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The S_p Property of a kind of Hankel Operators and Toeplitz Operators

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Abstract: For two kind of Möebius invariant subspace $A^{\alpha,2}(D)$ and $A^{\beta,2}(D)$ of $L^{\alpha,2}(D)$, define the Toeplitz operators T_f^s and Hankel operators H_f^r on $A^{\alpha,2}(D)\times A^{\beta,2}(D)$ with an arbitrary analytic "symbol function" f on a unit disk, and study their boundedness, compactness and Schatten-von Neumann properties.

Key words: Möebius group; weighted Bergman space; Toeplitz operator; Hankel operator; paracommutator; Schatten-von Neumann property.
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1. Introduction

The Möebius group SU (1,1) acts unitarily on a weighted Bergman space $A^{\alpha,2}(D)$ (or its complex conjugate $\overline{A}^{\alpha,2}(D)$). If we have two such spaces $A^{\alpha,2}(D)$ and $A^{\beta,2}(D)$, then the same group acts on the space of Hilbert-Schmidt operators from one space to another. We know that the study of linear operators from a space of analytic functions to a space of anti-analytic functions is equivalent to the study of bilinear forms between the original spaces. In the case when $\alpha=\beta$ the corresponding decomposition of the Hilbert space of these Hilbert-Schmidt operators into irreducible components has been considered by S. Janson, J. Peetre and C. Zhang[1−3]. In this paper we will study the case $\alpha\neq\beta$, find the irreducible decomposition of the space of the forms, and establish their Schatten-von Neumann properties.

Let D be the unit disk in the complex plane equipped with the Lebesgue measure $\mathrm{d}m(z)$.

The Möebius group SU (1,1) consists of the following 2×2 matrices

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix},\quad a,b,c,d\in C$$

with $c=\overline{b}, d=\overline{a}$ and $ad-bc=1$. It acts on D via the transformation

$$z\to gz=g(z)=\frac{az+b}{cz+d}.$$

Suppose α and β are nonnegative integers. $L^{\alpha,2}(D)$ is the space consisting of all functions

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on D square integrable with respect to the measure

$$d\mu_\alpha(z) = \frac{\alpha+1}{\pi}\left(1+|z|^2\right)^\alpha \mathrm{d}m(z), \quad \text{i.e.,} \quad L^{\alpha,2}(D) \equiv \left\{ f|\int\limits_D f^2(x)\,\mathrm{d}\mu_\alpha(z) < +\infty \right\}.$$

$A^{\alpha,2}(D)$ and $A^{\beta,2}(D)$ are the weighted Bergman space of $L^{\alpha,2}(D)$. Then for the tensor product $A^{\alpha,2}(D) \otimes A^{\beta,2}(D)$ of two such spaces we can consider the same problem, i.e., we can define the Toeplitz operators T_f^s from $A^{\beta,2}(D)$ to $A^{\alpha,2}(D)$

$$T_f^s g(z) \equiv \int\limits_D K_f^s(z,w)\,g(w)\,\mathrm{d}\mu_\beta(w), \quad g \in A^{\beta,2}(D) \tag{1.1}$$

and the kernel of T_f^s is

$$K_f^s(z,w) = (1-z\bar{w})^{-s} \int\limits_{\partial D} \frac{\bar{b}^{\nu_1-s} f(b)}{\left(1-\bar{b}z\right)^{\nu_1-s}(1-b\bar{w})^{\nu_2-s}}\mathrm{d}b, \tag{1.2}$$

where $\nu_1 = \alpha+2, \nu_2 = \beta+2, s = \frac{\alpha+\beta}{2}+2, \frac{\alpha+\beta}{2}+1, \cdots, \alpha+1$; and Hankel operators H_f^r on $A^{\alpha,2}(D) \times A^{\beta,2}(D)$,

$$H_f^r(f_1,f_2) = \iint\limits_{DD} \overline{K_f^r(z,w)} f_1(z)\,f_2(w)\,\mathrm{d}\mu_\alpha(z)\,\mathrm{d}\mu_\beta(w),$$

where

$$f_1 \in A^{\alpha,2}(D)\,, f_2 \in A^{\beta,2}(D)\,, \tag{1.3}$$

and the kernel function of H_f^r is

$$K_f^r(z,w) = (z-w)^r \int\limits_{\partial D} \frac{f(b)}{(b-z)^{\nu_1+r}(b-w)^{\nu_2+r}}\mathrm{d}b \tag{1.4}$$

and it is an eigenfunction of Casimir operator C[4].

2. Main results

Theorem 2.1 *Let $\frac{1}{p} < t$, then $T_f^s \in S_p$ if and only if $f \in B_p^{\frac{1}{p}-t}$; Let $\frac{1}{p} \geq t$, then $T_f^s \in S_p$ if and only if $f = 0$; (where $t = s - \frac{\alpha+\beta}{2} - 1$).*

Theorem 2.2 *For $p > 0$, then $H_f^r \in S_p$ if and only if $f \in B_p^\theta$, where $\theta = \frac{\nu_1+\nu_2+2(r-1)}{2} + \frac{1}{p}$, S_p is Schatten-von Neumann class, and B_p^θ is analytic Besov's space.*

3. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1 For Toeplitz operators (1.1), by its kernel (1.2) we can calculate this integral as follows:

$$T_f^s g(z) = \sum_{j=0}^{\sigma} \binom{\sigma}{j} \frac{(\nu_2-s)_{\sigma-j}}{(\nu_2)_{\sigma-j}} D^j f(z) D^{\sigma-j} g(z), \tag{2.1}$$

where $\sigma=\alpha+1-s$ and as $\beta>-1$. Since $\{e^{\alpha}_{nl}\}_{n=0}^{\infty}$ and $\left\{e^{\beta}_{m}\right\}_{m=0}^{\infty}$ are orthonormal basises of $A^{\alpha,2}(D)$ and $A^{\beta,2}(D)$ respectively, the "matrix coefficient" of the operators T^s_f is

$$\begin{aligned}\langle T^s_{z^k}e^{\beta}_m,e^{\alpha}_{nl}\rangle &= \int_D T^s_{z^k}e^{\beta}_m(z)\,\overline{e^{\alpha}_{nl}(z)}\mathrm{d}\mu_\alpha(z)\\ &= c_l\sqrt{\frac{(\nu_1-l)_n}{(l+1)_n}}\int_D T^s_{z^k}\left(\frac{z^m}{\|z^m\|_\beta}\right)\overline{p_{nl}\left(\frac{|z|^2}{1-|z|^2}\right)z^n}\mathrm{d}\mu_\alpha(z),\end{aligned}\tag{2.2}$$

where $e^{\beta}_m,e^{\alpha}_{nl},p_{nl}$ may see [5]. By (2.1) and properties of the hypergeometric function ${}_3F_2$, we can obtain

$$\begin{aligned}&\langle T^s_{z^k}e^{\beta}_m,e^{\alpha}_{nl}\rangle=\frac{(-1)^{\nu_1-s-1}(\nu_2-s)_{\alpha+1-s}(-m)_{\alpha+1-s}}{(\nu_2)_{\alpha+1-s}}c_l\sqrt{\frac{(\nu_1-l)_n}{(l+1)_n}}\delta_{m+k+s-\nu_1+1}\times\\ &{}_3F_2(s-\alpha-1,s-\alpha-\beta-2,-k;3s-\sigma-\beta-2,m+s-\alpha;1)\,F(-l,l-\alpha-1;-\alpha;1)\times\\ &\sqrt{\frac{\Gamma(m+\beta+2)}{\Gamma(m+1)\Gamma(\beta+1)}}\frac{\Gamma(\alpha+2)\Gamma(n+l+1)}{\Gamma(n+\alpha+2)}.\end{aligned}\tag{2.3}$$

Hence the singular number of the operators $T^s_{z^k}$ is $\langle T^s_{z^k}e^{\beta}_m,e^{\alpha}_{nl}\rangle\approx m^{1+\frac{\alpha+\beta}{2}-s}$, where the notation $u\approx v$ means that the ratio $\frac{u}{v}$ is bounded above and below by constans independent of n and m. Using a similar method as in [6] we can prove Theorem 2.1.

Proof of Theorem 2.2 It is easy to check that the function $f_1(z,w)=(z-w)^r$ is an eigenfunction of the Casimir operator C,

$$\begin{aligned}Cf&=-(z-w)^2\frac{\partial^2 f}{\partial z\partial w}-\nu_1(z-w)\frac{\partial f}{\partial w}+\nu_2(z-w)\frac{\partial f}{\partial z}+(\nu_1+\nu_2)(\nu_1+\nu_2-2)f\\ &=4\left(c_1^2+c_2^2+c_3^2\right),\end{aligned}\tag{2.4}$$

where c_1,c_2 and c_3 are defined [4] with the giving eigenvalue

$$\lambda=(\nu_1+\nu_2+r-1)r+(\nu_1+\nu_2)(\nu_1+\nu_2-2).$$

Take $z_0\in D$ and let $g\in SU(1,1)$ be the transformation $g(z)=\frac{z-z_0}{1-\overline{z_0}z}$. Since the operator C commutes with the group action SU(1,1), we see that the function

$$[g(z)-g(w)]^r\,[g'(z)]^{\nu_1/2}\,[g'(w)]^{\nu_2/2}$$

is also an eigenfunction of C with the same eigenvalue λ. A direct calculation shows

$$[g(z)-g(w)]^r\,[g'(z)]^{\nu_1/2}\,[g'(w)]^{\nu_2/2}=\frac{(z-w)^r\left(1-|z_0|^2\right)^{r+\frac{\nu_1+\nu_2}{2}}}{(1-\overline{z_0}z)^{\nu_1+r}(1-\overline{z_0}w)^{\nu_2+r}}.$$

Hence the function is a constant multiplier of the following function

$$\tilde f(z,w)=\frac{(z-w)^r}{(1-\overline{z_0}z)^{\nu_1+r}(1-\overline{z_0}w)^{\nu_2+r}}.$$

Now let z_0 approach a boundary point b, then this function $\tilde{f}$ approaches the function

$$f(z,w)=\frac{(z-w)^r}{\left(1-\overline{b}z\right)^{\nu_1+r}\left(1-\overline{b}w\right)^{\nu_2+r}}\quad\text{(in distribution sense).}$$

Therefore, the function f is also an eigenfunction of the Casimir operator C with the same eigenvalue λ, and so the kernel function K_f^r (1.4) of the operator H_f^r is also an eigenfunction of the Casimir operator C.

Since the group action SU(1,1) on these operators is equivalent to the following action on the symbols

$$f(z)\mapsto f[g(z)]\{g'(z)\}^{-\left(r-1+\frac{\nu_1+\nu_2}{2}\right)},$$

it is also easy to check that the operators H_f^r has finite Hilbert-Schmidt norm for

$$f(z)=z^{\alpha+\beta+2r+4}.$$

Therefore, by the well-known Arazy-Fisher theoty of Möebius invariant function spaces[7], we see that for an analytic function f has

$$\left\|H_f^r\right\|_2^2=c\left\|f\right\|^2_{B_2^{\frac{\nu_1+\nu_2}{2}+r-\frac{1}{2}}}$$

with a suitable constant c (here B_2^s is the usual scale of Besov's spaces). It follows that for each nonnegative integer r, the Hankel operators H_f^r constitute an irreducible component, which we denote by

$$V_r=\left\{H_f^r, f\in B_2^{\frac{\nu_1+\nu_2}{2}+r-\frac{1}{2}}\right\}.$$

Similar to the calculation in [1], we see that V_r is an irreducible SU(1,1)-module of the lowest weight $\nu_1+\nu_2+2(r-1)$, and has the direct sum decomposition

$$A^{\alpha,2}(D)\otimes A^{\beta,2}(D)=\mathop{\oplus}_{r=0}^{\infty}V_r. \tag{2.5}$$

Mapping D to the upper half plane and performing a Fourier transform, we see that the Hankel operators H_f^r is unitarily equivalent to the following paracommutator on $H^2(R)\times H^2(R)$ (on the sense of disregarding a constant)

$$(f_1,f_2)\mapsto\iint\limits_{RR}\overline{\hat{f}(\xi_1+\xi_2)J(\xi_1,\xi_2)}\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\xi_1^{-(\alpha+1)}\xi_2^{-(\beta+1)}\mathrm{d}\xi_1\mathrm{d}\xi_2,$$

where

$$J(\xi_1,\xi_2)=\sum_{j=0}^{\alpha+\beta+r+2}\begin{pmatrix}\alpha+\beta+r+2\\ j\end{pmatrix}\begin{pmatrix}r\\ \alpha+r-j+1\end{pmatrix}(-1)^j\xi_1^j\xi_2^{\alpha+\beta+r+2-j}.$$

Therefore the S_p-result follows from the general theory of S. Janson, J. Peetre[1] and L. Peng[8]. We omit the details here.

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一类 Hankel 算子与 Toeplitz 算子的 S_p 性质

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摘要: 对于 $L^{\alpha,2}(D)$ 的两类 Möebius 不变子空间 $A^{\alpha,2}(D)$ 和 $A^{\beta,2}(D)$, 我们定义了它们之间的 Toeplitz 算子 T_f^s 与其乘积空间上的 Hankel 算子 H_f^r, 并且研究了它们的有界性、紧性及 Schatten-von Neumann 性质.

关键词: 加权的 Bergman 空间; Toeplitz 算子; Hankel 算子; 仿交换子; Schatten-von Neumann 性质.