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An Alternative Approach about Several Theorems in Calculus

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Abstract: In this paper we change the angle of view to consider the connections among several important theorems in Calculus, and give alternative proofs and new implications about these theorems.

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There has been much discussion on the Increasing Function Theorem (IFT) and related results such as the Mean Value Theorem (MVT)[1,2] recent years in the Calculus reform in the USA. One viewpoint is using the IFT to replace the MVT while the many ones have reservations. Motivated by these articles, in this paper we will change the angle of view to consider the connections among these theorems in Calculus. Our point of view is that the MVT needs to be retained, but its proof (and the proofs of other fundamental theorems in Calculus) can be modernized. Our treatment of several well-known theorems in Calculus, including the starting point, the implication, and the inference, are different from that available in the current literature. Our process implies the equivalence of the MVT and Strictly IFT (and therefore IFT) in the sense that the theoretical corpus of Calculus can be derived using either IFT or Rolle's Theorem (and thus MVT) as the starting point.

In addition, we find that the Heine-Borel Lemma (i.e., the Heine-Borel Covering Theorem on closed intervals), easily proved using the bisection scheme (i.e., Bolzano method), is convenient for extending some local properties of a function to the entire closed interval being considered. Consider a collection $\sum$ of open intervals . If for every point x in $[a,b]$ there is an open interval of $\sum$ containing x, we say $\sum$ covers $[a,b]$. The Heine-Borel Lemma states that if a closed interval $[a,b]$ is covered by a collection $\sum$ of open intervals, then there must be finite open intervals selected from $\sum$ by which $[a,b]$ is covered. The Lemma can be proven by the bisection scheme and therefore is not too advanced for an elementary calculus textbook. Using the Heine-Borel Lemma, we can in turn prove the Intermediate Value Theorem (IVT), the Extreme Value Theorem (EVT), etc. for continuous functions. However, in this paper we will use the Heine-Borel Lemma to prove the following main lemma, Lemma 1 (see [3, Theorem 5.17 p.112).

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Next, as very easy corollaries of Lemma 1, the Strictly Incresing Function Theorem (Theorem 1), the Rolle's Theorem (Theorem 2) and the Intermediate-Value Theorem for Derivatives (Theorem 3) are obtained. We thus give alternative proofs and new implications about several well known theorems in the Calculus.

Lemma 1 *Let f be a real differentiable function on the interval (a,b). If $f'(x) \neq 0$ for all x in (a,b) then f is strictly monotone on (a,b).*

Proof Let x' and x'' be any two points in the interval (a,b) with $x' < x''$. Then for every $x \in [x', x'']$ we have $\lim_{z\to x} \frac{f(z)-f(x)}{z-x} = f'(x) \neq 0$, and so there exists $\delta_x > 0$ such that

$$z \in I_x = (x-\delta_x, x+\delta_x) \text{ and } z \in (a,b) \text{ implies}$$

$$\frac{f(z)-f(x)}{z-x} > 0, \quad \text{if} \quad f'(x) > 0;$$

$$\frac{f(z)-f(x)}{z-x} < 0, \quad \text{if} \quad f'(x) < 0. \tag{1}$$

The collection of all such intervals I_x forms an open cover of $[x', x'']$. By the Heine-Borel Covering Theorem, there is a finite number of these I_x , say $I_{x_1}, \cdots, I_{x_n}$ where $x' < x_1 < x_2 < \cdots < x_n < x''$, such that

$$[x', x''] \subseteq \bigcup_{i=1}^{n} I_{x_i}. \tag{2}$$

Clearly, we can assume that $I_{x_i} \cap I_{x_{i+1}} \neq \emptyset (i = 1, \cdots, n-1)$ by deleting points for which it is not true.

Now we need consider the following two cases:

(i) For all $i = 1, \cdots, n, f'(x_i)$ have same sign.

We may assume without loss of generality that $f'(x_i) > 0$ $(i = 1, \cdots, n)$. In this case, for each $i \in \{1, \cdots, n-1\}$ choose $z_i \in I_{x_i} \cap I_{x_{i+1}}$ such that $x_i < z_i < x_{i+1}$. By (1), we have

$$\frac{f(z_i)-f(x_i)}{z_i - x_i} > 0 \quad \text{and} \quad \frac{f(z_i)-f(x_{i+1})}{z_i - x_{i+1}} > 0.$$

Thus

$$f(x_i) < f(z_i) < f(x_{i+1}), \quad i = 1, \cdots, n-1; \tag{3}$$

and from $x' \in I_{x_1}(x' < x_1)$ and $x'' \in I_{x_n}(x_n < x'')$ we have

$$f(x') < f(x_1) \text{ and } f(x'') < f(x_n). \tag{4}$$

Combining the inequalities (3) and (4) we obtain

$$f(x') < f(x_1) < \cdots < f(x_n) < f(x'').$$

This shows that for every pair of points x' and x'' in (a,b), $x' < x''$ implies $f(x') < f(x'')$. Hence f is strictly increasing on (a,b).

(ii) There exist x_i and $x_{i+1}(i \in \{1, \cdots, n-1\})$ such that $f'(x_i) \cdot f'(x_{i+1}) < 0$.

We may assume without loss of generality that $f'(x_i) > 0$ and $f'(x_{i+1}) < 0$. Then by (1) we have

$$\frac{f(z) - f(x_i)}{z - x_i} > 0 \text{ when } z \in I_{x_i},$$

$$\frac{f(z) - f(x_{i+1})}{z - x_{i+1}} < 0 \text{ when } z \in I_{x_{i+1}}.$$

Choose $z' \in I_{x_i} \cap I_{x_{i+1}}$ such that $x_i < z' < x_{i+1}$ then

$$\frac{f(z') - f(x_i)}{z' - x_i} > 0 \text{ and } \frac{f(z') - f(x_{i+1})}{z' - x_{i+1}} < 0.$$

Thus

$$f(z') > f(x_i) \text{ and } f(z') > f(x_{i+1}).$$

Due to the fact that a continuous function assumes its maximum and minimum on a closed interval, the last two inequalities imply that there exists $\overline{z} \in (x_i, x_{i+1})$ such that $f(\overline{z})$ is a maximum of f on $[x_i, x_{i+1}]$.

Since $f'(\overline{z})$ is a real mumber, it must be positive , negative, or zero. We shall show that $f'(\overline{z})$ cannot be positive or negative and , therefore $f'(\overline{z})$ must be zero, which contradicts the condition that $f'(x) \neq 0$ for all $x \in (a, b)$. This contradiction shows that only case (i) holds.

Let $g(h) = (f(\overline{z} + h) - f(\overline{z}))/h$, and define $g(0) = f'(\overline{z})$, since f is continuous, g is a continuous function of h for $h \neq 0$. Also, since $\lim_{h \to 0} g(h) = f'(\overline{z}) = g(0), g$ is also continuous at $h = 0$. Now assume that $f'(\overline{z})$ is positive. Thus $g(0)$ is positive and by the sign preserving property of continuous functions ,$g(h)$ is positive over some interval centered at $h = 0$. Therefore, $f(\overline{z} + h) < f(\overline{z})$ when $h < 0$ and $f(\overline{z} + h) > f(\overline{z})$ when $h > 0$. However, this contradicts that f takes on an extremum at the point $\overline{z}$. A similar argument can be made if it is assumed that $f'(\overline{z})$ is negative. Since $f'(\overline{z})$ cannot be positive or negative , we have $f'(\overline{z}) = 0$.

Thus the Lemma is proved.

From first part of the proof of the Lemma 1 we obtain easily the following Theorem 1 which be called that the Strictly Increasing Function Theorem (SIFT) in [1].

Theorem 1 *Assume f is differentiable and $f'(x) > 0$ (resp. $f'(x) < 0$) for each x in (a, b). Then f is strictly increasing (resp. strictly decreasing) on (a, b).*

From the Theorem 1 by a perturbing for the function f , we have at once that the Increasing Function Theorem (IFT), that is ,if f is differentiable and $f'(x) \geqslant 0$ (resp. $f'(x) \leq 0$) for each x in (a, b), then f is non-decreasing (resp. non-increasing) on (a, b)[1].

Theorem 2 (the Rolle's Theorem) *Let a real function f be continuous on $[a, b]$, differentiable on (a, b) and such that $f(a) = f(b)$. Then there exists $c \in (a, b)$ with $f'(c) = 0$.*

Proof Suppose $f'(x) \neq 0$ for all $x \in (a, b)$, then f is strictly monotone on (a, b) by the Lemma 1. From the continuity of f on $[a, b]$ it follows easily that $f(a) \neq f(b)$ which contradicts the condition that $f(a) = f(b)$.

Remark From the above proof we see that Rolle's Theorem (and further the MVT) can is derived directly from the SIFT (or IFT). This show the SIFT (or IFT) and MVT are equivalent.

Theorem 3 (the Intermediate-Value Theorem for Derivatives) *Let f be a real differentiable function on $[a,b]$. If $f'_+(a) < \lambda < f'_-(b)$ then there exists a point $\xi \in (a,b)$ such that $f'(\xi) = \lambda$ (A similar result holds if $f'_+(a) > f'_-(b)$).*

Proof Define g on (a_0, b_0), where $-\infty < a_0 < a < b < b_0 < +\infty$ as follows:

$$g(x) = f(x) - \lambda x, \quad \text{when} \quad x \in [a,b];$$

$$g(x) = f(a) - \lambda x + f'_+(a)(x-a), \quad \text{when} \quad a_0 < x < a;$$

$$g(x) = f(b) - \lambda x + f'_-(b)(x-b), \quad \text{when} \quad b < x < b_0.$$

The function g is differentiable on (a_0, b_0). Suppose $g'(x) \neq 0$ for all $x \in (a_0, b_0)$, then either $g'(x) > 0$ or $g'(x) < 0$ for all $x \in (a_0, b_0)$ by the Lemma 1.

If $g'(x) > 0$ for all $x \in (a_0, b_0)$ then $f'_+(a) - \lambda > 0$; if $g'(x) < 0$ for all $x \in (a_0, b_0)$ then $f'_-(b) - \lambda < 0$. Whatever consequences follow, it contradicts that the condition that $f'_+(a) < \lambda < f'_-(b)$.

Therefore there exists a point $\xi \in (a_0, b_0)$ such that $g'(\xi) = 0$ and so $f'(\xi) = \lambda$. Furthermore $\xi \in (a,b)$ since $g'(x) = f_+(a) - \lambda < 0$ when $a_0 < x \leq a$ and $g'(x) = f'_-(b) - \lambda > 0$ when $b \leq x < b_0$.

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微积分中几个定理的另类处理

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摘要: 在本文中，我们转换视角考察微积分中几个重要定理之间的关系，给出这几个定理的非传统的另类证明和新的蕴涵关系以及彼此等价的结论.

关键词: 函数；连续；可微；单调.