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Orthogonal Laurent Polynomials and Their Zeros

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Abstract: In this note, we investigate the necessity for the measure $\mathrm{d}\psi$ being a strong distribution, which is associated with the coefficients of the recurrence relation satisfied by the orthogonal Laurent polynomials. We also give out a representation of the greatest zeros of orthogonal Laurent polynomials in the case of $d\psi$ being a strong distribution.

Key words: orthogonal Laurent polynomials; quadrature rule; zeros of orthogonal Laurent polynomials.
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1. Introduction

According to [1], the distribution $\mathrm{d}\psi(x)$ is said to be a strong distribution in $(a,b) \subset (0,+\infty)$ if $\psi(x)$ is a real bounded nondecreasing function on (a,b) with infinitely many points of increase there, and furthermore, all the moments

$$\mu_k = \int_a^b x^k \mathrm{d}\psi(x), \quad k = 0, \pm 1, \pm 2, \cdots \tag{1}$$

are finite.

For any given strong distribution in $(a,b) \subset (0,+\infty)$ there exists a unique sequence, up to a nonzero constant factor normalization, of polynomials $\{B_n\}_{n=0}^{+\infty}$ such that B_n is a polynomial of precise degree n and B_n satisfies the relations

$$\int_a^b x^{-n+k} B_n(x) \mathrm{d}\psi(x) = 0, \quad k = 0, \cdots, n-1. \tag{2}$$

If B_n are normalized to be monic, i.e., to have leading coefficients one, they satisfy the recurrence relation $B_{-1} = 0$, $B_0 = 1$,

$$B_{n+1}(x) = (x - \beta_n)B_n - \alpha_n x B_{n-1}(x), \quad n \geq 0, \tag{3}$$

where β_n and α_n, $n = 0, 1, \cdots$, are all positive. In addition, all the zeros of B_n are real, distinct and lie in (a,b). According to [2], there exists a unique quadrature formula of the form $\int_a^b f(x)\mathrm{d}\psi(x) \approx \sum_{i=1}^n A_i f(x_i)$ which is exact for every f for which $x^n f(x) \in \Pi_{2n-1}$ (Π_k denotes

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the set of polynomials of degree $\leq k$). Moreover, the nodes x_i, $i=1,\cdots,n$, of this quadrature rule coincide with the zeros of B_n and its weights A_i are positive.

In this note, we investigate the necessity for $\mathrm{d}\psi$ being a strong distribution, which is associated with α_n and β_n, $n=0,1,\cdots$ in (3). We also give out a representation of the greatest zeros of B_n in the case of $\mathrm{d}\psi$ being a strong distribution.

2. Necessity for $\mathrm{d}\psi$ being a strong distribution

Assume that $d\psi$ is a strong distribution in $(a,b)\subset(0,+\infty)$. It is easy to see that $x^i\mathrm{d}\psi(i=\pm1,\pm2,\cdots)$ are also strong distributions in (a,b). Denote by $\{B_n^{(i)}\}_{n=0}^{+\infty}$ the unique monic polynomial sequences satisfying

$$\int_a^b x^{-n+k}B_n^{(i)}(x)x^i\mathrm{d}\psi(x)=0,\quad k=0,\cdots,n-1,\quad i=0,\pm1,\pm2,\cdots, \tag{4}$$

and

$$B_{n+1}^{(i)}(x)=(x-\beta_n^{(i)})B_n^{(i)}(x)-\alpha_n^{(i)}xB_{n-1}^{(i)}(x),\quad B_{-1}^{(i)}(x)=0,\quad ,B_0^{(i)}(x)=1, \tag{5}$$

$$\beta_n^{(i)}>0,\quad \alpha_n^{(i)}>0,\quad n=0,1,\cdots,\quad i=0,\pm1,\pm2,\cdots. \tag{6}$$

We have the following result.

Theorem 2.1 *If $\mathrm{d}\psi$ is a strong distribution in $(a,b)\subset(0,+\infty)$, then $\{\alpha_n^{(i)}\}_{n=0}^{+\infty}$ and $\{\beta_n^{(i)}\}_{n=0}^{+\infty}$ $(i=0,\pm1,\pm2,\cdots)$ defined as above must satisfy*

$$\beta_n^{(i)}+\alpha_{n+1}^{(i)}>1,\quad n=2,3,\cdots,\quad i=0,\pm1,\pm2,\cdots,$$

or equivalently

$$\beta_n^{(i)}+\alpha_n^{(i)}>1,\quad n=2,3,\cdots,\quad i=0,\pm1,\pm2,\cdots.$$

Proof For convenience, we always write

$$\varphi_i(x^k)=\int_a^b x^kx^i\mathrm{d}\psi(x),\quad i=0,\pm1,\pm2,\cdots,$$

where φ_i is linear on $\mathrm{span}\{\cdots,x^{-1},1,x,\cdots\}$.

According to (5), we have for $n=0,1,\cdots$,

$$B_{n+1}^{(i)}(x)+\beta_n^{(i)}B_n^{(i)}(x)=x(B_n^{(i)}(x)-\alpha_n^{(i)}B_{n-1}^{(i)}(x)),$$

$$x^{-n+k}(B_{n+1}^{(i)}(x)+\beta_n^{(i)}B_n^{(i)}(x))=x\cdot x^{-n+k}(B_n^{(i)}(x)-\alpha_n^{(i)}B_{n-1}^{(i)}(x)),\quad k=0,\cdots,n-1.$$

φ_i being acted on the above, we have by (4),

$$\varphi_{i+1}(x^{-n+k}(B_n^{(i)}(x)-\alpha_n^{(i)}B_{n-1}^{(i)}(x)))=0,\quad k=0,\cdots,n-1.$$

Therefore,

$$B_n^{(i+1)}(x)=B_n^{(i)}(x)-\alpha_n^{(i)}B_{n-1}^{(i)}(x).$$

Substituting the above into (5) for $i+1$, we get

$$\begin{aligned}B_{n+1}^{(i)}(x) =&B_n^{(i)}(x)(\alpha_{n+1}^{(i)} + x - \beta_n^{(i+1)}) - B_{n-1}^{(i)}(x)(\alpha_n^{(i)}(x - \beta_n^{(i+1)}) + \alpha_n^{(i+1)}x)+\\ &\alpha_n^{(i+1)}\alpha_{n-1}^{(i)}xB_{n-2}^{(i)}(x).\end{aligned}$$

Applying (5) to the above again for all $n \geq 1$ results in

$$(\beta_n^{(i)} + \alpha_{n+1}^{(i)} - \beta_n^{(i+1)})B_n^{(i)}(x) + (-\alpha_n^{(i+1)}x + \alpha_n^{(i)}\beta_n^{(i+1)})B_{n-1}^{(i)}(x) + \alpha_n^{i+1}\alpha_{n-1}^{(i)}xB_{n-2}^{(i)}(x) = 0. \quad (7)$$

We can assert that $\beta_n^{(i)} + \alpha_{n+1}^{(i)} - \beta_n^{(i+1)} \neq 0$, since otherwise, $\alpha_n^{(i+1)} = 0$ $(n = 1, 2, \cdots)$, leading to a contradition. Comparing the coefficients in (7) with those in (5) for i, we obtain

$$\beta_n^{(i)} + \alpha_{n+1}^{(i)} - \beta_n^{(i+1)} = \alpha_n^{(i+1)}, \quad (8)$$

$$\frac{\alpha_n^{(i)}\beta_n^{(i+1)}}{\alpha_n^{(i+1)}} = \beta_{n-1}^{(i)}, \quad \alpha_n^{(i)}\beta_n^{(i+1)} = \beta_{n-1}^{(i)}(\beta_n^{(i)} + \alpha_{n+1}^{(i)} - \beta_n^{(i+1)}).$$

Therefore,

$$\beta_n^{(i+1)} = \frac{\beta_{n-1}^{(i)}(\beta_n^{(i)} + \alpha_{n+1}^{(i)})}{\alpha_n^{(i)} + \beta_{n-1}^{(i)}}, \quad \alpha_n^{(i+1)} = \frac{\alpha_n^{(i)}(\beta_n^{(i)} + \alpha_{n+1}^{(i)})}{\alpha_n^{(i)} + \beta_{n-1}^{(i)}},$$

or

$$\alpha_n^{(i)} = \frac{\beta_{n-1}^{(i+1)} + \alpha_{n-1}^{(i+1)}}{\beta_n^{(i+1)} + \alpha_n^{(i+1)}}, \quad \beta_n^{(i)} = (\beta_n^{(i+1)} + \alpha_n^{(i+1)})(1 - \frac{1}{\beta_{n+1}^{(i+1)} + \alpha_{n+1}^{(i+1)}}).$$

So,

$$\beta_{n+1}^{(i)} + \alpha_{n+1}^{(i)} > 1, \quad n = 1, 2, \cdots, \quad i = 0, \pm 1, \pm 2, \cdots,$$

or equivalently by (8)

$$\beta_n^{(i)} + \alpha_{n+1}^{(i)} > 1, \quad n = 2, 3, \cdots, \quad i = 0, \pm 1, \pm 2, \cdots. \qquad \square$$

3. A representation for the greatest zero of B_n

Let $\mathrm{d}\psi$ be a strong distribution in $(a, b) \subset (0, +\infty)$, and $\{B_n\}_{n=0}^{+\infty}$ be the unique monic polynomial sequence satisfying (2) and (3). Assume that $\{x_i\}_{i=1}^n$ are the n zeros of B_n, and $X_n = \max\{x_i : i = 1, \cdots, n\}$. Then we have the following representation of X_n.

Theorem 3.1

$$X_n = \max_{p \in \Pi_{n-1}, p \neq 0} \frac{\int_a^b x^{-n+1}p^2(x)\mathrm{d}\psi(x)}{\int_a^b x^{-n}p^2(x)\mathrm{d}\psi(x)}. \quad (9)$$

Proof According to the statements in the introduction, for each $p \in \Pi_{n-1}$ and $p \neq 0$, we have

$$\begin{aligned}\int_a^b x^{-n+1}p^2(x)\mathrm{d}\psi(x) &= \sum_{i=1}^n x_i x_i^{-n}p^2(x_i)A_i \leq X_n \sum_{i=1}^n x_i^{-n}p^2(x_i)A_i\\ &= X_n \int_a^b x^{-n}p^2(x)\mathrm{d}\psi(x).\end{aligned}$$

Therefore,

$$X_n \geq \max_{p\in\Pi_{n-1},p\neq 0} \frac{\int_a^b x^{-n+1}p^2(x)\mathrm{d}\psi(x)}{\int_a^b x^{-n}p^2(x)\mathrm{d}\psi(x)}.$$

On the other hand, let us take $L_i(x) \in \Pi_{n-1}$ satisfying $L_i(x_k) = \begin{cases} 1, & k=i \\ 0, & k\neq i \end{cases}$, $i,k=1,\cdots,n$. Then

$$\frac{\int_a^b x^{-n+1}L_n^2(x)\mathrm{d}\psi(x)}{\int_a^b x^{-n}L_n^2(x)\mathrm{d}\psi(x)} = X_n.$$

So (9) holds. □

Similar to the proof of (9), we can attain the following estimates on X_n.

Corollary 3.1

$$X_n \geq \max_{p\in\Pi_{n-2},p\neq 0} \frac{\int_a^b x^{-n+3}p^2(x)\mathrm{d}\psi(x)}{\int_a^b x^{-n+2}p^2(x)\mathrm{d}\psi(x)}.$$

Corollary 3.2

$$X_n^2 \geq \max_{p\in\Pi_{n-2},p\neq 0} \frac{\int_a^b x^{-n+3}p^2(x)\mathrm{d}\psi(x)}{\int_a^b x^{-n+1}p^2(x)\mathrm{d}\psi(x)}.$$

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Laurent 多项式及其零点

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摘要: 本文给出了测度 dψ 为强分布的一个必要条件，并得到了 dψ 为强分布时的 Laurent 多项式最大零点的一个表示.

关键词: Laurent 多项式； 求积公式； Laurent 多项式的零点.