第26卷第3期 数 学 研 究 与 评 论 Vol.26, No.3
2006年8月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Aug., 2006

Article ID: 1000-341X(2006)03-0502-07 **Document code**: A

Diameters of Altered Graphs

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Abstract: Let $P(t,n)$ and $C(t,n)$ denote the minimum diameter of a connected graph obtained from a single path and a circle of order n plus t extra edges, respectively, and $f(t,k)$ the maximum diameter of a connected graph obtained by deleting t edges from a graph with diameter k. This paper shows that for any integers $t \geq 4$ and $n \geq 5$, $P(t,n) \leq \frac{n-8}{t+1}+3$, $C(t,n) \leq \frac{n-8}{t+1}+3$ if t is odd and $C(t,n) \leq \frac{n-7}{t+2}+3$ if t is even; $\lceil\frac{n-1}{5}\rceil \leq P(4,n) \leq \lceil\frac{n+3}{5}\rceil$, $\lceil\frac{n}{4}\rceil - 1 \leq C(3,n) \leq \lceil\frac{n}{4}\rceil$; and $f(t,k) \geq (t+1)k-2t+4$ if $k \geq 3$ and is odd, which improves some known results.

Key words: diameter; altered graph; edge addition; edge deletion.
MSC(2000): 05C12
CLC number: O157.5

1. Introduction

We follow [1] for graph-theoretical terminology and notation not defined here. Let $G = (V,E)$ be a simple undirected graph with a vertex-set $V = V(G)$ and an edge-set $E = E(G)$. Let $P(t,n)$ and $C(t,n)$ be the minimum diameter of a graph obtained by adding t extra edges to a path and a cycle of order n, respectively. Let $f(t,k)$ denote the maximum diameter of a connected graph obtained by deleting t edges from a graph with diameter k. For given integers t, n and k, the problems determining $P(t,n)$, $C(t,n)$ and $f(t,k)$, proposed by Chung et al.[2], are of important interest in designing and analysis of interconnection networks[5].

For some small t and special n, the values of $P(t,n)$ and $C(t,n)$ have been determined. It is easy to verify that $P(1,n) = C(1,n) = \lfloor\frac{n}{2}\rfloor$ for $n \geq 3$; Schoone et al.[4] determined $P(2,n) = \lceil\frac{n}{3}\rceil$ and $C(2,n) = \lceil\frac{n+2}{4}\rceil$ for $n \geq 4$, and $P(3,n) = \lceil\frac{n+1}{4}\rceil$ for $n \geq 5$; For general $t \geq 3$, $n \geq 5$, Chung and Garey et al.[2] obtained the following results: $\frac{n}{t+1} - 1 \leq P(t,n) < \frac{n}{t+1}+3$, $\frac{n}{t+1} - 1 \leq C(t,n) < \frac{n}{t+1}+3$ if t is odd and $\frac{n}{t+2} - 1 \leq C(t,n) < \frac{n}{t+2}+3$ if t is even; Deng and Xu et al.[3] determined $P(t,(2k-1)(t+1)+2) = 2k$ for any positive integer k, $\left\lceil\frac{n-1}{t+1}\right\rceil \leq P(t,n) \leq \left\lceil\frac{n-1}{t+1}\right\rceil + 1$ for $t = 4,5$ and $n \geq 5$, and, in general, $\left\lceil\frac{n-1}{t+1}\right\rceil \leq P(t,n) \leq \left\lfloor\frac{n-3}{t+1}\right\rfloor + 3$. As to $f(t,k)$ Schoone et al.[4] determined:

$$(t+1)k \geq f(t,k) \geq \begin{cases} (t+1)k-t, & \text{if } k \text{ is even;} \\ (t+1)k-2t+2, & \text{if } k \geq 3 \text{ and is odd.} \end{cases}$$

Received date: 2004-12-29
Foundation item: the National Natural Science Foundation of China (10271114)

In this paper, we improve these upper bounds by proving that $P(t,n) \le \frac{n-8}{t+1}+3$ and $C(t,n) \le \frac{n-8}{t+1}+3$ if t is odd and $C(t,n) \le \frac{n-7}{t+2}+3$ if t is even for any integers $t \ge 4$ and $n \ge 5$. For special cases, we have $\lceil \frac{n-1}{5} \rceil \le P(4,n) \le \lceil \frac{n+3}{5} \rceil$ and $\lceil \frac{n}{4} \rceil - 1 \le C(3,n) \le \lceil \frac{n}{4} \rceil$ for $n \ge 5$. Finally we give $f(t,k) \ge (t+1)k-2t+4$ if $k \ge 3$ and is odd.

2. Several lemmas

Lemma 2.1 *$P(t,n) \le k$ if $n \le k(t+1)-2t+5$ for integers $k \ge 1$ and $t \ge 4$.*

Proof It is clear that $P(t,n) \le P(t,k(t+1)-2t+5)$ for $n \le k(t+1)-2t+5$. To prove the lemma, we only need to construct an altered graph G from a single path P of order $k(t+1)-2t+5$ by adding t extra edges such that the diameter of G is at most k.

Let $P=(x_1,x_2,\ldots,x_{k(t+1)-2t+5})$ be a single path. We construct G from P by adding t edges as follows:

$$\begin{aligned}
&e_1=(x_{2k},x_1)\\
&e_2=(x_k,x_{3k})\\
&e_j=(x_{2k},x_{k(j+1)-2j+5}), j=3,5,\ldots,2\lceil \tfrac{t}{2} \rceil-1\\
&e_i=(x_k,x_{k(i+1)-2i+5}), i=4,6,\ldots,2\lfloor \tfrac{t}{2} \rfloor
\end{aligned}$$

See Fig.1 for an example, where $k=5$, $t=8$ and $n=34$.

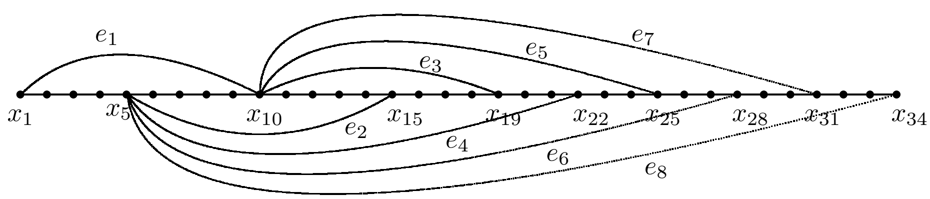

Fig.1 Illustration of Lemma 2.1 for $k=5$, $t=8$ and $n=34$.

Let $P'=(x_{2k},x_{2k+1},\cdots,x_{4k-1})$ and $H=P'+e_3$. It is easy to see that H is a cycle of length $2k$, and so $d(H)=k$.

Thus, let $P''=(x_{2k+1},x_{2k+2},\cdots,x_{4k-2})$, where $P'' \subset P'$. We have

$$d_G(x_i,x_k)+d_G(x_i,x_{2k})=\begin{cases} k+1, & \text{if } x_i \in V(P'');\\ k, & \text{if } x_i \notin V(P''). \end{cases}$$

So, for any two distinct vertices x_a and x_b in G, if $x_a,x_b \in V(P')$, then $d_G(x_a,x_b) \le d_H(x_a,x_b) \le k$; Otherwise,

$$d_G(x_a,x_k)+d_G(x_a,x_{2k})+d_G(x_b,x_k)+d_G(x_b,x_{2k}) \le (k+1)+k=2k+1,$$

which implies

$$2(d_G(x_a,x_b)) \le d_G(x_a,x_k)+d_G(x_b,x_k)+d_G(x_a,x_{2k})+d_G(x_b,x_{2k}) \le 2k+1,$$

that is, $d_G(x_a, x_b) \le k$. Thus, we get $d(G) \le k$.

Lemma 2.2 *$P(t,n) \le 2k$ if $n \le 2k(t+1) - t + 1$ for integers $k \ge 1$ and $t \ge 4$.*

Proof Similar to the proof of Lemma 2.1, we construct an altered graph G from a single path $P = (x_1, x_2, \ldots, x_{2k(t+1)-t+1})$ by adding t extra edges:

$$e_i = (x_{k+1}, x_{(2i+1)k-i+2}),\ i = 1, 2, \ldots, t.$$

See Fig.2 for an example, where $k = 3$,$t = 6$ and $n = 37$.

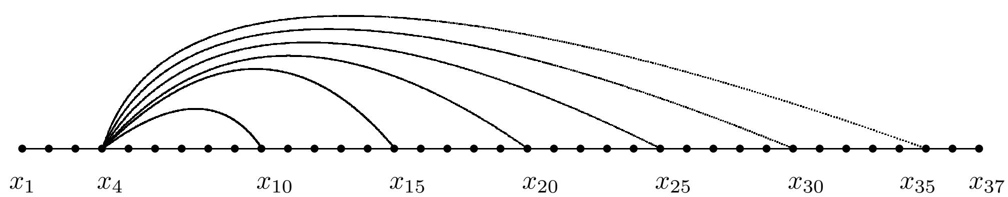

Fig.2 Illustration of Lemma 2.2 for $k = 3$, $t = 6$ and $n = 37$.

It is easy to know $d_G(x_i, x_k) \le k$ for any $i = 1, 2, \cdots, 2k(t+1) - t + 1$. Thus

$$d_G(x_i, x_j) \le d(x_i, x_k) + d(x_k, x_j) \le 2k \text{ for } 1 \le i \ne j \le 2k(t+1) - t + 1,$$

which means that $d(G) \le 2k$. □

Lemma 2.3 *Let both t and k be integers. If $t \ge 4$, then*

$$C(t,n) \le \begin{cases} k & \text{for } n \le k(t+1) - 2t + 5,\ k \ge 3; \\ 2k & \text{for } n \le 2k(t+1) - t + 1,\ k \ge 1. \end{cases}$$

Proof If we add one edge joining two end vertices of the path $P_{k(t+1)-2t+5}$ and add other t edges in the same way as one used in the proof of Lemma 2.1, then we could get an altered graph G from a single cycle of order $k(t+1) - 2t + 5$ by adding t extra edges such that the diameter of G is not more than k. Thus we have

$$C(t,n) \le k \text{ for } n \le k(t+1) - 2t + 5 \text{ and } k \ge 3.$$

In a way similar to one used in the proof of Lemma 2.2, we get another altered graph from a single cycle of order $2k(t+1) - t + 1$ by adding t extra edges such that the diameter at most $2k$. It means that

$$C(t,n) \le 2k \text{ for } n \le 2k(t+1) - t + 1 \text{ and } k \ge 1$$

as required.

Lemma 2.4 *Let t and k be integers. If t is even and $t \geq 4$, then $C(t,n) \leq k$ for $n \leq k(t+2)-2t+2$ and $k \geq 3$.*

Proof Again we need to construct an altered graph G from a single cycle $C_n = (x_1, x_2, \cdots, x_n, x_1)$ by adding t extra edges, where $n = k(t+2) - 2t + 2$.

Now we let G_p be the altered graph of diameter k in the proof of Lemma 2.1 obtained from a single path of order $k(t+2)-2(t+1)+5$ by adding $t+1$ extra edges. Assume the $t+1$ added edges are $e_1, e_2, \cdots, e_t, e_{t+1}$.

Notices that $k(t+2)-2(t+1)+5 = n+1$ and if t is even, $e_{t+1} = (x_{2k}, x_{n+1})$. So if we alter the graph G_p by deleting the vertex x_{n+1} and the edge e_{t+1} and adjoining the vertices x_1 and x_n, we get another graph G_c, which is an altered graph obtained from a single cycle of order n by adding t extra edges.

See Fig.3 for an example, where $k=5$, $n=30$ and $t=6$.

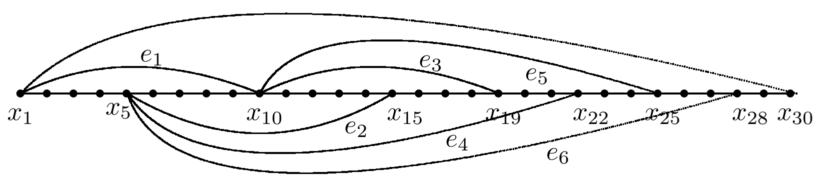

Fig.3 Illustration of Lemma 2.4 for $k=5$, $t=6$ and $n=30$.

It is clear that $d_{G_c}(x_i, x_k) + d_{G_c}(x_i, x_{2k}) = d_{G_p}(x_i, x_k) + d_{G_p}(x_i, x_{2k})$ for any vertex $x_i \in G_c$. Similar to the proof of Lemma 2.1, we can verify that $d_{G_c}(x_i, x_j) \leq d_{G_p}(x_i, x_j) \leq k$ for any two vertices x_i and x_j in G_c, which implies $d(G_c) \leq k$. And hence $C(t,n) \leq k$ for $n \leq k(t+2)-2t+2$ as required.

3. Proof of main results

Theorem 3.1 *For any integers $t \geq 4$ and $n \geq 5$, $P(t,n) \leq \frac{n-8}{t+1} + 3$; furthermore, $P(t,n) \leq \left\lceil \frac{n+t-6}{2t+2} \right\rceil + \left\lceil \frac{n+t-1}{2t+2} \right\rceil$.*

Proof Firstly, when t is fixed, for any $n \geq 5$ there exists an integer $k \geq 0$ such that

$$(k-1)(t+1) - 2t + 6 \leq n \leq k(t+1) - 2t + 5.$$

It follows from Lemma 2.1 that

$$p(t,n) \leq k \leq \frac{n+2t-6}{t+1} + 1 = \frac{n-8}{t+1} + 3.$$

Secondly, let $m(k) = 2k(t+1) - t + 1$ for any $n \geq 3$. Then there exists an integer $k \geq 0$ such that $m(k)+1 \leq n \leq m(k+1)$.

If $m(k)+1 \le n \le m(k)+5 = (2k+1)(t+1)-2t+5$, then, from Lemma 2.1, we have

$$P(t,n) \le 2k+1 = k+(k+1) = \left\lceil \frac{n+t-6}{2t+2} \right\rceil + \left\lceil \frac{n+t-1}{2t+2} \right\rceil.$$

If $m(k)+6 \le n \le m(k+1) = 2(k+1)(t+1)-t+1$, then, from Lemma 2.2, we have

$$P(t,n) \le 2(k+1) = (k+1)+(k+1) = \left\lceil \frac{n+t-6}{2t+2} \right\rceil + \left\lceil \frac{n+t-1}{2t+2} \right\rceil.$$

The theorem follows.

Remarks It is clear that for $t \ge 4$

$$P(t,n) \le \left\lceil \frac{n+t-6}{2t+2} \right\rceil + \left\lceil \frac{n+t-1}{2t+2} \right\rceil \le \frac{n-8}{t+1} + 3.$$

In fact, if let $2m = \left\lceil \frac{n+t-1}{t+1} \right\rceil = \left\lceil \frac{n-2}{t+1} \right\rceil + 1$, just when

$$2m-2 < \frac{n-2}{t+1} \le 2m-1 \Longleftrightarrow (2m-2)(t+1)+3 \le n \le (2m-1)(t+1)+2,$$

we have

$$P(t,n) \le \left\lceil \frac{n+t-6}{2t+2} \right\rceil + \left\lceil \frac{n+t-1}{2t+2} \right\rceil \le \ m+m = 2m \le \frac{n-3}{t+1} + 2.$$

Thus, we get that

$$P(t,n) \le \left\lceil \frac{n+t-6}{2t+2} \right\rceil + \left\lceil \frac{n+t-1}{2t+2} \right\rceil \le \begin{cases} \dfrac{n-8}{t+1} + 3, & \text{if } \left\lceil \frac{n-2}{t+1} \right\rceil \text{ is even} \\ \dfrac{n-3}{t+1} + 2, & \text{if } \left\lceil \frac{n-2}{t+1} \right\rceil \text{ is odd} \end{cases},$$

which is a better bound.

Corollary 3.1 $\lceil \frac{n-1}{5} \rceil \le P(4,n) \le \lceil \frac{n+3}{5} \rceil$ *for any integer* $n \ge 5$.

Proof On the one hand, by $P(t,n) \ge \left\lceil \frac{n-1}{t+1} \right\rceil$, due to Deng and Xu[3] and the statement in Introduction, we have

$$P(4,n) \ge \left\lceil \frac{n-1}{5} \right\rceil.$$

On the other hand, by Theorem 3.1,

$$P(4,n) \le \frac{n-8}{5} + 3 = \frac{n+2}{5} + 1.$$

Since $P(4,n)$ is an integer, we have

$$P(4,n) \le \left\lfloor \frac{n+2}{5} \right\rfloor + 1 = \left\lceil \frac{n-2}{5} \right\rceil + 1 = \left\lceil \frac{n+3}{5} \right\rceil$$

as required.

Theorem 3.2 *For any integers $t \geq 4$ and $n \geq 5$,*

$$C(t,n) \leq \begin{cases} \dfrac{n-7}{t+2}+3 & \text{if } t \text{ is even;} \\ \left\lceil \dfrac{n+t-6}{2t+2} \right\rceil + \left\lceil \dfrac{n+t-1}{2t+2} \right\rceil \leq \dfrac{n-8}{t+1}+3 & \text{if } t \text{ is odd.} \end{cases}$$

Proof If t is even and fixed, then for any $n \geq 5$ there exists am integer $k \geq 3$ such that

$$(k-1)(t+2)-2t+3 \leq n \leq k(t+2)-2t+2.$$

From Lemma 2.4, we have

$$C(t,n) \leq k \leq \frac{n+2t-3}{t+2}+1 = \frac{n-7}{t+2}+3.$$

If t is odd and fixed, then for any $n \geq 5$ there exists an integer $k \geq 3$ such that

$$(k-1)(t+1)-2t+6 \leq n \leq k(t+1)-2t+5.$$

From Lemma 2.3, we have

$$C(t,n) \leq k \leq \frac{n+2t-6}{t+1}+1 = \frac{n-8}{t+1}+3.$$

Furthermore, similar to the proof of Theorem 3.1, from Lemma 2.3 we have

$$C(t,n) \leq \left\lceil \frac{n+t-6}{2t+2} \right\rceil + \left\lceil \frac{n+t-1}{2t+2} \right\rceil,$$

which is a better bound.

Theorem 3.3 $\lceil \frac{n}{4} \rceil - 1 \leq C(3,n) \leq \lceil \frac{n}{4} \rceil$ *for any integer $n \geq 5$.*

Proof On the one hand, by $C(t,n) \geq \frac{n}{t+1} - 1$ if t is odd, due to Chung and Garey[2] and statement in Introduction, we have

$$C(3,n) \geq \left\lceil \frac{n}{4} \right\rceil - 1.$$

On the other hand, let $k = \lceil \frac{n}{4} \rceil$. It is easy to verify that the diameter of the altered graph obtained from a cycle $C_n = (x_1, x_2, \cdots, x_n)$ by adding the three edges

$$e_1 = (x_1, x_{2k+1}),\ e_2 = (x_3, x_{2k+3}),\ e_3 = (x_{k+2}, x_{3k+1}),$$

is k. Thus

$$C(3,n) \leq k = \left\lceil \frac{n}{4} \right\rceil$$

as required.

Theorem 3.4 $f(t,k) \geq (t+1)k - 2t + 4$ *if k is an odd integer and $k \geq 3$.*

Proof For any $k \geq 2$, we can delete t edges from the altered graph G constructed in the proof of Lemma 2.1 whose diameter is k to get a path of diameter $(t+1)k - 2t + 4$. So we have

$$f(t,k) \geq (t+1)k - 2t + 4,$$

which, of course, holds if k is an odd integer and $k \geq 3$.

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变更图的直径

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摘要: $P(t,n)$ 和 $C(t,n)$ 分别表示在阶为 n 的路和圈中添加 t 条边后得到的图的最小直径; $f(t,k)$ 表示从直径为 k 的图中删去 t 条边后得到的连通图的最大直径. 这篇文章证明了当 $t \geq 4$ 且 $n \geq 5$ 时， $P(t,n) \leq \frac{n-8}{t+1} + 3$; 若 t 为奇数，则 $C(t,n) \leq \frac{n-8}{t+1} + 3$; 若 t 为偶数，则 $C(t,n) \leq \frac{n-7}{t+2} + 3$. 特别地， $\lceil \frac{n-1}{5} \rceil \leq P(4,n) \leq \lceil \frac{n+3}{5} \rceil$, $\lceil \frac{n}{4} \rceil - 1 \leq C(3,n) \leq \lceil \frac{n}{4} \rceil$. 最后，证明了：若 $k \geq 3$ 且为奇数，则 $f(t,k) \geq (t+1)k - 2t + 4$. 这些改进了某些已知结果.

关键词: 直径; 变更图; 边增加; 边减少.