第26卷第1期 数 学 研 究 与 评 论 Vol.26, No.1
2006年2月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Feb., 2006

Article ID: 1000-341X(2006)01-0023-04 **Document code**: A

A Note on Vanishing Mean Oscillation

YANG Wei[1,2], *WANG Jie*[1], *LIU Yu-tang*[1]

(1. Dept. of Math., School of Science, Beijing Institute of Technology, Beijing 100081, China;
2. Division of Population Genetics and Prevention, Cardiovascular Institute and Fu Wai Hospital, Beijing 100037, China)
(E-mail: youngwill2003@hotmail.com)

Abstract: A certain inequality for maximal operators is obtained. From this result, a necessary condition for a function f in $\mathrm{VMO}(\mathbf{R}, \mathrm{d}x)$ is generalized.

Key words: maximal operator; vanishing mean oscillation; Radon measure.
MSC(2000): 42B25
CLC number: O174.2

0. Preliminaries

Let $(\mathbf{R}, \mu, \mathcal{R})$ be a measure space and μ is a positive Radon measure on $\mathcal{R}$ satisfying doubling condition. A measurable function $\varphi(t)$ on $\mathbf{R}$ is locally integrable, namely $\varphi \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}(\mathbf{R}, \mathrm{d}\mu)$, if $|\varphi|$ is integrable over any compact set. If $\varphi \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}(\mathbf{R}, \mathrm{d}\mu)$ and if I is a bounded interval, write

$$\varphi_I = \frac{1}{|I|}\int_I \varphi(t)\,\mathrm{d}\mu, \quad \text{where } |I| = \mu(I)$$

for any average of φ over I. If $\varphi \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}(\mathbf{R}, \mathrm{d}\mu)$, and if

$$\|\varphi\|_* = \sup_I \frac{1}{|I|}\int_I |\varphi - \varphi_I|\,\mathrm{d}\mu < \infty, \tag{1}$$

where the supremum is over all bounded intervals, we say φ is of bounded mean oscillation, denoted by $\varphi \in$BMO. The bounded $\|\varphi\|_*$ in (1) is the BMO norm of φ. Let $\varphi \in \mathbf{L}^{\mathbf{1}}_{\mathbf{loc}}(\mathbf{R}, \mathrm{d}\mu)$. For $\delta > 0$, write

$$rmM_\delta(\varphi) = \sup_{|I|\le\delta} \frac{I}{|I|}\int_I |\varphi - \varphi_I|\,\mathrm{d}\mu. \tag{2}$$

Then $\varphi \in$BMO if and only if $\mathrm{M}_\delta(\varphi)$ is bounded and $\|\varphi\|_* = \lim_{\delta\to\infty}\mathrm{M}_\delta(\varphi)$.

We say that φ has vanishing mean oscillation, $\varphi \in$VMO, if

(i) $\varphi \in$BMO and

(ii) $\mathrm{M}_0(\varphi) = \lim_{\delta\to 0}\mathrm{M}_\delta(\varphi) = 0$.

1. Main results

In the early 1980s the author of [2] proved that for a function $\varphi \in$BMO, the condition $\varphi \in$VMO is equivalent to the following condition:

Received date: 2003-07-15

If $\varphi_x(t) = \varphi(t-x)$ is the translation of φ by x units, then

$$\lim_{x\to 0} \|\varphi_x - \varphi\|_* = 0.$$

In this paper, we generalize the result partly from the formerly Lebesgue measure space to Radon measure space $(\mathbf{R}, \mu, \mathcal{R})$. To prove our results, we need the following lemma.

Lemma 1 *Let* $\delta > 0$, *and partition* $\mathbf{R}$ *into intervals*:

$$I_z = (z\frac{\delta}{2}, (z+1)\frac{\delta}{2}) \quad (x \in \mathbf{Z}).$$

Then, we have

$$|\varphi_{I_z} - \varphi_{I_{z+1}}| \le 4\mathrm{M}_\delta(\varphi).$$

Define $h(t) = \sum_{z\in\mathbf{Z}} \varphi_{I_z} \chi_{I_z}(t)$, we have our first result as

Theorem 1 *For some constant* c,

$$\|\varphi - h\|_* \le c\mathrm{M}_\delta(\varphi).$$

Our next main result is as follows,

Theorem 2 *If* $\varphi \in \mathrm{VMO}$, *then*

$$\lim_{\delta\to 0} \|\varphi_\delta - \varphi\|_* = 0.$$

2. Proofs of main results

Lemma 1 follows from

$$\begin{aligned}
|\varphi_{I_z} - \varphi_{I_{z+1}}| &= \left|\varphi_{I_z} - \varphi_{I_z\cup I_{z+1}} + \varphi_{I_z\cup I_{z+1}} - \varphi_{I_{z+1}}\right| \\
&\le \frac{1}{|I_z|}\int_{I_z\cup I_{z+1}} \left|\varphi - \varphi_{I_z\cup I_{z+1}}\right| \mathrm{d}\mu + \frac{1}{|I_{z+1}|}\int_{I_z\cup I_{z+1}} \left|\varphi - \varphi_{I_z\cup I_{z+1}}\right| \mathrm{d}\mu \\
&\le 4\frac{1}{|I_z\cup I_{z+1}|}\int_{I_z\cup I_{z+1}} \left|\varphi - \varphi_{I_z\cup I_{z+1}}\right| \mathrm{d}\mu \\
&\le 4\mathrm{M}_\delta(\varphi).
\end{aligned}$$

For Theorem 1, it is enough to prove that for $\forall\,\mathrm{I}$,

$$\frac{1}{|I|}\int_I |\varphi - h - (\varphi - h)_I| \mathrm{d}\mu < c\mathrm{M}_\delta(\varphi).$$

Since

$$\begin{aligned}
\frac{1}{|I|}\int_I |\varphi - h - (\varphi-h)_I|\,\mathrm{d}\mu &\le \frac{1}{|I|}\int_I |\varphi - \varphi_I|\,\mathrm{d}\mu + \frac{1}{|I|}\int_I |h - h_I|\,\mathrm{d}\mu \\
&\le \mathrm{M}_\delta(\varphi) + \frac{1}{|I|}\int_I |h - h_I|\,\mathrm{d}\mu,
\end{aligned}$$

again, it is enough to prove

$$\frac{1}{|I|}\int_I |h-h_I|\mathrm{d}\mu < (c-1)\mathrm{M}_\delta(\varphi).$$

This is done in three steps.

STEP I. Suppose $I=(a,b)$ and $|I|\le \delta/2$.

If there exists a integer z such that $I\subset I_z$, we have

$$\frac{1}{|I|}\int_I |h(t)-h_I(t)|\,\mathrm{d}\mu = 0 \le \mathrm{M}_\delta(\varphi).$$

Or if there exists such z that $I\subset I_z\cup I_{z+1}$ and $a\in I_z$ and $b\in I_{z+1}$ respectively, we have

$$\begin{aligned}
&\frac{1}{|I|}\int_I |h(t)-h_I|\,\mathrm{d}\mu(\mathrm{t})\\
&=\frac{1}{|I|}\int_{I\cap I_z}\left|\varphi_{I_z}-\frac{|I\cap I_z|}{|I|}\varphi_{I_z}-\frac{|I\cap I_{z+1}|}{|I|}\varphi_{I_{z+1}}\right|\,\mathrm{d}\mu(t)+\\
&\quad\frac{1}{|I|}\int_{I\cap I_{z+1}}\left|\varphi_{I_{z+1}}-\frac{|I\cap I_z|}{|I|}\varphi_{I_z}-\frac{|I\cap I_{z+1}|}{|I|}\varphi_{I_{z+1}}\right|\,\mathrm{d}\mu(t)\\
&=\frac{1}{|I|}\int_{I\cap I_z}\frac{|I\cap I_{z+1}|}{|I|}|\varphi_{I_{z+1}}-\varphi_{I_z}|\,\mathrm{d}\mu(t)+\frac{1}{|I|}\int_{I\cap I_{z+1}}\frac{|I\cap I_z|}{|I|}|\varphi_{I_{z+1}}-\varphi_{I_z}|\,\mathrm{d}\mu(t)\\
&\le\frac{|I\cap I_z|+|I\cap I_{z+1}|}{|I|}|\varphi_{I_{z+1}}-\varphi_{I_z}|\\
&=|\varphi_{I_{z+1}}-\varphi_{I_{z+1}}|\le 4\mathrm{M}_\delta(\varphi).
\end{aligned}$$

STEP II. If $\delta/2<|I|\le\delta$ and there exists z such that $I\subset I_z\cup I_{z+1}\cup I_{z+2}$, it follows that

$$\begin{aligned}
&\frac{1}{|I|}\int_I |h(t)-h_I(t)|\,\mathrm{d}\mu\\
&=\frac{1}{|I|^2}\int_{I\cap I_z}\left|\int_{I\cap I_{z+1}}(\varphi_{I_{z+1}}-\varphi_{I_z})\,\mathrm{d}\mu+\int_{I\cap I_{z+2}}(\varphi_{I_{z+2}}-\varphi_{I_z})\,\mathrm{d}\mu\right|\,\mathrm{d}\mu+\\
&\quad\frac{1}{|I|^2}\int_{I\cap I_{z+1}}\left|\int_{I\cap I_z}(\varphi_{I_z}-\varphi_{I_{z+1}})\,\mathrm{d}\mu+\int_{I\cap I_{z+2}}(\varphi_{I_{z+2}}-\varphi_{I_{z+1}})\,\mathrm{d}\mu\right|\,\mathrm{d}\mu+\\
&\quad\frac{1}{|I|^2}\int_{I\cap I_{z+2}}\left|\int_{I\cap I_z}(\varphi_{I_z}-\varphi_{I_{z+2}})\,\mathrm{d}\mu+\int_{I\cap I_{z+1}}(\varphi_{I_{z+1}}-\varphi_{I_{z+2}})\,\mathrm{d}\mu\right|\,\mathrm{d}\mu\\
&\le|\varphi_{I_{z+1}}-\varphi_{I_z}|+|\varphi_{I_{z+2}}-\varphi_{I_z}|+|\varphi_{I_z}-\varphi_{I_{z+1}}|+|\varphi{Iz+2}-\varphi{Iz+1}|+\\
&\quad|\varphi_{I_z}-\varphi_{I_{z+2}}|+|\varphi_{I_{z+1}}-\varphi_{I_{z+2}}|\\
&\le 32\mathrm{M}_\delta(\varphi).
\end{aligned}$$

STEP III. If $|I|>\delta$ and if J is the union of those $I_j(j=1,2,\ldots,N)$ such that $I_j\cap I\ne\varnothing$, viz. $J=I_1\cup I_2\cup\cdots\cup I_N$, clearly, $|J|\le 2|I|$. We have

$$\begin{aligned}
\frac{1}{|I|}\int_I|\varphi-h-(\varphi-h)_I|\,\mathrm{d}\mu&\le\frac{2}{|I|}\int_I|\varphi-h|\,\mathrm{d}\mu\le\frac{4}{|J|}\sum_{j=1}^N\int_{I_j\cap I}|\varphi-\varphi_{I_j}|\,\mathrm{d}\mu\\
&\le\frac{4}{|J|}\sum_{j=1}^N|I_j\cap I|\mathrm{M}_\delta(\varphi)\le 4\mathrm{M}_\delta(\varphi).
\end{aligned}$$

Thus Theorem 1 is proven.

With Theorem 1 proven, if $|x| \leq \delta$, we have

$$\begin{aligned} \|\varphi - \varphi_x\|_* \leq & \|\varphi - h\|_* + \|h - h_x\|_* + \|\varphi_x - h_x\|_* \\ \leq & 2c\mathrm{M}_\delta(\varphi) + \|h - h_x\|_*. \end{aligned}$$

Obviously, $\|h - h_x\|_* \leq 4\mathrm{M}_\delta(\varphi)$ and $\mathrm{M}_\delta(\varphi) \to 0$ $(\delta \to 0)$, thus Theorem 2 is proven.

It is worth noting that Theorem 2 can be generalized to the weighted measurable space $(\mathbf{R}, \mu, \mathcal{R})$.

Acknowledgement Thanks for those who have always being supporting for our work, and those who have imparted us with most invaluable discussion and information.

References:

[1] JOHN F, NIRENBERG L. *On function of bounded mean oscillation* [J]. Comm. Pure. Appl. Math., 1961, **14**: 415–426.
[2] SARASON D. *Functions of vanishing mean oscillation* [J]. Trans. Amer. Soc., 1975, **207**: 391–405.
[3] UERI U. *Some properties of functions with bounded mean oscillation* [J]. Studia Math., 1977, **61**: 63–75.
[4] FEFFERMAN C, STEIN E M. $\mathbf{H^p}$ *spaces of several variables* [J]. Acta Math., 1972, **129**: 137–193.
[5] GARNETT J B. *Bounded Analytic Functions* [M]. Pure and Applied Mathematics, 96. Academic Press, Inc. New York-London, 1981

关于消失平均振动的一个注记

杨 威[1,2], 王 杰[1], 刘 玉堂[1]
(1. 北京理工大学理学院数学系，北京 100081;
2. 北京阜外心血管医院群体遗传研究室，北京 100037)

摘要: 本文得到了极大算子的几个不等式,并从这些结果推广了函数 $f \in \mathbf{L}^1_{\mathrm{loc}}(\mathbf{R}, \mathrm{d}x)$ 属于 $\mathrm{VMO}(\mathbf{R}, \mathrm{d}x)$ 的一个必要条件.

关键词: 极大算子; VMO 空间; Radon 测度.