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On a New Strengthened Hardy-Hilbert's Inequality

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Abstract: By means of a sharpening of Hölder's inequality, Hardy-Hilbert's integral inequality with parameters is improved. Some new inequalities are established.

Key words: Hardy-Hilbert integral inequality; Hölder's inequality; weight function; beta function.
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1. Introduction

Let $p>1$, $1/p+1/q=1$, $f,g>0$. If $0<\int_0^\infty f^p(t)\mathrm{d}t<+\infty, 0<\int_0^\infty g^q(t)\mathrm{d}t<+\infty$, then

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y<\frac{\pi}{\sin(\pi/p)}(\int_0^\infty f^p(t)\mathrm{d}t)^{1/p}(\int_0^\infty g^q(t)\mathrm{d}t)^{1/q}, \tag{1.1}$$

where the constant $\frac{\pi}{\sin(\pi/p)}$ is best possible. The Inequality (1.1) is well known in the literature as the Hardy-Hilbert's integral inequality. In recent years, some improvements and extensions of Hilbert's inequality and Hardy-Hilbert's inequality have been given in [2–6]. Gao[3] gave an improvement of Hilbert's inequality as follows:

$$[\int_0^\infty\int_0^\infty\frac{f(s)g(t)}{s+t}\mathrm{d}s\mathrm{d}t]^2<\pi^2\int_0^\infty f^2(t)\mathrm{d}t\int_0^\infty g^2(t)\mathrm{d}t-G(\xi,\eta,\delta). \tag{1.2}$$

The main purpose of this paper is to build a few new inequalities, which include the generalization of the Inequality (1.2) and the extensions and the improvements of corresponding results from [2–4].

2. Lemmas

For convenience, we firstly introduce some notations:

$$(f^r,g^s)=\int_a^b f^r(x)g^s(x)\mathrm{d}x,\quad \|f\|_p=(\int_a^b f^p(x)\mathrm{d}x)^{1/p},$$

$$\|f\|_2=\|f\|,\quad S_r(H,x)=(H^{r/2},x)\|H\|_r^{-r/2},$$

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where x is a parametric variable unit vector. In general case, it is properly chosen such that the specific problems discussed are simplified.

Clearly, $S_r(H,x)=0$ when the vector x selected is orthogonal to $H^{r/2}$. Throughout this paper, the exponent m is taken to be $m=\min\{1/p,1/q\}$, and $a<b\le\infty$.

In order to verify our assertions, we need to point out the following lemmas.

Lemma 1 *Let $f(x), g(x)>0$, $1/p+1/q=1$ and $p>1$. If $0<\|f\|_p<+\infty$ and $0<\|g\|_q<+\infty$, then*

$$(f,g)<\|f\|_p\|g\|_q(1-R)^m,$$

where $R=(S_p(f,h)-S_q(g,h))^2<1$, $\|h\|=1$, $f^{p/2}(x), g^{q/2}(x)$ and $h(x)$ are linearly independent.

Proof First, we discuss the case $p\neq q$. Without loss of generality, suppose that $p>q>1$. Since $1/p+q/q=1$, we have $p>2$. Let $A=p/2, B=p/(p-2)$. Then $1/A+1/B=1$. By Hölder's inequality we obtain

$$\begin{aligned}(f,g)&=\int_a^b f(x)g(x)\mathrm{d}x=\int_a^b(f\cdot g^{q/p})g^{1-(q/p)}\mathrm{d}x\\ &\le(\int_a^b(f\cdot g^{q/p})^A\mathrm{d}x)^{1/A}(\int_a^b(g^{1-(q/p)})^B\mathrm{d}x)^{1/B}\\ &=(f^{p/2},g^{q/2})^{2/p}\|g\|_q^{q(1-2/p)}.\end{aligned}\tag{2.2}$$

The equality in (2.1) holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent. In fact, the equality in (2.1) holds if and only if there exists c_1 such that $(f\cdot g^{q/p})^A=c_1(g^{1-(q/p)})^B$. It is easy to deduce that $f^{p/2}=c_1g^{q/2}$.

In our previous paper [3], with the help of the positive definiteness of Gram matrix, an important inequality is established as follows:

$$(\alpha,\beta)^2\le\|\alpha\|^2\|\beta\|^2-(\|\alpha\|x-\|\beta\|y)^2=\|\alpha\|^2\|\beta\|^2(1-\bar{\gamma}),\tag{2.3}$$

where $\overline{\gamma}=(\frac{y}{\|\alpha\|}-\frac{x}{\|\beta\|})^2, x=(\beta,\gamma), y=(\alpha,\gamma)$ with $\|\gamma\|=1$ and $xy\ge0$.

The equality in (2.3) holds when and only when α and β are linearly dependent; Or the vector γ is a linear combination of α and β, and $xy=0$ but $x\neq y$. If α, β and γ in (2.3) are replaced by $f^{p/2}$, $g^{q/2}$ and h respectively, then we get

$$(f^{p/2}g^{q/2})^2\le\|f\|_p^p\,\|g\|_q^q\,(1-R),\tag{2.4}$$

where $R=(S_p(f,h)-S_q(g,h))^2$ with $||h||=1$. The equality in (2.4) holds when and only when $f^{p/2}$ and $g^{q/2}$ are linearly dependent; Or h is a linear combination of $f^{p/2}$ and $g^{q/2}$, and $(f^{p/2},h)(g^{q/2},h)=0$, but $(f^{p/2},h)\neq(g^{q/2},h)$. Because $f^{p/2}$ and $g^{q/2}$ are linearly independent, it is impossible to have equality in (2.4). Substituting (2.4) into (2.1), we obtain after simplifications

$$(f,g)<\|f\|_p\,\|g\|_q\,(1-R)^{1/p}.\tag{2.5}$$

Then $R \neq 0$, provided that $h(x)$ is properly chosen. (The choice of $h(x)$ is quite flexible, so long as condition $\|h\| = 1$ is satisfied, on which we can refer to [3,4] etc.) Noticing the symmetry of p and q, Inequality (2.1) follows from (2.5).

Next, we discuss the case of $p = q$. According to the hypothesis, when f, g and h are linearly independent, we immediately obtain from (2.3) the following result:

$$(f, g) < \|f\| \, \|g\| \, (1 - \overline{r})^{1/2},$$

where $\overline{r} = (\frac{(f,h)}{\|f\|} - \frac{(g,h)}{\|g\|})^2$, and $\|h\| = 1$.

It remains to verify the inequality $R < 1$. According to Cauchy inequality $(f, g) \leq \|f\| \cdot \|g\|$, and noting the equality holds when and only when f and g are linearly dependent, we have $(H^{r/2}, x) \leq \|H^{r/2}\| . \|x\| = \|H\|_r^{r/2}$. So $0 < S_r(H, x) = (H^{r/2}, x)\|H\|_r^{-r/2} \leq 1$, and $0 < S_p(f, h) \leq 1$. Since $f^{p/2}(x)$ and $h(x)$ are linearly independent, $0 < S_p(f, h) < 1$. In a similar way, we can get : $0 < S_q(g, h) < 1$. Hence $R = (S_p(f, h) - S_q(g, h))^2 < 1$.

Thus the lemma is proved.

Lemma 2 *Let $r > 1$, $1/r < t \leq 1$. Define the weight function as follows:*

$$\omega_t(a, b, r, x) = \int_a^b \frac{1}{x^t + y^t}(\frac{x}{y})^{1/r} \mathrm{d}y, \quad x \in [a, b]. \tag{2.6}$$

Then we have

$$\omega_t(a, b, r, x) \leq x^{1-t}[\frac{\pi}{t \sin((1 - 1/r)\pi/t)} - \varphi(r)], \tag{2.7}$$

$$\omega_t(a, \infty, r, x) \leq x^{1-t}[\frac{\pi}{t \sin((1 - 1/r)\pi/t)} - W(r)], \tag{2.8}$$

where $\varphi(r) = \int_0^{a/b} \frac{u^{(1/r)+t-2} + u^{-(1/r)}}{1+u^t} \mathrm{d}u$, $W(r) = \int_0^{a/x} \frac{u^{-1/r}}{1+u^t} \mathrm{d}u, r = p, q$.

Lemma 3 *Let $r > 1$, $1/r < t \leq 1$, and*

$$\omega_t(0, b, r, x) = \int_0^b \frac{1}{x^t + y^t}(\frac{x}{y})^{1/r} \mathrm{d}y, \quad x \in [0, b].$$

Then

$$\omega_t(0, b, r, x) \leq x^{1-t}(\frac{\pi}{t \sin((1 - 1/r)\pi/t)} - (\frac{x}{b})^{t/r} \int_0^1 \frac{u^{1/r+t-2}}{1 + u^t} \mathrm{d}u). \tag{2.9}$$

The proofs of Lemmas 2 and 3 are given in [2], and we omit them here.

3. Main results

For convenience, we need the following notations

$$F = \frac{f(x)}{(x^t + y^t)^{1/p}}(\frac{x}{y})^{1/pq}, \quad G = \frac{g(y)}{(x^t + y^t)^{1/q}}(\frac{y}{x})^{1/pq},$$

$$S_p(F, h) = \{\int_a^b \int_a^b F^{P/2} \cdot h \mathrm{d}x \mathrm{d}y\}\{\int_a^b \int_a^b F^p \mathrm{d}x \mathrm{d}y\}^{-1/2},$$

$$S_q(G,h)=\{\int_a^b\int_a^b G^{q/2}\cdot h\mathrm{d}x\mathrm{d}y\}\{\int_a^b\int_a^b G^q\mathrm{d}x\mathrm{d}y\}^{-1/2},$$

where $h=h(x,y)$ is the unit vector satisfying the property

$$\|h\|=\{\int_a^b\int_a^b h^2\mathrm{d}x\mathrm{d}y\}^{1/2}=1,$$

and $F^{p/2},G^{q/2},h$ are linearly independent.

The first main result is incorporated in the following theorem.

Theorem 1 *Let* $p>1, 1/p+1/q=1, \max\{1/p,1/q\}<t\le 1, f,g>0$. *If* $f\in L^p[0,\infty), g\in L^q[0,\infty)$, *then*

$$\int_a^b\int_a^b\frac{f(x)g(y)}{x^t+y^t}\mathrm{d}x\mathrm{d}y<(\frac{\pi}{t\sin(\pi/(pt))}-\varphi(q))^{1/p}(\frac{\pi}{t\sin(\pi/(qt))}-\varphi(p))^{1/q}\times$$
$$(\int_a^b x^{1-t}f^p(x)\mathrm{d}x)^{1/p}(\int_a^b x^{1-t}g^q(x)\mathrm{d}x)^{1/q}[1-R(t,a,b)]^m, \tag{3.1}$$

$$\int_0^b\int_0^b\frac{f(x)g(y)}{x^t+y^t}\mathrm{d}x\mathrm{d}y<(\int_0^b(\frac{\pi}{t\sin(\pi/(pt))}-(\frac{x}{b})^{t/q}\psi(p))x^{1-t}f^p(x)\mathrm{d}x)^{1/p}\times$$
$$(\int_0^b(\frac{\pi}{t\sin(\pi/(qt))}-(\frac{x}{b})^{t/p}\psi(q))x^{1-t}g^q(x)\mathrm{d}x)^{1/q}[1-R(t,0,b)]^m, \tag{3.2}$$

$$\int_a^\infty\int_a^\infty\frac{f(x)g(y)}{x^t+y^t}\mathrm{d}x\mathrm{d}y<(\int_a^\infty(\frac{\pi}{t\sin(\pi/(pt))}-W(q))x^{1-t}f^p(x)\mathrm{d}x)^{1/p}\times$$
$$(\int_a^\infty(\frac{\pi}{t\sin(\pi/(qt))}-W(p))x^{1-t}g^q(x)\mathrm{d}x)^{1/q}[1-R(t,a,\infty)]^m, \tag{3.3}$$

where

$$\varphi(r)=\int_0^{b/a}\frac{u^{(1/r)+t-2}+u^{-1/r}}{1+u^t}\mathrm{d}u,\quad \psi(r)=\int_0^1\frac{u^{t-(1/r)-1}}{1+u^t}\mathrm{d}u,$$

$$W(r)=\int_0^{a/x}\frac{u^{-1/r}}{1+u^t}\mathrm{d}u,\quad (r=p,q)\ \text{ and }\ R(t,a,b)=(S_p(F,h)-S_q(G,h))^2<1$$

$$h(x,y)=\begin{cases}(\frac{2}{\pi})^{1/2}\frac{e^{a-x}}{(x+y-2a)^{1/2}}(\frac{x-a}{y-a})^{1/4} & b=\infty;\\ \frac{b-a}{(x-a)(y-a)}e^{(1-\frac{b-a}{2(x-a)}-\frac{b-a}{2(y-a)})} & b<\infty.\end{cases} \tag{3.4}$$

Proof By Lemma 1, we get

$$\begin{aligned}\int_a^b\int_a^b\frac{f(x)g(y)}{x^t+y^t}\mathrm{d}x\mathrm{d}y&=\int_a^b\int_a^b FG\mathrm{d}x\mathrm{d}y\\ &\le\{\int_a^b\int_a^b F^p\mathrm{d}x\mathrm{d}y\}^{1/p}\{\int_a^b\int_a^b G^q\mathrm{d}x\mathrm{d}y\}^{1/q}[1-R(t,a,b)]^m\\ &=(\int_a^b\omega_t(a,b,q,x)f^p(x)\mathrm{d}x)^{1/p}(\int_a^b\omega_t(a,b,p,x)g^q(x)\mathrm{d}x)^{1/q}[1-R(t,a,b)]^m,\end{aligned} \tag{3.5}$$

where $\omega_t(a,b,r,x)$ is the function defined by (2.6), $r=p,q$.

By substituting (2.7), (2.9) and (2.8) into (3.5) respectively, the Inequalities (3.1), (3.2) and (3.3) follow.

It remains to discuss the expression of $R(t,a,b)$. We may choose the function $h(x,y)$ defined by (3.4).

For $b=\infty$, set $s=x-a, t=y-a$, then

$$||h|| = (\int_a^\infty \int_a^\infty h^2(x,y)\mathrm{d}x\mathrm{d}y)^{1/2} = \{\frac{2}{\pi}\int_0^\infty e^{-2s}\mathrm{d}s \int_0^\infty \frac{1}{s+t}(\frac{s}{t})^{1/2}\mathrm{d}t\}^{1/2} = 1.$$

For $b<\infty$, set $\xi = \frac{b-a}{x-a}, \eta = \frac{b-a}{y-a}$, then we have

$$\begin{aligned}||h|| &= (\int_a^b \int_a^b h^2 \mathrm{d}x\mathrm{d}y)^{1/2} = \{\int_a^b \frac{b-a}{(x-a)^2}\, e^{(1-\frac{b-a}{x-a})}\mathrm{d}x \cdot \int_a^b \frac{b-a}{(y-a)^2}\, e^{(1-\frac{b-a}{y-a})}\mathrm{d}y\}^{1/2}\\ &= \{\int_1^\infty e^{1-\xi}\mathrm{d}\xi \cdot \int_1^\infty e^{1-\eta}\mathrm{d}\eta\}^{1/2} = 1.\end{aligned}$$

According to Lemma 1 and the given $h(x,y)$, we have $R(t,a,b) = (S_p(F,h) - S_q(G,h))^2$and $R(t,a,b) < 1$. It is obvious that $F^{p/2}, g^{q/2}$ and h are linearly independent, so it is impossible for the equality to hold in (3.5).

The Proof of Theorem is thus completed.

Remark 1 Clearly, Inequalities (3.1), (3.2) and (3.3) are the improvements of (1.3), (1.4) and (1.5) in [2] respectively.

The following result is a natural consequence of Theorem 1.

Corollary 1 *Suppose that $p>1, 1/p+1/q=1, 0\le a<b\le\infty$, $f(t), g(t)>0$. If $f\in L^p[0,\infty), g\in L^q[0,\infty)$, then*

$$\begin{aligned}\int_a^b \int_a^b \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y &< (\frac{\pi}{\sin(\pi/p)} - \int_0^{a/b} \frac{u^{-1/q}+u^{-1/p}}{1+u}\mathrm{d}u)\times\\ &(\int_a^b f^p(x)\mathrm{d}x)^{1/p}(\int_a^b g^q(x)\mathrm{d}x)^{1/q}[1-R(1,a,b)]^m,\end{aligned} \tag{3.6}$$

$$\begin{aligned}\int_0^b \int_0^b \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y &< \{\int_0^b (\frac{\pi}{\sin(\pi/p)} - (\frac{x}{b})^{1/q}\int_0^1 \frac{u^{-1/p}}{1+u}\mathrm{d}u) f^p(x)\mathrm{d}x\}^{1/p}\times\\ &\{\int_0^b (\frac{\pi}{\sin(\pi/p)} - (\frac{x}{b})^{1/p}\int_0^1 \frac{u^{-1/q}}{1+u}\mathrm{d}u) g^q(x)\mathrm{d}x\}^{1/q}[1-R(1,0,b)]^m,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\int_a^\infty \int_a^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y &< \{\int_a^\infty (\frac{\pi}{\sin(\pi/p)} - \int_0^{a/x} \frac{u^{-1/q}}{1+u}\mathrm{d}u) f^p(x)\mathrm{d}x\}^{1/p}\times\\ &\{\int_a^\infty (\frac{\pi}{\sin(\pi/p)} - \int_0^{a/x} \frac{u^{-1/p}}{1+u}\mathrm{d}u) g^q(x)\mathrm{d}x\}^{1/q}[1-R(1,a,\infty)]^m,\end{aligned} \tag{3.8}$$

where R is given by Theorem 1 with $t=1$.

Remark 2 Owing to $\int_0^1 \frac{u^{-1/2}}{1+u}\mathrm{d}u = \frac{\pi}{2}$, when $p=q=2$, Inequality (3.7) becomes an improvement of (7) in [5]. Consequently, Inequalities (3.2) and (3.7) are all the extensions of Inequality (7) in [5].

When $a \to 0$, (3.3) becomes

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^t+y^t}\mathrm{d}x\mathrm{d}y < \frac{\pi}{t(\sin(\pi/pt))^{1/p}(\sin(\pi/qt))^{1/q}} \times$$
$$\{\int_0^\infty x^{1-t}f^p(x)\mathrm{d}x\}^{1/p}\{\int_0^\infty x^{1-t}g^q(x)\mathrm{d}x\}^{1/q}(1-\overline{R})^m. \tag{3.9}$$

Especially, for $t=1$, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin(\pi/p)}(\int_0^\infty f^p(x)\mathrm{d}x)^{1/p}(\int_0^\infty g^q(x)\mathrm{d}x)^{1/q}(1-\overline{r})^m. \tag{3.10}$$

When $p=q=2$, Inequality (3.10) reduces, after simple computation, to Inequality (1.2). As a result, Inequalities (3.1)–(3.3) and (3.6)–(3.10) are all the extensions of Inequality (1.2).

Theorem 2 *Under the same conditions as in Theorem 1. We have*

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^t}\mathrm{d}x\mathrm{d}y < (B(1/p,t-1/p)-\overline{\varphi}(q))^{1/p}(B(1/q,t-1/q)-\overline{\varphi}(p))^{1/q}\times$$
$$(\int_a^b x^{1-t}f^p(x)\mathrm{d}x)^{1/p}(\int_a^b x^{1-t}g^q(x)\mathrm{d}x)^{1/q}[1-\overline{R}(a,b)]^m, \tag{3.11}$$

$$\int_0^b \int_0^b \frac{f(x)g(y)}{(x+y)^t}\mathrm{d}x\mathrm{d}y < (\int_0^b (B(1/p,t-1/p)-(\frac{x}{b})^{t/q}\overline{\psi}(p))x^{1-t}f^p(x)\mathrm{d}x)^{1/p}\times$$
$$(\int_0^b (B(1/q,t-1/q)-(\frac{x}{b})^{t/p}\overline{\psi}(q))x^{1-t}g^q(x)\mathrm{d}x)^{1/q}[1-\overline{R}(0,b)]^m, \tag{3.12}$$

$$\int_a^\infty \int_a^\infty \frac{f(x)g(y)}{(x+y)^t}\mathrm{d}x\mathrm{d}y < (\int_a^\infty (B(1/p,t-1/p)-\overline{W}(q))x^{1-t}f^p(x)\mathrm{d}x)^{1/p}\times$$
$$(\int_a^\infty (B(1/q,t-1/q)-\overline{W}(p))x^{1-t}g^q(x)\mathrm{d}x)^{1/q}[1-\overline{R}(a,\infty)]^m, \tag{3.13}$$

where $B(1/r,t-1/r)$ is the beta function and

$$\overline{\varphi}(r) = \int_0^{a/b} \frac{u^{(1/r)+t-2}+u^{-1/\gamma}}{(1+u)^t}\mathrm{d}u,$$

$$\overline{\psi}(r) = \int_0^1 \frac{u^{t-(1/r)-1}}{(1+u)^t}\mathrm{d}u, \quad \overline{W}(r) = \int_0^{a/x} \frac{u^{-1/r}}{(1+u)^t}\mathrm{d}u, \qquad (r=p,q).$$

$$\overline{R}(a,b) = (S_p(\overline{F},h)-S_q(\overline{G},h))^2 < 1, \quad \overline{F} = \frac{f(x)}{(x+y)^{t/p}}(\frac{x}{y})^{1/pq}, \quad \overline{G} = \frac{g(y)}{(x+y)^{t/q}}(\frac{y}{x})^{1/pq}.$$

Theorem 2 can be proved in a way similar way to that for Theorem 1. We omit the details.

Remark 3 Inequalities (3.11)–(3.13) are the improvements of Inequalities (1.12)–(1.14) in [2] respectively.

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关于 Hardy-Hilbert 不等式的一个新的加强

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摘要: 本文利用改进了的 Hölder's 不等式对带参数的 Hardy-Hilbert 积分不等式作了改进，建立了一些新的不等式.

关键词: Hardy-Hilbert 不等式； Hölder' 不等式；权函数； β 函数.