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Inclusive Regular Separation in L-Fuzzy Topological Spaces

ZHAO Bin

(Dept. of Math., Shaanxi Normal University, Xi'an 710062, China)
(E-mail: zhaobin@snnu.edu.cn)

Abstract: In this paper, we first introduce the concept of the inclusive regular separation in L-fuzzy topological spaces. Then we compare the inclusive regular separation with pointed regular separation and regular separation, and discuss the implicative and non-implicative relations among the above three separations. Finally, we illustrate that the inclusive regular separation is harmonic with the inclusive normal separation and inclusive completely regular separation.

Key words: L-fuzzy topological space; inclusive regular space; L-fuzzy unit interval; inclusive normal space; inclusive completely regular space; L-valued Zadeh function.
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1. Basic concepts

Throughout this paper, L denotes a completely distributive complete lattice which has an order-reversing involution "$'$", a smallest element 0 and a largest element 1 ($0 \neq 1$). An L-fuzzy topological space (briefly, L-fts) is a pair (L^X, δ) where L^X is the family consisting of all mappings (i.e., L-fuzzy sets[1]) from the nonempty crisp set X into L, and δ is a subfamily of L^X closed under the operations of finite intersections and arbitrary unions[2]. The elements in δ are called open sets and the elements in $\delta' = \{B' \mid B \in \delta\}$ are called closed sets. If $L = [0, 1]$, then L-fts (L^X, δ), L-fuzzy set, L-fuzzy point are briefly called fts (X, δ), F set, F point, respectively. An element a of L is said to be $\vee$-irreducible (or a molecule[3]) if $a \leq b \vee c$ implies that $a \leq b$ or $a \leq c$, where $b, c \in L$. The set consisting of all nonzero $\vee$-irreducible elements of L will be denoted by $M(L)$, and the set consisting of all nonzero $\vee$-irreducible elements of L^X will be denoted by $M^*(L^X)$, i.e., $M^*(L^X) = \{x_\lambda : x \in X, \lambda \in M(L)\}$. A^-, A^o, and A' will denote the closure, the interior and the pseudo-complement of $A \in L^X$, respectively. The family consisting of all remote neighborhoods[4] of x_λ will be denoted by $\eta(x_\lambda)$. Let $\eta^-(x_\lambda)$ be the family $\eta(x_\lambda) \cap \delta'$. The top and bottom elements in L^X are denoted by 1_X and 0_X, respectively. An L-fts (L^X, δ) is said to be fully stratified, if $[r] \in \delta$ for all $r \in L$, where $[r] \in L^X$ satisfying $[r](x) = r$ for all $x \in X$.

For other undefined notions and symbols in this paper, we refer to [3, 4, 5].

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Definition 1.1[4,6] *Let R be the real line, L be a fuzzy lattice, and $\sum$ be the set $\{\lambda : R \to L \mid \lambda$ is an order-reversing mapping and $\lambda(t) = 1$ for $t < 0, \lambda(t) = 0$ for $t > 1\}$. $\forall\ \lambda \in \sum$ and $\forall\ t \in R$, $\lambda(t+) = \vee\{\lambda(s) \mid s > t\}$, $\lambda(t-) = \wedge\{\lambda(s) \mid s < t\}$. For any $\lambda,\ \mu \in \sum$, we define an order on $\sum$ by $\lambda \leq \mu$ iff $\lambda(t-) \leq \mu(t-)$, $\lambda(t+) \leq \mu(t+)$ for $t \in R$, and an equivalence relation on $\sum$ by $\lambda \sim \mu$ iff $\lambda(t+) = \mu(t+)$, $\lambda(t-) = \mu(t-)$ for $t \in R$. Let $X = \sum / \sim$. $\forall\ t \in R$, l_t, $r_t : X \to L$ defined by $\forall\ x = [\lambda] \in X$, $l_t([\lambda]) = (\lambda(t-))'$, $r_t([\lambda]) = \lambda(t+)$, then $\mathcal{L} = \{l_t \mid t \in R\}$ and $\mathcal{R} = \{r_t \mid t \in R\}$ are two L-fuzzy topologies on L^X. Let φ be the L-fuzzy topology on L^X generated by $\mathcal{L} \bigcup \mathcal{R}$ as a subbase. Then we call $(L^X,\ \varphi)$ the L-fuzzy unit interval, and denote it by $I(L)$.*

Lemma 1.2 Let $s,\ t \in R$ and $s < t$, then $r_t \leq l'_t \leq r_s \leq l'_s$, and $l_s \leq r'_s \leq l_t \leq r'_t$, especially, $l'_1 \leq r_0$.

Lemma 1.3[4] *Let A be a nonempty subset of R, then (i) $l_{\sup A} = \sup\{l_t \mid t \in A\}$ if A has an upper bound; (ii) $r_{\inf A} = \sup\{r_t \mid t \in A\}$ if A has a lower bound.*

Definition 1.4 *A mapping $F : L^X \to L^Y$ is said to be an L-valued Zadeh function induced by a crisp mapping $f : X \to Y$ if $F(A)(y) = \vee\{A(x) \mid f(x) = y\}$ for every $A \in L^X$ and every $y \in Y$. For the sake of convenience, we denote the L-valued Zadeh function F by f. It can be easily checked that $f^{-1}(B) = \vee\{A \in L^X \mid f(A) \leq B\} = B \circ f$ for all $B \in L^Y$.*

Definition 1.5[4,7] *Let $(L^X,\ \delta)$ and (L^Y, η) be two L-fts. An L-valued Zadeh function $f : (L^X,\ \delta) \to (L^Y,\ \eta)$ is said to be continuous if $f^{-1}(B) \in \delta$ for every $B \in \eta$.*

Definition 1.6[7,8] *Let $\{(L^{X_t},\ \delta_t)\}_{t \in T}$ be a family of L-fts indexed by a set T, $X = \prod_{t \in T} X_t$, and δ be an L-fuzzy topology on X generated by the subbase $\{P_t^{-1}(A_t) \mid A_t \in \delta_t,\ t \in T\}$, where $P_t : L^X \to L^{X_t}$, called projection, is the L-valued Zadeh function induced by usual projection $P_t : X \to X_t (t \in T)$. Then the L-fts $(L^X,\ \delta)$ is called the product space of the family $\{(L^{X_t},\ \delta_t)\}_{t \in T}$ of L-fts, and $(L^{X_t},\ \delta_t)$ are called factor spaces of $(L^X,\ \delta)$. It is clearly that the set $\beta = \{\wedge_{t \in S} P_t^{-1}(A_t) \mid S \in 2^{(T)},\ \forall\ t \in S,\ A_t \in \delta_t\}$ is a base of δ, where $2^{(T)} = \{S \subset T \mid S$ is a finite subset of $T\}$.*

Definition 1.7[4] *Let $\{X_t\}_{t \in T}$ be a family of nonempty sets, $T \neq \emptyset$, $X = \prod_{t \in T} X_t$, $x = \{x_t\}_{t \in T} \in X$ and $s \in T$. Suppose that $\tilde{X}_s = \{y \in X \mid \forall\ t \in T,\ y_t = x_t$ if $t \neq s\}$, and $(L^X,\ \delta)$ be the product space of the family $\{(L^{X_t},\ \delta_t)\}_{t \in T}$ of L-fts, then the subspace $(L^{\tilde{X}_s},\ \delta \mid \tilde{X}_s)$ of $(L^X,\ \delta)$ is called the L-fuzzy plane which pass through the point x and parallel the factor space $(L^{X_s},\ \delta_s)$. It was proved in [6] that if for $s \in T$, $(L^{X_s},\ \delta_s)$ be fully stratified, then the L-fuzzy plane $(L^{\tilde{X}_s},\ \delta \mid \tilde{X}_s)$ is homeomorphic to the factor space $(L^{X_s},\ \delta_s)$.*

Definition 1.8 *Let $(L^X,\ \delta)$ be an L-fts. If $\forall\ x_\lambda \in L^X$, x_λ is a closed set of $(L^X,\ \delta)$, then $(L^X,\ \delta)$ is called ST_1 space. Where x_λ is defined by $\forall\ y \in X$, $x_\lambda(y) = 0$ for $y \neq x$, and $x_\lambda(x) = \lambda$.*

Definition 1.9[9] *Let* $(L^X,\ \delta)$ *be an L-fts. If* $\forall\ U \in \delta,\ \exists\ \mathcal{V} = \{V_i\}_{i\in I} \subset \delta$ *such that* $U = \vee_{i\in I} V_i = \vee_{i\in I} V_i^-$, *then* $(L^X,\ \delta)$ *is called an inclusive regular space. An inclusive regular space with* ST_1 *separation is called* T_3^* *space.*

Theorem 1.10 *Let* $(L^X,\ \delta)$ *be an L-fts. Then* $(L^X,\ \delta)$ *is an inclusive regular space if and only if* $\forall\ B \in \delta',\ \exists\ \{F_i\}_{i\in I} \subset \delta'$ *such that*

$$B = \wedge_{i\in I} F_i = \wedge_{i\in I} F_i^o.$$

The next theorem can be used to simplify the verifications of the inclusive regularity. It's proof is omitted.

Theorem 1.11 *Let* $(L^Y,\ \delta)$ *be an L-fts, and* β *a base of* δ. *If* $\forall\ U \in \beta,\ \exists\ \{V_i\}_{i\in T} \subset \delta$ *such that* $U = \vee_{i\in I} V_i = \vee_{i\in I} V_i^-$, *then* $(L^Y,\ \delta)$ *is an inclusive regular space.*

Example 1.12 L-fuzzy unit interval is an inclusive regular space. Actually, let $I(L) = (L^X,\ \varphi)$ be the L-fuzzy unit interval. By Definition 1.1 we know that the set

$$\beta = \{r_{t_1} \wedge \cdots \wedge r_{t_n} \wedge l_{s_1} \wedge \cdots \wedge l_{s_m} \mid r_{t_i} \in \mathcal{R}(i \le n),\ \ l_{s_j} \in \mathcal{L}(j \le m)\}$$

be a base of φ. Let $a = \max\{t_1, \cdots,\ t_n\}$, $b = \min\{s_1, \cdots,\ s_m\}$, by Lemma 1.2,

$$r_{t_1} \wedge \cdots \wedge r_{t_n} \wedge l_{s_1} \wedge \cdots \wedge l_{s_m} = r_a \wedge l_b.$$

This means that

$$\beta = \{r_a \wedge l_b \mid r_a \in \mathcal{R},\ l_b \in \mathcal{L}\}.$$

For $r_a \wedge l_b \in \beta$, we first show that $r_a \wedge l_b = \vee\{r_{a+\varepsilon} \wedge l_{b-\varepsilon} \mid \varepsilon > 0\}$. In fact, by Lemma 1.3 we know that $r_a = \vee\{r_{a+\varepsilon} \mid \varepsilon > 0\}$ and $l_b = \vee\{l_{b-\delta} \mid \delta > 0\}$. Then it follows from the completely distributive law that $r_a \wedge l_b = \vee\{r_{a+\varepsilon} \wedge l_{b-\delta} \mid \varepsilon > 0,\ \delta > 0\} \ge \vee\{r_{a+\varepsilon} \wedge l_{b-\varepsilon} \mid \varepsilon > 0\}$. On the other hand, take $r_{a+\varepsilon} \wedge l_{b-\delta}$, and let $\mu = \min\{\varepsilon,\ \delta\}$. Then by Lemma 1.2, $r_{a+\varepsilon} \wedge l_{b-\delta} \le r_{a+\mu} \wedge l_{b-\mu}$. Hence $r_a \wedge l_b = \vee\{r_{a+\varepsilon} \wedge l_{b-\varepsilon} \mid \varepsilon > 0\}$.

Now we prove that $\forall\, \varepsilon > 0$, $(r_{a+\varepsilon} \wedge l_{b-\varepsilon})^- \le r_a \wedge l_b$. In fact, by Lemma 1.2, $r_{a+\varepsilon} \le l'_{a+\varepsilon} \le r_a$, and $l_{b-\varepsilon} \le r'_{b-\varepsilon} \le l_b$. Thus,

$$r_{a+\varepsilon} \wedge l_{b-\varepsilon} \le l'_{a+\varepsilon} \wedge r'_{b-\varepsilon} \le r_a \wedge l_b.$$

By the fact that $l'_{a+\varepsilon}$ and $r'_{b-\varepsilon}$ are closed sets, we see that

$$(r_{a+\varepsilon} \wedge l_{b-\varepsilon})^- \le l'_{a+\varepsilon} \wedge r'_{b-\varepsilon} \le r_a \wedge l_b,$$

which means that

$$r_a \wedge l_b = \vee\,\{r_{a+\varepsilon} \wedge l_{b-\varepsilon} \mid \varepsilon > 0\} = \vee\{(r_{a+\varepsilon} \wedge l_{b-\varepsilon})^- \mid \varepsilon > 0\}.$$

This indicates that $I(L)$ is an inclusive regular space.

If $L = [0,\ 1]$, we have

Theorem 1.13 *Let $(Y,\ \delta)$ be a fts, then $(Y,\ \delta)$ is an inclusive regular space if and only if $\forall\ U \in \delta$ and $\forall\ F$ point y_λ, if $\lambda < U(y)$, then there is a $V \in \delta$ such that $y_\lambda \leq V \leq V^- \leq U$.*

Proof Suppose that $(Y,\ \delta)$ is an inclusive regular space, $U \in \delta$ and $0 < \lambda < U(y)$. Take $\{V_t\}_{t\in T} \subset \delta$ such that $U = \vee_{t\in T} V_t = \vee_{t\in T} V_t^-$. By $\lambda < U(y) = \vee_{t\in T} V_t(y)$ we know that there is a $t_o \in T$ such that $\lambda < V_{to}(y)$. Let $V = V_{to}$, then $y_\lambda \leq V \leq V^- \leq U$.

On the other hand, for any F point y_λ, if $\lambda < U(y)$, then there is a $V \in \delta$ such that $y_\lambda \leq V \leq V^- \leq U$. We denote the V by $V_{(y,\ \lambda)}$, then $U = \vee\{y_\lambda \mid 0 < \lambda < U(y),\ y \in Y\}$ and for y_λ satisfying $0 < \lambda < U(y)$, $y_\lambda \leq V_{(y,\ \lambda)} \leq V^-_{(y,\ \lambda)} \leq U$. This implies that

$$U = \vee\{V_{(y,\ \lambda)} \mid y \in Y,\ 0 < \lambda < U(y)\} = \vee\{V^-_{(y,\ \lambda)} \mid y \in Y,\ 0 < \lambda < U(y)\}.$$

Hence $(Y,\ \delta)$ is an inclusive regular space.

2. Comparison among three regularities

Definition 2.1[4] *Let $(L^X,\ \delta)$ be an L-fts. If $\forall\ U \in \delta$ and $\forall$ L-fuzzy point x_λ satisfying $x_\lambda \leq U$, $\exists\ V \in \delta$ such that $x_\lambda \leq V \leq V^- \leq U$, then $(L^X,\ \delta)$ is called the pointed regular space.*

Theorem 2.2 *A pointed regular space is an inclusive regular space.*

Proof Suppose that $(L^X,\ \delta)$ is a pointed regular space, then $\forall\ U \in \delta$, and $\forall\ y_\lambda \in M^*(L^X)$ satisfying $y_\lambda \leq U$, $\exists\ V \in \delta$ such that $y_\lambda \leq V \leq V^- \leq U$. Denote the V by $V_{(y,\ \lambda)}$, then $y_\lambda \leq V_{(y,\ \lambda)} \leq V^-_{(y,\ \lambda)} \leq U$. Hence

$$\begin{aligned} U &= \vee\{y_\lambda \mid y_\lambda \in M^*(L^X),\ y_\lambda \leq U\} = \vee\{V_{(y,\ \lambda)} \mid y_\lambda \in M^*(L^X),\ y_\lambda \leq U\} \\ &= \vee\{V^-_{(y,\ \lambda)} \mid y_\lambda \in M^*(L^X),\ y_\lambda \leq U\}. \end{aligned}$$

This shows that $(L^X,\ \delta)$ is an inclusive regular space.

Example 2.3 Inclusive regular separation does not imply pointed regular separation. Take $X = \{x,\ y\}$ and $L = [0,\ 1]$. Let $\delta = \{A \in [0,\ 1]^X \mid A(x) = 0 \Rightarrow A(y) = 0\}$. It is easy to check that δ is a fuzzy topology on X, and $\delta' = \{B \in [0,\ 1]^X \mid B(x) = 1 \Rightarrow B(y) = 1\}$. Then we can verify that:

(1) $(X,\ \delta)$ is an inclusive regular space.

In fact, $\forall\ U \in \delta$, we have

(i) $U(x) \neq 1,\ U(y) = 1$; (ii) $U(x) \neq 1,\ U(y) \neq 1$;

(iii) $U(x) = 1,\ U(y) = 1$; (iv) $U(x) = 1, U(y) \neq 1$.

If U belongs to (i), (ii), (iii), then $U \in \delta'$, which means that $U = \vee U = \vee U^-$. If U belongs to (iv), we define $\mathcal{V} = \{V_i \mid i \in (0,\ 1)\}$ satisfying

$$\forall\ i \in (0,\ 1),\ V_i(z) = \begin{cases} i, & z = x \\ U(y), & z = y \end{cases},$$

then $V_i \in \delta$ and $V_i \in \delta'$. Hence $U = \vee_{i\in(0,1)} V_i = \vee_{i\in(0,1)} V_i^-$. This shows that (X, δ) is an inclusive regular space.

(2) (X, δ) is not a pointed regular space.

In fact, take $U \in \delta$ by the following condition:

$$U(z) = \begin{cases} 1, & z = x \\ 0, & z = y \end{cases},$$

then $x_1 \leq U$. If A is a closed set and $x_1 \leq A$, then $A(y) = 1$ by $A(x) = 1$, *i.e.*, $A = 1_X$. Thus $A \not\leq U$, *i.e.*, there is not a $V \in \delta$ such that $x_1 \leq V \leq V^- \leq U$. Hence (X, δ) is not a pointed regular space.

Definition 2.4 *Let A be an L-fuzzy set on X. If $\exists\, a \in L$, $a \neq 0$ such that $A(x) \geq 0 \Leftrightarrow A(x) \geq a$ for $x \in X$. Then A is called a pseudo crisp set.*

Definition 2.5[4] *Let (L^X, δ) be an L-fts, $A \in L^X$ and $P \in \delta'$. If $\forall\, x \in X$, $A(x) > 0 \Longrightarrow A(x) \not\leq P(x)$, then P is called a closed remote neighborhood of A. If $Q \in L^X$ and $Q \leq P$, then Q is called a remote neighborhood of A. Denote $\eta(A) = \{Q \in L^X \mid Q$ is a remote neighborhood of $A\}$, and $\eta^-(A) = \{P \in L^X \mid P$ is a closed remote neighborhood of $A\}$.*

Definition 2.6[4] *Let (L^X, δ) be an L-fts. If for any nonzero pseudo crisp closed set A and $x_\lambda \in M^*(L^X)$ satisfying $x \notin$ supp A, there exist $P \in \eta(x_\lambda)$ and $Q \in \eta(A)$ such that $P \vee Q = 1_X$, then (L^X, δ) is called a regular space, where supp$A = \{y \in X \mid A(y) > 0\}$.*

Example 2.7 Inclusive regular separation does not imply regular separation. Take $X = \{x, y\}$, $L = [0, 1]$. Let $\delta = \{0_X, 1_X, A_1, A_2\}$, where

$$A_1(z) = \begin{cases} \frac{1}{3}, & z = x \\ 0, & z = y \end{cases}, \quad A_2(z) = \begin{cases} \frac{2}{3}, & z = x \\ 1, & z = y \end{cases}.$$

It is easy to check that δ is a fuzzy topology on X, and $\delta' = \{0_X, 1_X, A_2, A_1, \}$, i.e., the elements of δ is open and closed set. Hence (X, δ) is an inclusive regular space. But (X, δ) is not a regular space. In fact, take $A_1 \in \delta'$ and $y_{\frac{1}{2}}$, then A_1 is a pseudo crisp closed set and $y \notin$suppA_1. For any closed remote neighborhood P of $y_{\frac{1}{2}}$, and $\forall\, Q \in \eta^-(A_1)$, then $Q = 0_X$, which means that $P \vee Q \neq 1_X$. Hence (X, δ) is not a regular space.

Example 2.8 Regular separation does not imply inclusive regular separation. Take $X = \{x, y\}$, $L = [0, 1]$. Let $\delta = \{0_X, 1_X, A\}$, where

$$A(z) = \begin{cases} \frac{1}{2}, & z = x \\ \frac{2}{3}, & z = y \end{cases},$$

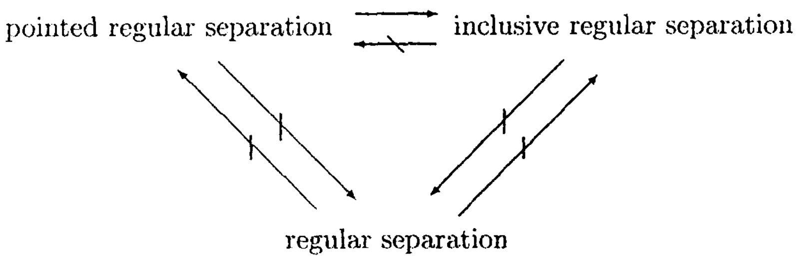
then δ is a fuzzy topology on X, and $\delta' = \{0, 1, B\}$, where

$$B(z) = \begin{cases} \frac{1}{2}, & z = x \\ \frac{1}{3}, & z = y \end{cases}.$$

For any nonzero closed set $C \in \delta'$, supp$C = X$, i.e., there is not fuzzy point x_λ such that $x \notin X$. Hence (X, δ) is a regular space. But (X, δ) is not an inclusive regular space. In fact, take $A \in \delta$, then $A \notin \delta'$. If $V \in \delta$, and $V \le A$, then $V = 0_X$ or $V = A$. This implies that $V^- = 0_X$ or $V^- = 1_X$, which shows that there are not open set family $\{V_i \mid i \in I\}$ such that $A = \vee_{i \in I} V_i = \vee_{i \in I} V_i^-$. Hence $(X,\ \delta)$ is not an inclusive regular space.

Remark 2.9 (i) There are examples[4] to illustrate that pointed regular separation does not imply regular separation, and regular separation does not imply pointed regular separation.

(ii) The relations among the above three separations can be summarized by the following graph:



3. Properties of inclusive regular separation

Definition 3.1[4] *Let $(L^X,\ \delta)$ be an L-fts. If for every $U \in \delta$ there is a family of L–fuzzy sets $\{W_t\}_{t \in T}$ such that $U = \vee_{t \in T} W_t$, and $\forall\, t \in T$, there exists a continuous L-valued Zadeh function $f_t : (L^X,\ \delta) \longrightarrow I(L)$ satisfying $W_t \le f_t^{-1}(l_1') \le f_t^{-1}(r_0) \le U$, then we call $(L^X,\ \delta)$ an inclusive completely regular space. An inclusive completely regular space with ST_1 separation is called $T^*_{3\frac{1}{2}}$ space.*

Theorem 3.2 *An inclusive completely regular space is an inclusive regular space.*

Proof Let $(L^X,\ \delta)$ be an inclusive completely regular space, then $\forall\, U \in \delta$, $\exists\, \{W_i\}_{i \in I} \subset L^X$ such that $U = \vee_{i \in I} W_i$, and $\forall\, i \in I$, there exists a continuous L-valued Zadeh function $f_i : (L^X,\ \delta) \to I(L)$ such that

$$W_i \le f_i^{-1}(l_1') \le f_i^{-1}(r_0) \le U.$$

By Lemma 1.2 and the inequality $0 < \frac{1}{3} < \frac{1}{2} < 1$, we know that $l_1' \le r_{\frac{1}{2}} \le l_{\frac{1}{3}}' \le r_0$. Let $V_i = f_i^{-1}(r_{\frac{1}{2}})$, then $V_i \in \delta$ and $V_i^- \le f_i^{-1}(l_{\frac{1}{3}}') \le f_i^{-1}(r_0) \le U$. Thus

$$\begin{aligned} U &= \vee_{i \in I} W_i \le \vee_{i \in I} f_i^{-1}(l_1') \le \vee_{i \in I} f_i^{-1}(r_{\frac{1}{2}}) = \vee_{i \in I} V_i \le \vee_{i \in I} V_i^- \le \vee_{i \in I} f_i^{-1}(l_{\frac{1}{3}}') \\ &\le \vee_{i \in I} f_i^{-1}(r_0) \le U. \end{aligned}$$

This shows that $U = \vee_{i \in I} V_i = \vee_{i \in I} V_i^-$. Hence $(L^X,\ \delta)$ is an inclusive regular space.

Definition 3.3[4,6] *Let $(L^X,\ \delta)$ be an L-fts. If for any open set U and closed set K such that $K \le U$, there exists an open set V such that $K \le V \le V^- \le U$, then we call $(L^X,\ \delta)$ an inclusive normal space. An inclusive normal space with ST_1 separation is called T_4^* space.*

Lemma 3.4[4,6] *Let $(L^X,\ \delta)$ be an L-fts.Then $(L^X,\ \delta)$ is an inclusive normal space if and only* *if for any closed set K and open set U such that $K \le U$, there exists a continuous L-valued Zadeh function $f: (L^X, \delta) \to I(L)$ such that $K \le f^{-1}(l_1') \le f^{-1}(r_0) \le U$.*

Corollary 3.5 $T_4^* \Rightarrow T_{3\frac{1}{2}}^* \Rightarrow T_3^*$.

Theorem 3.6 *An inclusive regular space with inclusive normal separation is an inclusive completely regular space.*

Proof Let (L^X, δ) be an inclusive regular space, and an inclusive normal space. $\forall\, U \in \delta$, by inclusive regular separation we know that $\exists\, \{V_i\}_{i\in I} \subset \delta$ such that $U = \vee_{i\in I} V_i = \vee_{i\in I} V_i^-$. For each $i \in I$, $V_i^- \le U$, by inclusive normal separation and Lemma 3.4 we see that there exists a continuous L-valued Zadeh function $f_i: (L^X, \delta) \to I(L)$ such that

$$V_i^- \le f_i^{-1}(l_1') \le f_i^{-1}(r_0) \le U.$$

By Definition 3.1, (L^X, δ) is an inclusive completely regular space.

Definition 3.7[4] *Let (L^X, δ) be an L-fts. If for any closed set A and open cover $\mathcal{U}$ of A, i.e., $A \le \vee\, \mathcal{U}$ and $\mathcal{U} \subset \delta$, there exists a finite subcover $\mathcal{V}$ of A, i.e., $\mathcal{V} \subset \mathcal{U}$, $\mathcal{V}$ has finite members and $A \le \vee\, \mathcal{V}$, then (L^X, δ) is called having strong finite-cover property.*

Theorem 3.8 *An inclusive regular space having strong finite-cover property is an inclusive normal space.*

Proof Let (L^X, δ) be an inclusive regular space and have the strong finite-cover property. $\forall\, K \in \delta'$, and $\forall\, U \in \delta$, if $K \le U$, then there exists $\mathcal{V} = \{V_i\}_{i\in I} \subset \delta$ such that

$$U = \vee_{i\in I} V_i = \vee_{i\in I} V_i^-.$$

Because (L^X, δ) has the strong finite-cover property and $K \le \vee_{i\in I} V_i$, there exist $V_1, V_2, \cdots, V_n \in \mathcal{V}$ such that $K \le V_1 \vee V_2 \vee \cdots \vee V_n$. Then

$$K \le V_1 \vee V_2 \vee \cdots \vee V_n \le (V_1 \vee V_2 \vee \cdots \vee V_n)^- = V_1^- \vee V_2^- \vee \cdots \vee V_n^- \le \vee_{i\in I} V_i^- = U.$$

Let $V = V_1 \vee V_2 \vee \cdots \vee V_n$, then $V \in \delta$ and $K \le V \le V^- \le U$. Hence (L^X, δ) is an inclusive normal space.

Corollary 3.9 *An inclusive regular space having strong finite-cover property is an inclusive completely regular space.*

Finally, we need to point that the hereditary, homeomorphic invariant, good L-extension and multiplicativity of the inclusive regular separation are all right[10].

4. Conclusion

(1) In this paper, the separation of Definition 2.1 is called the pointed regularity and is not called the inclusive regularity[4] due to these reasons :

(i) It is not harmonic with the inclusive normality and inclusive completely regularity. For example, it is difficult to prove that the inclusive completely regularity implies the pointed regularity, or $T^*_{3\frac{1}{2}} \Rightarrow T^*_3$ (pointed regularity with ST_1), and that the pointed regularity with inclusive normality implies inclusive completely regularity, etc. But these results is fundamental in general topology.

(ii) It is also difficult to prove that $L-$fuzzy unit interval $I(L)$ is a pointed regular space.

(2) The separation of Definition 1.9[9] is harmonic with the inclusive normality and inclusive completely regularity. Hence we call it the inclusive regular separation.

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L-fuzzy 拓扑空间中的包含式正则分离性

赵 彬

(陕西师范大学数学系，陕西 西安 710062)

摘要: 本文首先在一般 *L*-fuzzy 拓扑空间中引入了包含式正则分离性的概念. 其次将包含式正则分离性与点式正则分离性及正则分离性作了比较，讨论了它们之间的相互蕴含关系. 最后说明了包含式正则分离性与包含式正规分离性及包含式完全正则分离性之间的协调性.

关键词: *L*-fuzzy 拓扑空间；包含式正则空间； *L*-fuzzy 单位区间；包含式正规空间；包含式完全正则空间； *L*- 值 Zadeh 型函数.