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Composition Operators from $\mathcal{B}^0$ to $E(p,q)$ and $E_0(p,q)$ Spaces

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Abstract: When φ is an analytic map of the unit disk D into itself, and X is a Banach space of analytic functions on D, define the composition operator C_φ by $C_\varphi(f) = f \circ \varphi$, for $f \in X$. This paper deals with a collection of subclasses of Bloch space by means of composition operators from a subspace $\mathcal{B}^0$ of Q_q to $E(p,q)$ and $E_0(p,q)$ and gets a new characterization of spaces $E(p,q)$ and $E_0(p,q)$.

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1. Introduction

First, we introduce some basic notations, used in this paper. Throughout the paper, the unit disk and circle in the finite complex plane C will be denoted by $D = \{z \in C : |z| < 1\}$ and $\partial D = \{z \in C : |z| = 1\}$, respectively. $H(D)$ will denote the space of all analytic functions on D and dm Lebesgue measure on D, normalized so that $m(D) = 1$. For $a \in D$, $\sigma_a(z) = \frac{a-z}{1-\overline{a}z}$ is the Möbius transformation of D to itself and $g(z,a) = \log|\frac{1-\overline{a}z}{a-z}|$ is the Green function of D with singularity at a. It is easy to check that

$$1 - |\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}.$$

Every analytic self-map φ of the unit disk D induces through composition a linear composition operator C_φ from $H(D)$ to itself. It is a well-known consequence of Littlewood's subordination principle[1] that the formula $C_\varphi(f) = f \circ \varphi$ defines a bounded linear operator on the classical Hardy and Bergman spaces. So $C_\varphi : H^p \to H^p$ and $C_\varphi : A^p \to A^p$ are bounded operators. A problem that has received much attention recently is to relate function theoretic properties of φ to operator theoretic properties of the restriction of C_φ to various Banach spaces of analytic functions[2−7]. One goal here is to characterize these analytic function $\varphi \in E(p,q)$ or $E_0(p,q)$ that induce bounded or compact composition operators from a space to another space, where $E(p,q)$ and $E_0(p,q)$ were recently studied by Tan H.O. in [8] and [9], defined as follows:

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For $p>0$, $q\geq 0$ and $p+q>1$, define

$$E(p,q)=\{f: f\in H(D)\ \text{ and }\ \sup_{a\in D}\int_D |f'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z)<\infty\},$$

$$E_0(p,q)=\{f: f\in H(D)\ \text{ and }\ \lim_{|a|\to 1}\int_D |f'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z)=0\},$$

and $\mathcal{B}^0$ is a space of analytic functions f with $f'\in H^\infty$. Set

$$\|f\|_{\mathcal{B}^0}=|f(0)|+\|f'\|_\infty$$

and

$$\|f\|_{p,q}^p=\sup_{a\in D}\int_D |f'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z).$$

Tan H.O. in [8], [9] gives the characterizations of $E(p,q)$ and $E_0(p,q)$ via a Carleson measure condition. We will get a new characterization of spaces $E(p,q)$ and $E_0(p,q)$ by the boundedness or compactness of composition operators from a subspace $\mathcal{B}^0$ of Q_q to $E(p,q)$ and $E_0(p,q)$, where

$$Q_q=\{f: f\in H(D)\ \text{ and }\ \sup_{a\in D}\int_D |f'(z)|^2 g^q(z,a)\mathrm{d}m(z)<\infty\}, (q>0).$$

For $0<q<\infty$, we say that a positive measure μ defined on D is a bounded q-Carleson measure provided $\mu(S(I))=O(|I|^q)$ for all subarcs I of ∂D, and a positive measure μ defined on D is a compact q-Carleson measure provided $\mu(S(I))=o(|I|^q)$ $(|I|\to 0)$ for subarcs I of ∂D, where $|I|$ denotes the arc length of I and $S(I)$ denotes the usual Carleson box based on I,

$$S(I)=\left\{z\in D: 1-\frac{|I|}{2\pi}\leq |z|<1, \frac{z}{|z|}\in I\right\}.$$

Throughout this paper, the letter C denotes a positive constant which may vary at each occurrence but it is independent of the essential variables.

2. Preliminary material

Here we collect some Lemmas which will be used in the main results.

Lemma 1[8] *Suppose $p>0, q\geq 0$, $p+q>1$ and $\varphi\in H(D)$. Then the following statements are equivalent:*

(1) $\varphi\in E(p,q)$;

(2) $\mathrm{d}\mu_{\varphi,p,q}(z)$ *is a bounded q-Carleson measure, where* $\mathrm{d}\mu_{\varphi,p,q}(z)=|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)$.

Lemma 2[8] *Suppose $p>0, q\geq 0$, $p+q>1$ and $\varphi\in H(D)$. Then the following statements are equivalent:*

(1) $\varphi\in E_0(p,q)$;

(2) $\mathrm{d}\mu_{\varphi,p,q}(z)$ *is a compact q-Carleson measure.*

Lemma 3[10] *For $0 < q < \infty$, a positive measure μ on D is a bounded q-Carleson measure if and only if*

$$\sup_{a\in D}\int_D (\frac{1-|a|^2}{|1-\overline{a}z|^2})^q \mathrm{d}\mu(z) < \infty,$$

and μ is a compact q-Carleson measure if and only if

$$\lim_{|a|\to 1}\int_D (\frac{1-|a|^2}{|1-\overline{a}z|^2})^q \mathrm{d}\mu(z) = 0.$$

Lemma 4 *Suppose $p > 0, q > 0$, $p + q > 1$ and φ is an analytic self-map of D, then $C_\varphi : \mathcal{B}^0 \to E(p,q)$ is compact if and only if for any bounded sequence f_n in $\mathcal{B}^0$ with $\{f_n\} \to 0$ uniformly on compact subsets of D, $\|C_\varphi(f_n)\|_{p,q} \to 0$ as $n \to \infty$.*

Proof It is easy.

Lemma 5 *Suppose $p > 0, q \geq 0$, $p + q > 1$ and $\varphi \in E(p,q)$. Then*

$$\|\varphi\|_{p,q}^p \geq (\frac{\pi}{2})^q \frac{\int_{S(I)} |\varphi'(z)|^p (1-|z|^2)^{p+q-2}\mathrm{d}m(z)}{|I|^q}$$

for all I on D.

Proof $\forall z \in S(I)$, $1 - \frac{|I|}{2\pi} \leq |z| < 1$, take $b = (1 - \frac{|I|}{2\pi})\frac{z}{|z|}$, then $b \in D$,

$$1 - |b|^2 = 1 - |1 - \frac{|I|}{2\pi}|^2 = \frac{|I|}{\pi} - \frac{|I|^2}{4\pi^2} < \frac{|I|}{\pi},$$

$$1 - |b|^2 = \frac{|I|}{\pi} - \frac{|I|^2}{4\pi^2} = \frac{|I|}{\pi}(1 - \frac{|I|}{4\pi}) > \frac{|I|}{2\pi},$$

and

$$1 - \overline{b}z = 1 - (1 - \frac{|I|}{2\pi})|z| = 1 - |z| + \frac{|I|}{2\pi}|z| < \frac{|I|}{\pi},$$

so

$$|1 - \overline{b}z|^2 < \frac{|I|^2}{\pi^2}.$$

Thus

$$\begin{aligned}
\|\varphi\|_{p,q}^p &= \sup_{a\in D}\int_D |\varphi'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z)\\
&= \sup_{a\in D}\int_D (\frac{1-|\sigma_a(z)|^2}{1-|z|^2})^q |\varphi'(z)|^p (1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\
&= \sup_{a\in D}\int_D (\frac{1-|a|^2}{|1-\overline{a}z|^2})^q |\varphi'(z)|^p (1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\
&\geq \int_D (\frac{1-|b|^2}{|1-\overline{b}z|^2})^q |\varphi'(z)|^p (1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\
&\geq \int_{S(I)} (\frac{\pi}{2|I|})^q |\varphi'(z)|^p (1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\
&= (\frac{\pi}{2})^q \frac{\int_{S(I)} |\varphi'(z)|^p (1-|z|^2)^{p+q-2}\mathrm{d}m(z)}{|I|^q},
\end{aligned}$$

for all I on D. This completes the proof.

3. The proof of main results

We have three main theorems, which will appear as Theorems 1, 2 and 3 in this section.

Theorem 1 *Suppose $p > 0, q > 0$, $p+q > 1$, and φ is an analytic self-map of D. Then the composition operator $C_\varphi : \mathcal{B}^0 \to E(p,q)$ is bounded if and only if $\varphi \in E(p,q)$.*

The proof of the theorem is based on the above several lemmas.

Proof We first prove that the condition is sufficient. If $\varphi \in E(p,q)$, by Lemma 1, $\mathrm{d}\mu_{\varphi,p,q}(z) = |\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)$ is a bounded q-Carleson measure. Therefore,

$$M = \sup_{a\in D}\int_D \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^q \mathrm{d}\mu_{\varphi,p,q}(z) < \infty.$$

So, for all $f \in \mathcal{B}^0$,

$$\begin{aligned}\|f\circ\varphi\|_{p,q}^p &= \sup_{a\in D}\int_D |f'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q\mathrm{d}m(z)\\ &\le \|f\|_{\mathcal{B}^0}^p \sup_{a\in D}\int_D \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^q \mathrm{d}\mu_{\varphi,p,q}(z)\\ &\le M\|f\|_{\mathcal{B}^0}^p.\end{aligned}$$

Conversely, we assume that $C_\varphi : \mathcal{B}^0 \to E(p,q)$ is bounded. Then $C_\varphi(f) \in E(p,q)$ for all $f \in \mathcal{B}^0$. Taking $f(z) = z$ gives $\varphi \in E(p,q)$, as desired.

Theorem 2 *Suppose $p > 0, q > 0$, $p+q > 1$, and φ is an analytic self-map of D. Then the composition operator $C_\varphi : \mathcal{B}^0 \to E(p,q)$ is compact if and only if $\varphi \in E(p,q)$ and for every $\varepsilon > 0$, there is a δ, $0 < \delta < 1$, such that*

$$\int_{S(I)} \chi_{D_\delta}(z)|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) < \varepsilon|I|^q, \tag{1}$$

for all arcs I on ∂D, where $D_\delta = \{z \in D : |\varphi(z)| > \delta\}$, $\chi_{D_\delta}(z)$ denotes the characteristic function of D_δ.

Proof If $C_\varphi : \mathcal{B}^0 \to E(p,q)$ is compact, then $C_\varphi : \mathcal{B}^0 \to E(p,q)$ is bounded, so $\varphi \in E(p,q)$ by Theorem 1. For any bounded sequence $\{f_n\}$ in $\mathcal{B}^0$ with $f_n \to 0$ uniformly on compact subsets of D, we have by Lemma 4 $\|C_\varphi(f_n)\|_{p,q}^p \to 0$ as $n \to \infty$. Set $f_n(z) = z^n/n$. Since $\{z^n/n\}$ is a bounded sequence in $\mathcal{B}^0$ and converges uniformly to 0 on compact subsets of D, we have $\|\frac{\varphi^n}{n}\|_{p,q}^p \to 0$ as $n \to \infty$. Hence, for any given $\varepsilon > 0$, there is an $N > 0$, such that if $n \ge N$, then

$$\begin{aligned}\left\|\frac{\varphi^n}{n}\right\|_{p,q}^p &= \sup_{a\in D}\int_D |\varphi^{n-1}|^p\varphi'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_a(z)|^2)^q\mathrm{d}m(z)\\ &= \sup_{a\in D}\int_D \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^q|\varphi(z)|^{(n-1)p}|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\ &< \varepsilon.\end{aligned}$$

From Lemma 5, for any given $\varepsilon > 0$, there is an $N > 0$, such that if $n \geq N$, then

$$\int_{S(I)} |\varphi(z)|^{pn-p}|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) < C\varepsilon|I|^q$$

for all I. For any δ, $0 < \delta < 1$, we have

$$\begin{aligned}&\delta^{pN-p}\int_{S(I)} \chi_{D_\delta}(z)|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\ &\leq \int_{S(I)} \chi_{D_\delta}(z)|\varphi(z)|^{pN-p}|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\ &< C\varepsilon|I|^q\end{aligned}$$

for all I, since $|\varphi(z)| > \delta$ on D_δ. Choosing δ so that $\delta^{pN-p} = C$, we obtain

$$\int_{S(I)} \chi_{D_\delta}(z)|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) < \varepsilon|I|^q$$

for all I.

To prove that $C_\varphi : \mathcal{B}^0 \to E(p,q)$ is compact, for any bounded sequence $\{f_n\}$ in $\mathcal{B}^0$ with $f_n \to 0$ uniformly on compact subsets of D, we must have that

$$\|C_\varphi(f_n)\|_{p,q} \to 0 \quad \text{as} \quad n \to \infty.$$

Given any $\varepsilon > 0$, there is a δ, $0 < \delta < 1$, such that (1) holds. Since $\varphi(D - D_\delta)$ is a relatively compact subset of D, $\{f_n\}$ and $\{f_n'\}$ converge uniformly to 0 on $\varphi(D - D_\delta)$, and there exists an N_1 such that for all $n \geq N_1$ and for all arcs I on ∂D

$$\frac{1}{|I|^q}\int_{S(I)} \chi_{D-D_\delta}(z)|f_n'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) \leq \|\varphi\|_{p,q}^p\varepsilon$$

and

$$\frac{1}{|I|^q}\int_{S(I)} \chi_{D_\delta}(z)|f_n'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) \leq \|f_n\|_{\mathcal{B}^0}^p\varepsilon.$$

Since $\{f_n\}$ is a bounded sequence in $\mathcal{B}^0$, there is a $C > 0$ such that $\|f_n\|_{\mathcal{B}^0} \leq C$. Thus, we get

$$\int_{S(I)} |f_n'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) \leq C\varepsilon|I|^q$$

for all I on ∂D. Thus for all a, $\delta < |a| < 1$,

$$\begin{aligned}&\int_D |(f_n\circ\varphi)'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q\mathrm{d}m(z)\\ &= \int_D |(f_n\circ\varphi)'(z)|^p(1-|z|^2)^{p-2}\Big(\frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}\Big)^q\mathrm{d}m(z)\\ &= \int_D \Big(\frac{1-|a|^2}{|1-\overline{a}z|^2}\Big)^q|(f_n\circ\varphi)'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) < C\varepsilon.\end{aligned}$$

Therefore, for all $n \geq N_1$

$$\sup_{\delta<|a|<1}\int_D |(f_n\circ\varphi)'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q\mathrm{d}m(z) < C\varepsilon.$$

$\forall a \in D$, $t \in (0,1)$ and $D_t = \{z \in D : |\varphi(z)| > t\}$, set

$$I_t(a) = \int_{D_t} |(f_n \circ \varphi)'(z)|^p (1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z),$$

then $I_t(a)$ is a continuous function of a. Since $f_n \circ \varphi \in E(p,q)$, $\lim_{t\to 1} I_t(a) = 0$. For a fixed $a \in D$, there is a t_a such that $I_{t_a}(a) < \varepsilon$. Using continuousness of $I_t(a)$, there is a neighborhood $U(a) \subset D$ of a such that $I_{t_a}(b) < \varepsilon$ for all $b \in U(a)$. Since $\{a \in D : |a| \le \delta\}$ is compact, there exists a $t_0 \in (0,1)$ such that when $|a| \le \delta$, $I_{t_0}(a) < \varepsilon$, that is,

$$\sup_{|a|\le\delta} \int_{D_{t_0}} |(f_n \circ \varphi)'(z)|^p (1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z) < \varepsilon.$$

Since $\{f_n' \circ \varphi\}$ converges uniformly to 0 on compact subsets $D - D_{t_0}$ of D, there exists an N_2 such that for all $n > N_2$

$$\begin{aligned}
&\sup_{|a|\le\delta} \int_{D-D_{t_0}} |f_n'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z) \\
&\le \varepsilon \sup_{|a|\le\delta} \int_D |\varphi'(z)|^p (1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z) \\
&\le \varepsilon \|\varphi\|_{p,q}^p.
\end{aligned}$$

So

$$\sup_{|a|\le\delta} \int_D |(f_n \circ \varphi)'(z)|^p (1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z) < C\varepsilon.$$

Therefore, there exists an N, for all $n > N$

$$\begin{aligned}
&\|C_\varphi(f_n)\|_{p,q}^p \\
&\quad = \sup_{a\in D} \int_D |(f_n \circ \varphi)'(z)|^p (1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z) + \\
&\quad = \sup_{|a|\le\delta} \int_D |(f_n \circ \varphi)'(z)|^p (1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z) + \\
&\qquad \sup_{\delta<|a|<1} \int_D |(f_n \circ \varphi)'(z)|^p (1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q \mathrm{d}m(z) \\
&\quad < C\varepsilon.
\end{aligned}$$

Thus,

$$\|C_\varphi(f_n)\|_{p,q} \to 0 \quad \text{as} \quad n \to \infty.$$

The proof of Theorem 2 is completed.

To give another result, we need the following notations[8]. Let $\theta \in [0, 2\pi)$, $h \in (0,1)$ and $S(h,\theta) = \{z \in D : 1-h \le |z| < 1, |\theta - \arg z| \le h\}$ be the Carleson box at $e^{i\theta}$. It is easy to see that that the measure μ defined on D is a bounded q-Carleson measure is equivalent to $\sup_{\theta\in[0,2\pi),h\in(0,1)} \frac{\mu(S(h,\theta))}{h^q} < \infty$, and that the measure μ defined on D is a compact q-Carleson measure is equivalent $\lim_{h\to 0} \frac{\mu(S(h,\theta))}{h^q} = 0$ uniformly on $\theta \in [0, 2\pi)$.

Now we establish the third result of this section.

Theorem 3 *Suppose $p>0, q>0$, $p+q>1$ and φ is an analytic self-map of D. Then the following statements are equivalent:*

(1) $\varphi \in E_0(p,q)$;

(2) The composition operator $C_\varphi : \mathcal{B}^0 \to E_0(p,q)$ is bounded;

(3) The composition operator $C_\varphi : \mathcal{B}^0 \to E_0(p,q)$ is compact.

Proof (1)$\to$ (2). If $\varphi \in E_0(p,q)$, by Theorem 1, the composition operator $C_\varphi : \mathcal{B}^0 \to E(p,q)$ is bounded. So it is enough to show that $C_\varphi(\mathcal{B}^0) \subset E_0(p,q)$. Since $\varphi \in E_0(p,q)$, by Lamma 2, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $f \in \mathcal{B}^0$

$$\begin{aligned}&\int_{S(I)} |(f\circ\varphi)'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\ &= \int_{S(I)} |f'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\ &\le \|f\|^p_{\mathcal{B}^0}\int_{S(I)} |\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\ &= \|f\|^p_{\mathcal{B}^0}\int_{S(I)} d\mu_{\varphi,p,q} < C\varepsilon|I|^q,\end{aligned}$$

provided $|I| < \delta$. So by Lemma 2, we get $f\circ\varphi \in E_0(p,q)$.

(2)$\to$ (1) is obvious.

(3)$\to$ (2) is easy.

Finally, we show that (1)$\to$ (3). Suppose that $\varphi \in E_0(p,q)$. To prove that the composition operator $C_\varphi : \mathcal{B}^0 \to E_0(p,q)$ is compact, we need to show that $C_\varphi(\mathcal{B}^0) \subset E_0(p,q)$ and the composition operator $C_\varphi : \mathcal{B}^0 \to E(p,q)$ is compact. The first inclusion is obvious since (1) implies (2). Now we prove compactness of C_φ. For any bounded sequence $\{f_n\}$ in $\mathcal{B}^0$ with $f_n \to 0$ uniformly on compact subsets of D, we must prove

$$\|C_\varphi(f_n)\|^p_{p,q} \to 0 \quad \text{as} \quad n \to \infty.$$

Since $\varphi \in E_0(p,q)$, from Lemma 2 for $\varepsilon > 0$, there is a δ, $0<\delta<1$, such that for $h<\delta$ and $\theta \in [0, 2\pi)$

$$\int_{S(h,\theta)} |\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) \le \varepsilon h^q. \tag{2}$$

For h, $h<\delta$, $\theta\in[0,2\pi)$ and $\|f_n\|_{\mathcal{B}^0} \le C$, we have

$$\begin{aligned}&\int_{S(h,\theta)} |(f_n\circ\varphi)'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\ &= \int_{S(h,\theta)} |f_n'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\ &\le \|f_n\|_{\mathcal{B}^0}\int_{S(h,\theta)} |\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) \le C\varepsilon h^q.\end{aligned}$$

For h, $h \ge \delta$, $\theta \in [0,2\pi)$, choose $h_0 < \delta$ and a positive integer m such that $2(m-1)h_0 < 2\pi \le 2mh_0$, so $m < \frac{\pi}{h_0} + 1$, to set

$$K = \{z \in D : 1-h \le |z| \le 1-h_0\},$$

and $\theta_j = 2(j-1)h_0$ $(j=1,2,\cdots,m)$. $\forall z \in S(h,\theta)$, if $z \overline{\in} K$, $\exists j \in \{1,2,\cdots,m\}$, such that $|\arg z - \theta_j| \le h_0$, and $1-h_0 < |z|$, so $z \in S(h_0,\theta_j)$. Consequently, there exist $\theta_1,\theta_2,\cdots,\theta_m \in [0,2\pi)$ and a compact subset K of D such that

$$S(h,\theta) \subset K \cup (\bigcup_{j=1}^{m} S(h_0,\theta_j)).$$

Since $\{f_n'\}$ converges uniformly to 0 on a compact subset K of D, then there exists an $N>0$, such that for all $n \ge N$ and $h \in (0,1)$

$$\int_K |(f_n\circ\varphi)'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) \le \varepsilon \int_K |\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) \le \varepsilon C h^q. \tag{3}$$

For $S(h_0,\theta_j)$, $j=1,2,\cdots,m$, using (2) we have

$$\int_{S(h_0,\theta_j)} |\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) \le \varepsilon h_0^q. \tag{4}$$

From (3) and (4), we have for $h \ge \delta$ and $n \ge N$

$$\begin{aligned}
&\int_{S(h,\theta)} |(f_n\circ\varphi)'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\
&= \int_{S(h,\theta)} |f_n'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\
&\le (\int_K + \int_{\bigcup_{j=1}^{m} S(h_0,\theta_j)})|(f_n\circ\varphi)'(z))|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\
&\le C\varepsilon h^q + \sum_{j=1}^{m}\int_{S(h_0,\theta_j)} |f_n'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\
&\le C\varepsilon h^q + C\varepsilon\sum_{j=1}^{m} h_0^q \le C\varepsilon h^q.
\end{aligned} \tag{5}$$

Combining (2) and (5), we get for all $n>N$, $h\in(0,1)$ and $\theta\in[0,2\pi)$ that

$$\int_{S(h,\theta)} |(f_n\circ\varphi)'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) \le C\varepsilon h^q.$$

Hence, we obtain

$$\int_{S(I)} |f_n'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z) \le C\varepsilon |I|^q$$

for all I on ∂D. As in the proof of Theorem 2, we have

$$\|C_\varphi(f_n)\|_{p,q} \to 0 \ \text{ as } \ n\to\infty.$$

The proof is finished.

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从 $\mathcal{B}^0$ 到 $E(p,q)$ 和 $E_0(p,q)$ 空间的复合算子

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摘要: 设 φ 是单位园盘 D 到自身的解析映射， X 是 D 上解析函数的 Banach 空间，对 $f \in X$，定义复合算子 C_φ : $C_\varphi(f) = f \circ \varphi$. 我们利用从 $\mathcal{B}^0$ 到 $E(p,q)$ 和 $E_0(p,q)$ 空间的复合算子研究了空间 $E(p,q)$ 和 $E_0(p,q)$, 给出了一个新的特征.

关键词: 有界算子；紧算子；复合算子；解析函数； q-Carleson 测度.