - \\ & x^d[x^d, x]yf(x). \end{aligned} \tag{27}$$

Left multiplication by x^d of (26) gives

$$\begin{aligned} 0 =& (x^d)^2 y[f(x), x] + x^d[f(x), x]yx^d - \\ & 2x^d f(x)yf(x) - x^d f(x)[y, x]x^d - (x^d)^2[y, x]f(x). \end{aligned} \tag{28}$$

Substracting (28) from (27) and using $[[f(x), x], x^d] = 0$ and $[x^d, f(x)] = 0$, we have

$$0 = f(x)[x^d, x]yx^d + x^d[x^d, x]yf(x) = f(x)^2 yx^d + x^d f(x)yf(x). \tag{29}$$

Right multiplication by $f(x)$ of (29) yields that

$$0 = f(x)^2 yx^d f(x) + x^d f(x)yf(x)^2, \quad x, y \in R. \tag{30}$$

Now we intend to prove that

$$x^d f(x) = 0, \quad x \in R. \tag{31}$$

Suppose on the contrary that $r^d f(r) \neq 0$ for some $r \in R$. By [7, Lemma], we obtain $f(r)^2 = 0$. Replacing x by r in (29), we obtain $r^d f(r)yf(r) = 0$, $y \in R$. Since R is a prime ring, we have $r^d f(r) = 0$. This is a contradiction to the assumption, which completes the proof of (31). Using $[x^d, f(x)] = 0$, we get

$$f(x)x^d = x^d f(x) = 0, \quad x \in R.$$

Then Theorem 1 implies that $d = 0$. The proof of Theorem 2 is completed.

By Theorem 2, we can give an alternative proof of the following result which was first proved by Lanski[9].

Theorem 3 *Let R be a 2-torsion free noncommutative prime ring and d, g be derivations of R. If $[x^d, x^g] = 0$ for all $x \in R$ and $d \neq 0$, then there exists $\lambda \in C$, such that $g = \lambda d$, where C is the extended centroid of R.*

Proof The linearization of $[x^d, x^g] = 0$ gives

$$0 = [(x+y)^d, (x+y)^g] = [x^d, y^g] + [y^d, x^g], \quad x, y \in R. \tag{32}$$

Substituting yx for y, we obtain

$$\begin{aligned} 0 =& [x^d, (yx)^g] + [(yx)^d, x^g] \\ =& [x^d, y^g]x + y^g[x^d, x] + [x^d, y]x^g + y[x^d, x^g] + \\ & y^d[x, x^g] + [y^d, x^g]x + y[x^d, x^g] + [y, x^g]x^d. \end{aligned}$$

Using (32) and $[x^d, x^g] = 0$, we get

$$y^g[x^d, x] + [x^d, y]x^g + y^d[x, x^g] + [y, x^g]x^d = 0. \tag{33}$$

Replacing y by yz, we have

$$\begin{aligned} 0 =& (yz)^g[x^d, x] + [x^d, yz]x^g + (yz)^d[x, x^g] + [yz, x^g]x^d \\ =& y^g z[x^d, x] + yz^g[x^d, x] + [x^d, y]zx^g + y[x^d, z]x^g + \\ & y^d z[x, x^g] + yz^d[x, x^g] + y[z, x^g]x^d + [y, x^g]zx^d. \end{aligned}$$

It follows from (33) that

$$0 = y^g z[x^d, x] + [x^d, y]zx^g + y^d z[x, x^g] + [y, x^g]zx^d, \quad x, y, z \in R. \tag{34}$$

Putting zx^d for z, we have

$$0 = y^g zx^d[x^d, x] + [x^d, y]zx^d x^g + y^d zx^d[x, x^g] + [y, x^g]zx^d x^d. \tag{35}$$

Right multiplication of (34) by x^d leads to

$$0 = y^g z[x^d, x]x^d + [x^d, y]zx^g x^d + y^d z[x, x^g]x^d + [y, x^g]zx^d x^d. \tag{36}$$

Substracting (36) from (35) and using $[x^g, x^d] = 0$, we get

$$y^g z[[x^d, x], x^d] + y^d z[[x, x^g], x^d] = 0, \quad x, y, z \in R. \tag{37}$$

Since $d \neq 0$, $[[a^d, a], a^d] \neq 0$ for some $a \in R$ by Theorem 2. By (37) and [6, pp20-23], we know that y^g, y^d are C-dependent, that is, there exists $\lambda(y) \in C$ such that $y^g = \lambda(y)y^d$, where C is the extended centroid of R.

In (34), we substitute a for x, y^g for $\lambda(y)y^d$ and a^g for $\lambda(a)a^d$, then

$$0 = \lambda(y)y^d z[a^d, a] + [a^d, y]z\lambda(a)a^d + y^d z[a, \lambda(a)a^d] + [y, \lambda(a)a^d]za^d$$

which reduces to

$$(\lambda(y) - \lambda(a))y^d z[a^d, a] = 0, \quad y, z \in R. \tag{38}$$

Since R is a prime ring and $[[a^d,a],a^d]\neq 0$, we know that $[a^d,a]\neq 0$. We now obtain $(\lambda(y)-\lambda(a))y^d=0$, hence $\lambda(y)y^d=\lambda(a)y^d$. Now we have $y^g=\lambda(y)y^d=\lambda(a)y^d$. The proof of Theorem 3 is completed.

In [2], Vukman has proved that if d is a derivation on a complex semisimple Banach algebra B and ad^3+d^2 is a derivation on B for some complex number a, then $d=0$. We now extend this result to the case of semiprime rings.

Theorem 4 *Let R be a 2-torsion free semiprime ring and d, g be derivations of R. If $\lambda x^{d^3}+x^{d^2}=x^g$ for all $x\in R$ and for some $\lambda\in C$, then $d=0$.*

Proof We define $H(x)=\lambda x^{d^3}+x^{d^2}=x^g$. Then

$$H(xy)=\lambda(xy)^{d^3}+(xy)^{d^2}=H(x)y+xH(y)+3\lambda x^{d^2}y^d+3\lambda x^dy^{d^2}+2x^dy^d \tag{39}$$

$$g(xy)=H(xy)=x^gy+xy^g=H(x)y+xH(y). \tag{40}$$

Substracting (40) from (39), we have $3\lambda x^{d^2}y^d+3\lambda x^dy^{d^2}+2x^dy^d=0$. Now we define $P(x)=3\lambda x^{d^2}+x^d$, then

$$P(x)y^d+x^dP(y)=0. \tag{41}$$

Replacing y by yx in (41), we obtain

$$\begin{aligned}0=&P(x)(yx)^d+x^dP(yx)\\=&P(x)y^dx+P(x)yx^d+x^dP(y)x+x^dyP(x)+6\lambda x^dy^dx^d.\end{aligned}$$

Which reduces to

$$0=P(x)yx^d+x^dyP(x)+6\lambda x^dy^dx^d. \tag{42}$$

Substituting yx^d for y in (42), we get

$$\begin{aligned}0&=P(x)yx^dx^d+x^dyx^dP(x)+6\lambda x^dy^dx^dx^d+6\lambda x^dyx^{d^2}x^d\\&=-x^dyP(x)x^d+x^dyx^dP(x)+6\lambda x^dyx^{d^2}x^d.\end{aligned} \tag{43}$$

It follows from (41) and (43) that

$$\begin{aligned}0&=x^dyx^dx^d+3\lambda x^dyx^dx^{d^2}-x^dyP(x)x^d+6\lambda x^dyx^{d^2}x^d\\&=x^dy(x^d+3\lambda x^{d^2})x^d-x^dyP(x)x^d+3\lambda x^dyx^dx^{d^2}+3\lambda x^dyx^{d^2}x^d\\&=3\lambda x^dyx^dx^{d^2}+3\lambda x^dyx^{d^2}x^d\\&=x^dyx^dP(x)+x^dyP(x)x^d-2x^dyx^dx^d\\&=-2x^dyx^dx^d.\end{aligned} \tag{44}$$

Left multiplication of (44) by x^d gives

$$2(x^d)^2y(x^d)^2=0,\quad x,y\in R. \tag{45}$$

Since R is 2-torsion free semiprime ring, $d = 0$ by the well-known Giambruno-Herstein theorem[4]. The proof of Theorem 4 is completed.

References:

[1] POSNER E C. *Derivations in prime rings* [J]. Proc. Amer. Math. Soc., 1957, **8**: 1093–1100.
[2] VUKMAN J. *A result concerning derivations in Banach algebras* [J]. Proc. Amer. Math. Soc., 1992, **116**: 971–975.
[3] BRESAR M. *Certralizing mappings and derivations in prime ring* [J]. J. Algebra, 1993, **156**: 385–394.
[4] BRESAR M. *Commuting traces of biadditive mappings, Commutativity-preserving mappings and Lie mapping* [J]. Trans. Amer. Math. Soc., 1993, **335**: 525–545.
[5] BRESAR M. *On a generalization of the notion of centralzing mappings* [J]. Proc. Amer. Math. Soc., 1992, **114**: 641–649.
[6] HERSTEIN I N. Derivations of prime rings having power central values[J]. Contemp. Math., 1982, **13**: 163–171.
[7] HERSTEIN I N. *Topics in Ring Theory* [M]. The University of Chicago Press, Chicago, Ill.-London 1969.
[8] I.N.Herstein. *Rings with Involution* [M]. The University of Chicago Press, Chicago, Ill.-London, 1976.
[9] LANSKI C. *Differential idetities of prime rings, Kharchenko's theoerm and applications* [J]. Contemp. Math., 1992, **124**: 111–128.
[10] GIAMBRUNO A, HERSTEIN I N. *Derivations with nilpotent values* [J]. Rend. Circ. Mat. Palermo, 1981, **30**: 199–206.

关于素环导子的一些结论

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摘要: 设 R 是一个特征不等于 2 的不可交换的素环, d 为 R 的一个导子, 如果 $[x^d, x]x^d = 0$ 对所有的 $x \in R$ 都成立, 那么 $d = 0$. 进一步, 如果对所有的 $x \in R$, 都有 $[[x^d, x], x^d] = 0$, 那么 $d = 0$.

关键词: 导子; 素环; 半素环.