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Deformation Retraction of Groups and Toeplitz Algebras

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Abstract: Let (G, G_+) be a quasi-partial ordered group such that $G_+^0 = G_+ \cap G_+^{-1}$ is a non-trivial subgroup of G. Let $[G]$ be the collection of left cosets and $[G_+]$ be its positive. Denote by $\mathcal{T}^{G_+}$ and $\mathcal{T}^{[G_+]}$ the associated Toeplitz algebras. We prove that $\mathcal{T}^{G_+}$ is unitarily isomorphic to a C^*-subalgebra of $\mathcal{T}^{[G_+]} \otimes C_r^*(G_+^0)$ if there exists a deformation retraction from G onto G_+^0. Suppose further that G_+^0 is normal, then $\mathcal{T}^{G_+}$ and $\mathcal{T}^{[G_+]} \otimes C_r^*(G_+^0)$ are unitarily equivalent.

Key words: Toeplitz algebra; quasi-partial ordered group.
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1. Introduction

Throughout this paper, G is a discrete group. Given a subset G_+ of G, we say that (G, G_+) is a quasi-partial ordered group, if $e \in G_+, G_+ \cdot G_+ \subseteq G_+$ and $G = G_+ \cdot G_+^{-1}$; if furthermore $G = G_+ \cup G_+^{-1}$, then (G, G_+) is referred to as a quasily ordered group. Note when $G_+^0 = G_+ \cap G_+^{-1} = \{e\}$, a quasi-partial ordered group (resp. quasily ordered group) (G, G_+) is known as a partially ordered (resp. ordered) group.

Let $\{\, \delta_g \,|\, g \in G\}$ be the usual orthonormal basis for $\ell^2(G)$. For any $E \subseteq G$, let $\ell^2(E)$ be the closed subspace of $\ell^2(G)$ generated by $\{\, \delta_g \,|\, g \in E\,\}$. The projection from $\ell^2(G)$ onto $\ell^2(E)$ is denoted by p^E. For $g \in G$, the left regular representation L_g on $\ell^2(G)$ is given by $L_g(\delta_h) = \delta_{gh}$ for $h \in G$. For any subset E of G, the C^*-algebra generated by $\{\, p^E L_g p^E \overset{\text{def}}{=} T_g^E \,|\, g \in G\,\}$ is denoted by $\mathcal{T}^E$, and is called the Toeplitz algebra with respect to E.

Toeplitz algebras on the quarter-planes have been studied by many mathematicians, see [1] and [2] for example. Associated with such Toeplitz algebras are the usual quasily ordered groups (Z^2, Z_α^2) for $\alpha \in R^1$, where $Z_\alpha^2 = \{\, (m,n) \in Z^2 \,|\, \alpha m + n \geq 0\,\}$. When α is a rational number, it is known that $\mathcal{T}^{Z_\alpha^2}(Z^2) \cong \mathcal{T}^{Z_+}(Z) \otimes C(T)$, where T is the unit circle in C^1 and $\mathcal{T}^{Z_+}(Z)$ is the classical Toeplitz algebra. Such a result was generalized by the author to the abelian quasily ordered groups; see [3, Theorem 3] for the details. Note that the original proof of this main result of [3] relies heavily on certain universal property of Toeplitz algebras over abelian quasily ordered groups. But as shown by [4, Theorem 4.3], such a universal property is no longer true for non-amenable groups. The object of this paper is to give a direct proof of [3, Theorem 3]; in

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fact, we will show that the same property also holds for a general quasi-partial ordered group, which needs to be neither abelian nor quasily ordered. Our technique is the construction of unitary operators between the underlying Hilbert spaces, and this leads us to consider certain generalized Toeplitz operators T_g for $g \in G$ defined as (2.1) below.

2. The main result

Throughout this section, (G, G_+) is a quasi-partial ordered group such that $G_+^0 = G_+ \cap G_+^{-1}$ is a non-trivial subgroup of G, which need not to be normal. For C^*-algebras A and B, we denote by $A \otimes B$ the completion of their algebraic tensor product $A \bigodot B$ with respect to the spatial, or minimal C^*-norm.

Let $G_+^* = G_+ \setminus G_+^0$. It is easy to show that

$$G_+^* \cdot G_+ = G_+^* = G_+ \cdot G_+^*.$$

So if we set $G_1 = G_+^* \cup \{e\}$, then (G, G_1) is a partially ordered group. Given any $g \in G$, since (G, G_+) is a quasi-partial ordered group, $g = st^{-1}$ for some $s, t \in G_+$. Then choose any $g_+^* \in G_+^*$,

$$g = (sg_+^*)(tg_+^*)^{-1} \in G_+^* \cdot (G_+^*)^{-1}.$$

Let $[G] = \{\, [g] \,|\, g \in G \,\}$ be the collection of left cosets, and $[G_+] = \{\, [g_+] \,|\, g_+ \in G_+ \,\}$ be its positive. Although $[G]$ may fail to be a group when G_+^0 is not normal, the partial isometry operator T_g acting on $\ell^2([G])$ can be however defined unanimously as

$$T_g(\delta_{[h]}) = \begin{cases} \delta_{[gh]}, & \text{if } [h] \in [G_+] \ \text{ and } [gh] \in [G_+], \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

for $g, h \in G$, where $\ell^2([G])$ is the Hilbert space of square integrable functions on the set $[G]$, and $\delta_{[h]}$ is the function that vanishes everywhere on $[G]$ except at the point $[h]$ where its value is one. It is easy to check that $T_g^* = T_{g^{-1}}$ for $g \in G$. Let $\mathcal{T}^{[G_+]}$ be the C^*-algebra generated by $\{\, T_g \,|\, g \in G \,\}$. For the convenience, we also call it the (generalized) Toeplitz algebra with respect to the pair $([G], [G_+])$.

Definition *Let (G, G_+) be a quasi-partial ordered group. A morphism of groups $\varphi : G \to G_+^0$ is said to be a deformation retraction if $\varphi(h) = h$ for all $h \in G_+^0$.*

Theorem *Let (G, G_+) be a quasi-partial ordered group such that G_+^0 is a non-trivial subgroup of G. If there is a deformation retraction $\varphi : G \to G_+^0$, then $\mathcal{T}^{G_+}$ is unitarily isomorphic to a C^*-subalgebra of $\mathcal{T}^{[G_+]} \otimes C_r^*(G_+^0)$, where $C_r^*(G_+^0)$ is the reduced group C^*-algebra of G_+^0. Suppose further that G_+^0 is normal, then $\mathcal{T}^{G_+}$ and $\mathcal{T}^{[G_+]} \otimes C_r^*(G_+^0)$ are unitarily equivalent.*

Proof Let $\ell^2([G]) \otimes \ell^2(G_+^0)$ be the Hilbert space tensor product of $\ell^2([G])$ and $\ell^2(G_+^0)$. Then the set $Y = \{\, \delta_{[g_1]} \otimes \delta_h \,|\, g_1 \in G, h \in G_+^0 \,\}$ forms an orthormal basis for $\ell^2([G]) \otimes \ell^2(G_+^0)$. Set up a morphism of sets λ from the basis

$$X = \{\, \delta_g \,|\, g \in G \,\} \text{ of } \ell^2(G) \text{ to } Y \text{ of } \ell^2([G]) \otimes \ell^2(G_+^0)$$

as

$$\lambda(\delta_g) = \delta_{[g]} \otimes \delta_{\varphi(g)} \quad \text{for} \quad g \in G.$$

For any $s, t \in G$, if $\delta_{[s]} \otimes \delta_{\varphi(s)} = \delta_{[t]} \otimes \delta_{\varphi(t)}$, then $\varphi(s) = \varphi(t)$ and $s = th$ for some $h \in G_+^0$. Since φ is a morphism of groups which is an identity map on G_+^0, we know that

$$\varphi(t) = \varphi(s) = \varphi(t)\varphi(h) = \varphi(t)h,$$

and hence $h = e$. Therefore φ is injective. On the other hand, given any $\delta_{[g_1]} \otimes \delta_h \in Y$, let $g = g_1\varphi(g_1)^{-1}h$, then

$$\lambda(\delta_g) = \delta_{[g_1]} \otimes \delta_h,$$

so λ is a bijection. It follows that there exists a unitary operator U from $\ell^2(G)$ to $\ell^2([G]) \otimes \ell^2(G_+^0)$ such that

$$U(\delta_g) = \delta_{[g]} \otimes \delta_{\varphi(g)} \quad \text{with} \quad U^*(\delta_{[g_1]} \otimes \delta_h) = \delta_{g_1\varphi(g_1)^{-1}h}. \tag{2.2}$$

For any $g, g_1 \in G$ and $h \in G_+^0$,

$$\begin{aligned} &UT_g^{G_+}U^*(\delta_{[g_1]} \otimes \delta_h) = UT_g^{G_+}\delta_{g_1\varphi(g_1)^{-1}h} \\ &= \begin{cases} U\delta_{gg_1\varphi(g_1)^{-1}h}, & \text{if } g_1\varphi(g_1)^{-1}h \in G_+ \quad \text{and} \quad gg_1\varphi(g_1)^{-1}h \in G_+, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \delta_{[gg_1]} \otimes \delta_{\varphi(g)h}, & \text{if } g_1\varphi(g_1)^{-1}h \in G_+ \quad \text{and} \quad gg_1\varphi(g_1)^{-1}h \in G_+, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.3}$$

On the other hand, for such g, g_1 and h,

$$\begin{aligned} &(T_g \otimes L_{\varphi(g)})(\delta_{[g_1]} \otimes \delta_h) = (T_g\delta_{[g_1]}) \otimes \delta_{\varphi(g)h} \\ &= \begin{cases} \delta_{[gg_1]} \otimes \delta_{\varphi(g)h}, & \text{if } [g_1] \in [G_+] \quad \text{and} \quad [gg_1] \in [G_+], \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.4}$$

Since $G_+ \cdot G_+^0 \subseteq G_+ \cdot G_+ \subseteq G_+$ and G_+^0 is a group, we know for any $s \in G$ and $t \in G_+^0$, $[s] \in [G_+] \Longleftrightarrow s \in G_+ \Longleftrightarrow st \in G_+$. The equality $UT_g^{G_+}U^* = T_g \otimes L_{\varphi(g)}$ then follows from (2.3) and (2.4), so $\mathcal{T}^{G_+}$ is unitarily isomorphic to a C^*-subalgebra of $\mathcal{T}^{[G_+]} \otimes C_r^*(G_+^0)$.

Suppose further that G_+^0 is normal, then $([G], [G_+])$ becomes a partially ordered group, and the operator T_g defined as (2.1) equals to the usual Toeplitz operator $p^{[G_+]}L_{[g]}p^{G_+}$. In this case, for any $g_1 \in G$ and $h \in G_+^0$, let $g = g_1\varphi(g_1)^{-1}h$. Then

$$UT_g^{G_+}U^* = T_g \otimes L_{\varphi(g)} = T_{g_1} \otimes L_h.$$

Therefore $\mathcal{T}^{G_+}$ and $\mathcal{T}^{[G_+]} \otimes C_r^*(G_+^0)$ are unitarily equivalent. □

Remark When G is abelian, by [5, Proposition 7.1.6] we know that

$$C_r^*(G_+^0) \cong C(\widehat{G_+^0}),$$

where $\widehat{G_+^0}$ is the dual group of G_+^0, and thus get [3, Theorem 3] even if $G \neq G_+ \cup G_+^{-1}$.

3. Some examples

In this section, we will give some examples of quasi-partial ordered groups which satisfy the conditions given in the preceding theorem.

Example 1 Any two different planes passing through the original point will divide the space into four parts, each part can induce a partial or quasi-partial order on $G = Z^3$. For instance, let

$$G_+ = \{(m_1, m_2, m_3) \in Z^3 \,|\, m_1 + m_2 \geq 0, \quad \text{and} \quad m_2 + m_3 \leq 0\,\}.$$

Then $G_+^0 = \{(m, -m, m) \,|\, m \in Z\,\} \cong Z$, and $\varphi : (m_1, m_2, m_3) \to (m_1, -m_1, m_1)$ is a deformation retraction.

Example 2 Perhaps Free groups are the best candidates for non-abelian groups. The equality that $G = G_+ \cdot G_+^{-1}$ is however not true for any free group, so we turn to the matrix algebras over the real numbers, which are not commutative with respect to the multiplication of matrices. Let

$$G = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \,\middle|\, a_{11} > 0, a_{22} > 0 \right\},$$
$$G_+ = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \,\middle|\, a_{11} > 0, a_{22} > 0 \text{ and } a_{12} \geq 0 \right\}.$$

For

$$g = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in G, \quad g^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & -\frac{a_{12}}{a_{11}a_{22}} \\ 0 & \frac{1}{a_{22}} \end{pmatrix},$$

so (G, G_+) is actually a quasily ordered group with $G_+^0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \,\middle|\, a > 0, b > 0 \right\}$. The deformation retraction φ can be defined naturally by

$$\varphi\left(\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \right) = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}.$$

Example 3 Let

$$G = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \,\middle|\, a_{11} > 0, a_{22} > 0 \text{ and } a_{33} > 0 \right\},$$
$$G_+ = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \,\middle|\, \begin{matrix} a_{11} > 0, & a_{22} > 0, & a_{33} > 0, \\ a_{12} \geq 0, & a_{13} \geq 0, & a_{23} \geq 0 \end{matrix} \right\}.$$

Given any $g = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in G$, first we choose two positive numbers b_{12} and b_{23} with $a_{11}b_{12} + a_{12} > 0$ and $a_{22}b_{23} + a_{23} > 0$. After doing that, we then choose another positive number b_{13} large enough so that $a_{11}b_{13} + a_{12}b_{23} + a_{13} > 0$. Then $g = ts^{-1} \in G_+ \cdot G_+^{-1}$, where $s = \begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix}$. It follows that (G, G_+) is a quasi-partial ordered group.

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群的形变收缩及 Toeplitz 代数

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摘要: 设 G 为一个离散群，(G, G_+) 为一个拟偏序群使得 $G_+^0 = G_+ \cap G_+^{-1}$ 为 G 的非平凡子群. 令 $[G]$ 为 G 关于 G_+^0 的左倍集全体， $|G_+|$ 为 $[G]$ 的正部. 记 $\mathcal{T}^{G_+}$ 和 $\mathcal{T}^{[G_+]}$ 为相应的 Toeplitz 代数. 当存在一个从 G 到 G_+^0 上的形变收缩映照时, 我们证明了 $\mathcal{T}^{G_+}$ 酉同构于 $\mathcal{T}^{[G_+]} \otimes C_r^*(G_+^0)$ 的一个 C^*- 子代数. 若进一步， G_+^0 还为 G 的一个正规子群，则 $\mathcal{T}^{G_+}$ 与 $\mathcal{T}^{[G_+]} \otimes C_r^*(G_+^0)$ 酉同构.

关键词: Toeplitz 代数；拟偏序群.