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Multiple Positive Solutions to a Nonlinear Two-point Boundary Value Problem with p-Laplacian

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Abstract: By a simple application of a new three functionals fixed point theorem, sufficient conditions are obtained to guarantee the existence of at least three positive solutions for p-Laplacian equation: $(\varphi_p(u'))' + a(t)f(t,u(t)) = 0$ subject to nonlinear boundary value conditions. An example is presented to illustrate the theory.

Key words: boundary value problem; p-Laplacian operator; positive solution; the three functionals fixed point theorem.
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1. Introduction

This paper deals with the p-Laplacian equation

$$(\varphi_p(u'))' + a(t)f(t,u(t)) = 0, \quad t \in (0,1) \tag{1}$$

subject to the following nonlinear boundary conditions

$$u(0) - B_0(u'(0)) = 0, \quad u(1) + B_1(u'(1)) = 0 \tag{2}$$

where $\varphi_p(x) = |x|^{p-2}x, p > 1$.

In [1], using the three functionals fixed point theorem due to Avery and Henderson[4], HE Xiao-ming and GE Wei-gao obtained at least two positive solutions of (1)(2) when $f(t,u) = f(u)$; In [2], the authors yielded at least triple positive solutions to BVP (1)(2) by applying the five functionals fixed point theorem on cone; In [3], by Leggett-Williams fixed point theorem, the authors obtained three positive solutions of three-point BVPs when $p = 2$. But the conditions for the methods and techniques mentioned above are difficult to check. This paper will apply a new three functionals three fixed points theorem proved in literature[5] to study the existence of three positive solutions of BVP(1)(2). Our results are new and different from those in [1–3] and very easy to check.

The following conditions are satisfied throughout this paper.

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(H_1) $f:\ [0,1]\times[0,\infty)\to[0,\infty)$ is continuous;

(H_2) $B_0(v)$ and $B_1(v)$ are both nondecreasing continuous odd functions defined on $(-\infty,+\infty)$, and satisfy that there are nonnegative numbers l and L such that

$$lv\le B_i(v)\le Lv,\ v\ge 0,\ i=0,1;$$

(H_3) $a(t)$ is a nonnegative measurable function defined on (0,1), and $a(t)$ is not identical zero on any compact subinterval of (0,1). Furthermore, $a(t)$ satisfies

$$0<\int_0^1 a(t)\mathrm{d}t<+\infty.$$

2. Some definitions and lemmas

In this section,we provide some background definitions cited from cone theory in Banach spaces.

Definition 2.1 *Let $(E,\|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P\subset E$ is said to be a cone provided the following are satisfied:*

(i) If $y\in P$ and $\lambda\ge 0$, then $\lambda y\in P$;

(ii) If $y\in P$ and $-y\in P$, then $y=0$.

If $P\subset E$ is a cone, we denote the order induced by P on E by $\le$, that is, $x\le y$ if and only if $y-x\in P$.

Definition 2.2 *Given a cone P in a real Banach space E, a functional $\psi:P\to R$ is said to be increasing on P, provided $\psi(x)\le\psi(y)$ for all $x,y\in P$ with $x\le y$.*

Definition 2.3 *Given a nonnegative continuous functional γ on a cone P of E, we define for each $d>0$ the set*

$$P(\gamma,d)=\{x\in P:\gamma(x)<d\}.$$

The following fixed point theorem is fundamental and important to the proofs of our main results.

Lemma 2.1[5] *Let P be a cone in a Banach space E. Let α,β and γ be three increasing, nonnegative and continuous functionals on P, satisfying for some $c>0$ and $M>0$ such that*

$$\gamma(x)\le\beta(x)\le\alpha(x),\quad ||x||\le M\gamma(x)$$

for all $x\in\overline{P(\gamma,c)}$. Suppose there exists a completely continuous operator $T:\overline{P(\gamma,c)}\to P$ and $0<a<b<c$ such that

(i) $\gamma(Tx)<c$, for all $x\in\partial P(\gamma,c)$;

(ii) $\beta(Tx)>b$, for all $x\in\partial P(\beta,b)$;

(iii) $P(\alpha, a) \neq \emptyset$, *and* $\alpha(Tx) < a$, *for all* $x \in \partial P(\alpha, a)$.
Then T *has at least three fixed points* $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ *such that*

$$0 \leq \alpha(x_1) < a < \alpha(x_2),\ \ \beta(x_2) < b < \beta(x_3), \gamma(x_3) < c.$$

3. Main results

Let the Banach space $E = C([0,1])$ be endowed the norm $||x|| = \max_{t\in[0,1]} |x(t)|$. And choose the cone $P \subset E$ defined by

$$P = \{x \in E : x(t) \ \text{ is nonnegative concave on } [0,1]\}.$$

It follows from (H_2) that there exists $\delta \in (0, \frac{1}{2})$ such that

$$0 < \int_\delta^{1-\delta} a(t)\mathrm{d}t < +\infty \tag{3}$$

and hence the function

$$y(x) := \varphi_q(\int_\delta^x a(t)\mathrm{d}t) + \varphi_q(\int_x^{1-\delta} a(t)\mathrm{d}t), \quad \delta \leq x \leq 1-\delta$$

is continuous and positive on $[\delta, 1-\delta]$, where $\varphi_q(x) := |x|^{1/(p-1)}\mathrm{sgn}x$.

We define the following nonnegative, increasing and continuous functionals

$$\gamma(u) = \frac{1}{2}(u(\delta) + u(1-\delta)),$$

$$\beta(u) = \max_{\delta\leq t\leq 1-\delta} u(t),$$

$$\alpha(u) = \max_{0\leq t\leq 1} u(t).$$

Obviously, for every $u \in P$, we have $\gamma(u) \leq \beta(u) \leq \alpha(u)$.

Lemma 3.1[1] *Let* $u \in P$ *and* $\delta \in (0, 1/2)$, *then* $u(t) \geq \delta\|u\|$, *for all* $t \in [\delta, 1-\delta]$.

From Lemma 3.1 and the definition of $\gamma(u)$, one has $\gamma(u) \geq \delta\|u\|$. Thus

$$\|u\| \leq \frac{1}{\delta}\gamma(u), \quad \text{for all } u \in P.$$

We shall use the following notations:

$$K = \min_{\delta\leq x\leq 1-\delta} y(x),$$

$$\eta = \max\{(L+1-\delta)\varphi_q(\int_0^\delta a(r)\mathrm{d}r), (L+1-\delta)\varphi_q(\int_{1-\delta}^1 a(r)\mathrm{d}r)\},$$

$$\xi = \min\left\{\begin{array}{l} l\varphi_q(\int_\delta^1 a(r)\mathrm{d}r) + \delta\varphi_q(\int_\delta^{1-\delta} a(r)\mathrm{d}r), \\ l\varphi_q(\int_0^{1-\delta} a(r)\mathrm{d}r) + \delta\varphi_q(\int_\delta^{1-\delta} a(r)\mathrm{d}r), \\ \frac{l[\varphi_q(\int_0^\delta a(r)\mathrm{d}r) + \varphi_q(\int_{1-\delta}^1 a(r)\mathrm{d}r)] + K\delta}{2} \end{array}\right\},$$

$$\lambda = (L+1-\delta)\varphi_q(\int_0^1 a(r)\mathrm{d}r).$$

Now we give the main result of this paper.

Theorem 3.1 *Assume that (H$_1$)–(H$_3$) hold, and that there exist positive constants* $0 < a < b < \frac{\xi a}{\eta} < \delta^2 c$ *such that*

(H$_4$) $f(t,\omega) < \varphi_p(\frac{c}{\lambda})$, *if* $\delta \le t \le 1-\delta$, $\delta c \le \omega \le \frac{c}{\delta}$;

(H$_5$) $f(t,\omega) > \varphi_p(\frac{b}{\xi})$, *if* $0 \le t \le 1, 0 \le \omega \le \frac{b}{\delta}$;

(H$_6$) $f(t,\omega) < \varphi_p(\frac{a}{\eta})$, *if* $0 \le t \le 1, 0 \le \omega \le a$.

Then, the BVP (1)+(2) has at least three positive solutions u_1, u_2 *and* u_3 *such that*

$$0 \le \alpha(u_1) < a < \alpha(u_2),\ \beta(u_2) < b < \beta(u_3), \gamma(u_3) < c.$$

Proof We define an operator $T : P(\gamma, c) \to E$ by

$$(Tu)(t) = \begin{cases} B_0 \circ \varphi_q(\int_0^\sigma a(r)f(r,u(r))\mathrm{d}r) + \int_0^t \varphi_q(\int_s^\sigma a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s, & 0 \le t \le \sigma \\ B_1 \circ \varphi_q(\int_\sigma^1 a(r)f(r,u(r))\mathrm{d}r) + \int_t^1 \varphi_q(\int_\sigma^s a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s, & \sigma \le t \le 1 \end{cases}$$

for each $u \in P$, where $\sigma = 0$ if $(Tu)'(0) = 0$; $\sigma = 1$ if $(Tu)'(1) = 0$; otherwise, σ is a solution of the equation

$$z_0(x) = z_1(x),$$

where

$$z_0(x) = B_0 \circ \varphi_q(\int_0^x a(r)f(r,u(r))\mathrm{d}r) + \int_0^x \varphi_q(\int_s^x a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s,\ 0 \le x < 1,$$

$$z_1(x) = B_1 \circ \varphi_q(\int_x^1 a(r)f(r,u(r))\mathrm{d}r) + \int_x^1 \varphi_q(\int_x^s a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s,\ 0 < x \le 1.$$

It is shown in [6] that σ exists and the operator $T : P(\gamma, c) \to E$ is well defined. In particular, if $u \in \overline{P(\gamma, c)}$, we also have $Tu \in P$. So $T : \overline{P(\gamma, c)} \to P$.

It is easy to prove that $T : \overline{P(\gamma, c)} \to P$ is completely continuous.

We now show that all the conditions of Lemma 2.1 are satisfied. To make use of property (i) of Lemma 2.1, we choose $u \in \partial P(\gamma, c)$, then $\gamma(u) = \frac{1}{2}(u(\delta) + u(1-\delta)) = c$. If we recall that $\|u\| \le \frac{1}{\delta}\gamma(u)$, we have

$$\delta c \le \delta\|u\| \le u(t) \le \frac{c}{\delta},\quad \delta \le t \le 1-\delta.$$

Then assumption (H$_4$) implies

$$f(s, \omega(s)) < \varphi_p(\frac{c}{\lambda}),\quad \delta \le s \le 1-\delta.$$

Therefore,

$$\begin{aligned}\gamma(Tu) =& \frac{1}{2}(Tu(\delta) + Tu(1-\delta)) \le Tu(\delta) \\ =& B_1 \circ \varphi_q(\int_\sigma^1 a(r)f(r,u(r))\mathrm{d}r) + \int_\delta^1 \varphi_q(\int_\sigma^s a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s \\ \le& L\varphi_q(\int_0^1 a(r)f(r,u(r))\mathrm{d}r) + \int_\delta^1 \varphi_q(\int_0^1 a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= (L+1-\delta)\varphi_q(\int_0^1 a(r)f(r,u(r))\mathrm{d}r)\\
&< (L+1-\delta)\varphi_q(\int_0^1 a(r)\mathrm{d}r)\cdot\frac{c}{\lambda}\\
&= c, \quad \text{if} \quad \sigma<\delta,
\end{aligned}
$$

$$
\begin{aligned}
\gamma(Tu) &= \frac{1}{2}(Tu(\delta)+Tu(1-\delta)) \le Tu(1-\delta)\\
&= B_0\circ\varphi_q(\int_0^\sigma a(r)f(r,u(r))\mathrm{d}r) + \int_0^{1-\delta}\varphi_q(\int_s^\sigma a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s\\
&\le L\varphi_q(\int_0^1 a(r)f(r,u(r))\mathrm{d}r) + \int_0^{1-\delta}\varphi_q(\int_0^1 a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s\\
&= (L+1-\delta)\varphi_q(\int_0^1 a(r)f(r,u(r))\mathrm{d}r)\\
&< (L+1-\delta)\varphi_q(\int_0^1 a(r)dr)\cdot\frac{c}{\lambda}\\
&= c, \quad \text{if} \quad \sigma>1-\delta,
\end{aligned}
$$

$$
\begin{aligned}
2\gamma(Tu) =&(Tu(\delta)+Tu(1-\delta))\\
=&B_0\circ\varphi_q(\int_0^\sigma a(r)f(r,u(r))\mathrm{d}r) + \int_0^{\delta}\varphi_q(\int_s^\sigma a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s+\\
&B_1\circ\varphi_q(\int_\sigma^1 a(r)f(r,u(r))\mathrm{d}r) + \int_{1-\delta}^{1}\varphi_q(\int_\sigma^s a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s\\
\le&L\varphi_q(\int_0^1 a(r)f(r,u(r))\mathrm{d}r) + \int_0^{\delta}\varphi_q(\int_0^1 a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s+\\
&L\varphi_q(\int_0^1 a(r)f(r,u(r))\mathrm{d}r) + \int_{1-\delta}^{1}\varphi_q(\int_0^1 a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s\\
<&(2L+2\delta)\varphi_q(\int_0^1 a(r)\mathrm{d}r)\cdot\frac{c}{\lambda}\\
<&(2L+2-2\delta)\varphi_q(\int_0^1 a(r)\mathrm{d}r)\cdot\frac{c}{\lambda}\\
=&2c, \quad \text{if} \quad \delta\le\sigma\le 1-\delta.
\end{aligned}
$$

Hence, condition (i) is satisfied .

Secondly, we show that (ii) of Lemma 2.1 is fulfilled. For this, we select $u\in\partial P(\beta,b)$. Then $\beta(u)=\max\limits_{\delta\le t\le 1-\delta}u(t)=b$. Noticing that $\|u\|\le\frac{1}{\delta}\gamma(u)\le\frac{1}{\delta}\beta(u)=\frac{b}{\delta}$, we have

$$0\le u(t)\le\frac{b}{\delta}, \quad 0\le t\le 1.$$

By (H_5), we have

$$f(s,\omega(s))>\varphi_p(\frac{b}{\xi}), \quad 0\le s\le 1$$

and so

$$\begin{aligned}
\beta(Tu) = &\max_{\delta\le t\le 1-\delta} Tu(t) \ge Tu(1-\delta)\\
&= B_1\circ\varphi_q(\int_\sigma^1 a(r)f(r,u(r))\mathrm{d}r) + \int_{1-\delta}^1 \varphi_q(\int_\sigma^s a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s\\
&\ge l\varphi_q(\int_\delta^1 a(r)f(r,u(r))\mathrm{d}r) + \int_{1-\delta}^1 \varphi_q(\int_\delta^{1-\delta} a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s\\
&> [l\varphi_q(\int_\delta^1 a(r)\mathrm{d}r) + \delta\varphi_q(\int_\delta^{1-\delta} a(r)\mathrm{d}r)]\cdot\frac{b}{\xi}\\
&\ge \xi\cdot\frac{b}{\xi}\\
&= b,\quad \text{if}\quad \sigma<\delta,
\end{aligned}$$

$$\begin{aligned}
\beta(Tu) = &\max_{\delta\le t\le 1-\delta} Tu(t) \ge Tu(\delta)\\
&= B_0\circ\varphi_q(\int_0^\sigma a(r)f(r,u(r))\mathrm{d}r) + \int_0^\delta \varphi_q(\int_s^\sigma a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s\\
&\ge l\varphi_q(\int_0^{1-\delta} a(r)f(r,u(r))\mathrm{d}r) + \int_0^\delta \varphi_q(\int_\delta^{1-\delta} a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s\\
&> [l\varphi_q(\int_0^{1-\delta} a(r)\mathrm{d}r) + \delta\varphi_q(\int_\delta^{1-\delta} a(r)\mathrm{d}r)]\cdot\frac{b}{\xi}\\
&\ge \xi\cdot\frac{b}{\xi}\\
&= b,\quad \text{if}\quad \sigma>1-\delta,
\end{aligned}$$

$$\begin{aligned}
2\beta(Tu) = &2\max_{\delta\le t\le 1-\delta} Tu(t) \ge Tu(\delta)+Tu(1-\delta)\\
&= B_0\circ\varphi_q(\int_0^\sigma a(r)f(r,u(r))\mathrm{d}r) + \int_0^\delta \varphi_q(\int_s^\sigma a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s\\
&\quad B_1\circ\varphi_q(\int_\sigma^1 a(r)f(r,u(r))\mathrm{d}r) + \int_{1-\delta}^1 \varphi_q(\int_\sigma^s a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s+\\
&\ge l\varphi_q(\int_0^\delta a(r)f(r,u(r))\mathrm{d}r) + \int_0^\delta \varphi_q(\int_\delta^\sigma a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s+\\
&\quad 1\varphi_q(\int_{1-\delta}^1 a(r)f(r,u(r))\mathrm{d}r) + \int_{1-\delta}^1 \varphi_q(\int_\sigma^{1-\delta} a(r)f(r,u(r))\mathrm{d}r)\mathrm{d}s\\
&> \{l[\varphi_q(\int_0^\delta a(r)\mathrm{d}r) + \varphi_q(\int_{1-\delta}^1 a(r)\mathrm{d}r)] + \delta[\varphi_q(\int_\delta^\sigma a(r)\mathrm{d}r) + \varphi_q(\int_\sigma^{1-\delta} a(r)\mathrm{d}r)]\}\cdot\frac{b}{\xi}\\
&\ge [l\varphi_q(\int_0^\delta a(r)\mathrm{d}r) + l\varphi_q(\int_{1-\delta}^1 a(r)\mathrm{d}r)] + \delta K]\cdot\frac{b}{\xi}\\
&\ge 2\xi\cdot\frac{b}{\xi}\\
&= 2b,\quad \text{if}\quad \delta\le\sigma\le 1-\delta.
\end{aligned}$$

Hence, condition (ii) is satisfied.

Finally, we verify that (iii) of Lemma 2.1 is also satisfied. We note that $u(t) \equiv \frac{a}{4}, 0 \le t \le 1$, is a member of $P(\alpha, a)$ and $\alpha(u) = \frac{a}{4} < a$. So $P(\alpha, a) \neq \emptyset$. Now , let $u \in \partial P(\alpha, a)$, then $\alpha(u) = \max_{0 \le t \le 1} u(t) = a$. This means that

$$0 \le u(t) \le a, \quad 0 \le t \le 1.$$

From assumption (H_6), we have

$$f(s, \omega(s)) < \varphi_p(\frac{a}{\eta}), \quad 0 \le s \le 1.$$

As before, we get

$$\begin{aligned}
\alpha(Tu) = \max_{0 \le t \le 1} Tu(t) = Tu(\sigma) \\
&= B_0 \circ \varphi_q(\int_0^\sigma a(r) f(r, u(r))\mathrm{d}r) + \int_0^\sigma \varphi_q(\int_s^\sigma a(r) f(r, u(r))\mathrm{d}r)\mathrm{d}s \\
&\le L\varphi_q(\int_0^\sigma a(r) f(r, u(r))\mathrm{d}r) + \int_0^\delta \varphi_q(\int_s^\delta a(r) f(r, u(r))\mathrm{d}r)\mathrm{d}s \\
&< (L+\delta)\varphi_q(\int_0^\delta a(r)\mathrm{d}r) \cdot \frac{a}{\eta} \\
&< (L+1-\delta)\varphi_q(\int_0^\delta a(r)\mathrm{d}r) \cdot \frac{a}{\eta} \\
&\le a, \quad \text{if} \quad \sigma < \delta,
\end{aligned}$$

$$\begin{aligned}
\alpha(Tu) = Tu(\sigma) \\
&= B_1 \circ \varphi_q(\int_\sigma^1 a(r) f(r, u(r))\mathrm{d}r) + \int_\sigma^1 \varphi_q(\int_\sigma^s a(r) f(r, u(r))\mathrm{d}r)\mathrm{d}s \\
&\le L\varphi_q(\int_{1-\delta}^1 a(r) f(r, u(r))\mathrm{d}r) + \int_{1-\delta}^1 \varphi_q(\int_{1-\delta}^s a(r) f(r, u(r))\mathrm{d}r)\mathrm{d}s \\
&= (L+\delta)\varphi_q(\int_{1-\delta}^1 a(r) f(r, u(r))\mathrm{d}r) \\
&< (L+\delta)\varphi_q(\int_{1-\delta}^1 a(r)\mathrm{d}r) \cdot \frac{a}{\eta} \\
&< (L+1-\delta)\varphi_q(\int_0^\delta a(r)\mathrm{d}r) \cdot \frac{a}{\eta} \\
&\le a, \quad \text{if} \quad \sigma > 1-\delta,
\end{aligned}$$

$$\begin{aligned}
2\alpha(Tu) =& 2Tu(\sigma) \\
&= B_0 \circ \varphi_q(\int_0^\sigma a(r) f(r, u(r))\mathrm{d}r) + \int_0^\sigma \varphi_q(\int_s^\sigma a(r) f(r, u(r))\mathrm{d}r)\mathrm{d}s + \\
&\quad B_1 \circ \varphi_q(\int_\sigma^1 a(r) f(r, u(r))\mathrm{d}r) + \int_\sigma^1 \varphi_q(\int_\sigma^s a(r) f(r, u(r))\mathrm{d}r)\mathrm{d}s \\
&\le L\varphi_q(\int_0^{1-\delta} a(r) f(r, u(r))\mathrm{d}r) + \int_o^{1-\delta} \varphi_q(\int_0^{1-\delta} a(r) f(r, u(r))\mathrm{d}r)\mathrm{d}s + \\
&\quad L\varphi_q(\int_\delta^1 a(r) f(r, u(r))\mathrm{d}r) + \int_\delta^1 \varphi_q(\int_\delta^s a(r) f(r, u(r))\mathrm{d}r)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&=(L+1-\delta)[\varphi_q(\int_0^{1-\delta} a(r)f(r,u(r))\mathrm{d}r)+\varphi_q(\int_\delta^1 a(r)f(r,u(r))\mathrm{d}r)] \\
&<(L+1-\delta)[\varphi_q(\int_0^{1-\delta} a(r)\mathrm{d}r)+\varphi_q(\int_\delta^1 a(r)\mathrm{d}r)]\cdot\frac{a}{\eta} \\
&\le 2a, \quad \text{if} \quad \sigma\in[\delta,1-\delta].
\end{aligned}
$$

Therefore, BVP(1)(2) has at least three positive solutions u_1, u_2 and u_3 such that

$$0\le\alpha(u_1)<a<\alpha(u_2), \quad \beta(u_2)<b<\beta(u_3), \quad \gamma(u_3)<c.$$

Remark If we add the condition of $a(t)f(t,u)\not\equiv 0$, $t\in[0,1]$, to Theorem 3.1, we can get positive solutions u_1, u_2 and u_3 such that

$$0<\alpha(u_1)<a<\alpha(u_2), \quad \beta(u_2)<b<\beta(u_3), \gamma(u_3)<c.$$

4. An example

In this section, we present a simple example to explain our results.

Consider the boundary value problem

$$(\varphi_{\frac{3}{2}}(u'))'+a(t)f(t,u)=0, \tag{4}$$

$$u(0)-B_0(u'(0))=0, \quad u(1)+B_1(u'(1))=0, \tag{5}$$

where

$$f(t,u)=\begin{cases} 60, & 0\le t\le 1,\ 0\le u\le 5, \\ u+55, & 0\le t\le 1,\ 5\le u\le 36, \\ 91, & 0\le t\le 1,\ 36\le u\le 10^4, \\ 91+\frac{u-10^4}{\sqrt{u}}, & 0\le t\le 1,\ u\ge 10^4. \end{cases}$$

$$a(t)=\begin{cases} t, & 0\le t\le\frac{1}{2} \\ 1-t, & \frac{1}{2}\le t\le 1 \end{cases}$$

In the example, we notice that $B_0(v)=B_1(v)=\frac{1}{2}v,\ l=L=\frac{1}{2},\ p=\frac{3}{2}$, and $q=3$. Choose $\delta=\frac{1}{4}$. It follows from a direct calculation that

$$K=\min_{\frac{1}{4}\le x\le 1-\frac{1}{4}} y(x)=\frac{18}{32^2},$$

$$\eta=\max\{(\frac{1}{2}+1-\frac{1}{4})\varphi_3(\int_0^{\frac{1}{4}} r\mathrm{d}r),(\frac{1}{2}+1-\frac{1}{4})\varphi_3(\int_{\frac{3}{4}}^1(1-r)\mathrm{d}r)\}=\frac{5}{4}\times(\frac{1}{32})^2$$

$$\xi=\min\{\frac{67}{2}\times(\frac{1}{32})^2,\ \frac{67}{2}\times(\frac{1}{32})^2,\ \frac{11}{4}\times(\frac{1}{32})^2\}=\frac{11}{4}\times(\frac{1}{32})^2,$$

$$\lambda=(\frac{1}{2}+1-\frac{1}{4})\varphi_3(\int_0^1 a(r)\mathrm{d}r)=\frac{5}{64}.$$

Here we choose $a=5,\ b=9,\ c=2500$, then we get

$$f(t,u)=60<64=\varphi_{\frac{3}{2}}(4\times 32^2)=\varphi_{\frac{3}{2}}(\frac{a}{\eta}), \quad \frac{1}{4}\le t\le 1-\frac{1}{4},\ 0\le u\le 5,$$

$$f(t,u) \geq 60 > \frac{192}{\sqrt{11}} = \varphi_{\frac{3}{2}}(\frac{9}{\frac{11}{4}\times(\frac{1}{32})^2}) = \varphi_{\frac{3}{2}}(\frac{b}{\xi}),\ 0 \leq t \leq 1,\ 0 \leq u \leq 36,$$

$$f(t,u) = 91 < 80\sqrt{5} = \varphi_{\frac{3}{2}}(\frac{2500}{\frac{5}{64}}) = \varphi_{\frac{3}{2}}(\frac{c}{\lambda}),\quad 0 \leq t \leq 1,\ 625 \leq u \leq 10^4.$$

Then all the conditions of Lemma 2.1 are satisfied. So, by Theorem 3.1, we know BVP(4)(5) has at least three positive solutions.

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p-Laplace 非线性两点边值问题多个正解的存在性

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摘要: 本文利用一种新的三个泛函不动点定理得到了 p-Laplacian 方程在具有非线性边值条件时至少存在三个正解的充分条件, 并且举了一个简单例子来说明得到的结论.

关键词: 边值问题; p-Laplacian 算子; 正解; 三个泛函不动点定理.