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A Conceivable Inequality for Analytic Functions and Its Application

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Abstract: This brief note presents an easily conceivable inequality for analytic functions. As an application, a function-theoretic proposition involving the Fundamental Theorem of Algebra (FTA) is deduced immediately.

Key words: conceivable inequality; analytic function; Fundamental Theorem of Algebra.
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Without loss of generality, a non-constant analytic function $f(z)$ defined for $|z| < \rho$ in the Gaussian plane $\mathbf{C}$ may be written in the form

$$f(z) = a + bz^m + cz^{m+1} + \cdots, \tag{1}$$

where $b \neq 0$ and $m \geq 1$.

Proposition 1 *Let $f(t)$ be defined by (1) with $f(0) = a \neq 0$, then for every sufficiently small $\delta > 0$ there holds the inequality*

$$|f((-a\delta/b)^{1/m})| < |f(0)|, \tag{2}$$

where $(-a\delta/b)^{1/m}$ may take any of the m distinct roots for $m \geq 2$.

Proof Solving the equation $bz^m = -a\delta$, we get $z = (-a\delta/b)^{1/m}$. Rewrite $f(z)$ in the form

$$f(z) = a + bz^m + bz^m g(z), \tag{1*}$$

where $g(z) = (c/b)z + \cdots$ is also an analytic function so that $|g(z)| \to 0$ as $|z| \to 0$. Thus for every sufficiently small δ with $0 < \delta < 1$, we could have

$$|g((-a\delta/b)^{1/m})| < \frac{1}{2}.$$

Consequently, the following estimation holds via (1)*

$$\begin{aligned}|f((-a\delta/b)^{1/m})| &\leq |a - a\delta| + |(-a\delta)g((-a\delta/b)^{1/m})| < |a|(1-\delta) + |a|\delta \cdot \frac{1}{2}\\ &= |a|(1 - \frac{\delta}{2}) < |a| = |f(0)|.\end{aligned}$$

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Hence the Inequality (2) is always valid for small $\delta > 0$. □

Evidently, Inequality (2) may be stated in a more general form: If $f(z)$ is non-constant and analytic in a neighborhood of $z_0 \in \mathbf{C}$, with $f(z_0) = a \neq 0$, then $f(z_0 + z)$ may be written in a similar form, as that of (1),

$$f(z_0 + z) = f(z_0) + bz^m + cz^{m+1} + \cdots, \quad (m \geq 1), \tag{3}$$

and consequently the following inequality

$$|f(z_0 + (-a\delta/b)^{1/m})| < |f(z_0)| \tag{4}$$

holds for every sufficiently small $\delta (0 < \delta < 1)$.

In what follows suppose that $F(z)$ is non-constant and analytic in a domain $D \subset \mathbf{C}$. Then the form of Inequality (4) shows that for every $z_0 \in D$ with $F(z_0) \neq 0$, the value $|F(z_0)| > 0$ can never become an absolute minimum $\min_z |F(z)|$. This implies that if $\min_z |F(z)|$ really exists and is attained at $z = z_0 \in D$, then it must be that $\min_z |F(z)| = |F(z_0)| = 0$. Accordingly, we get the following useful and well-known proposition as a consequence of (4).

Proposition 2 *Let F(z)be a non-constant analytic function in $D \subset C$, and let $|F(z)|$ attain an absolute minimum at $z_0 \in D$. Then it must be that*

$$\min_z |F(z)| = |F(z_0)| = 0.$$

In words, z_0 must be a zero of $F(z)$.

In particular, if $F(z)$ is a polynomial in z of degree $n (n \geq 1)$, then the obvious fact that $|F(z)| \to \infty (|z| \to \infty)$ implies that $|F(z)|$ should attain $\min_z |F(z)|$ at certain $z_0 \in \mathbf{C}$. Thus by Proposition 2 we must have $F(z_0) = 0$. This is what so-called the well-known existence theorem first proved by Gauss (1799):

FTA *Any polynomial equation $F(z) = 0$ of degree $n (n \geq 1)$ has at least a root $z_0 \in \mathbf{C}$, viz.* $F(z_0) = 0$.

Note that for the example $F(z) = e^z$ we have $|e^z| = |e^{x+iy}| = e^x > 0$ for $z = x + iy \in \mathbf{C}$. This shows that the second condition in Proposition 2 cannot be omitted.

Remark 1 Recall that in the complex analysis a limit process such as $f(z) \to A$ $(z \to a \in \mathbf{C})$ involves that both z and $f(z)$ could tend to their limits in various possible directions in $\mathbf{C}$. In particular, if $f(z) \to f(a) \neq 0$ $(z \to a)$ and if the mode of passage $z \to a$ could be so chosen that $|f(z)| \uparrow |f(a)|$, then we shall have $|f(z)| < |f(a)|$ for all those z sufficiently close to a. Observing in this way, we see that (2) and (4) are geometrically comprehensible.

Remark 2 For the polynomial equation $F(z) = 0$ of degree $n (\geq 1)$, the truth of (FTA) is just based on the fact that $\min_z |F(z)|$ really exists and cannot take any positive value such as $\min_z |F(z)| = |F(z_0)| > 0$, in view of (4). Looking in this way, we may say that the truth of (4) or (2) is the basic source for the truth of (FTA).

Remark 3 As usual, for the nth degree polynomial $|F(z)|$ we may denote $|F(z)| = |F(x+iy)| = |u(x,y)+iv(x,y)| = \sqrt{u(x,y)^2 + v(x,y)^2}$. Thus the assertion of (FTA), $\min_z |F(z)| = |F(z_0)| = |F(x_0+iy_0)| = 0$, just means that (x_0, y_0) is the intersection point of the two plane curves defined by $u(x,y)$=0 and $v(x,y)$=0, respectively. As is known, Gauss' famous doctoral thesis (1799) first proved this fact[1,2]. Certainly, the existence of such a point $(x_0, y_0) \leftrightarrow x_0 + iy_0 = z_0$ can also be inferred from Proposition 2 or (4) directly.

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关于解析函数的一个易想象的不等式及其应用

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摘要: 本文关于解析函数给出了一个可作几何理解的不等式，由此易得出一个有关解析函数零点的命题，而代数学基本定理成为它的直接推论.

关键词: 易想象不等式；解析函数；代数学基本定理 (FTA).