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Generalized Noetherian Property of Rings

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Abstract: It is well-known that a ring R is right Noetherian if and only if every direct sum of injective right R-modules is injective. In this paper, we will characterize Ne-Noetherian rings and U-Noetherian rings by Ne-injective modules and U-injective modules.

Key words: Ne-injective module; Ne-Noetherian ring; U-injective module; U-Noetherian ring.
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1. Introduction

Throughout this paper all rings are associative with identity, all modules are right unitary. For a module M, $N \leq M$ means that N is a submodule of M. The injective envelop of M is denoted by $E(M)$. A submodule L of M is essential, if $L \cap N \neq 0$ for every non-zero submodule N of M, otherwise L is non-essential. A submodule N of M is uniform, if N is uniform, that is any submodule K of N is essential in N. According to [2], a module M is said to be Ne-Noetherian (resp. U-Noetherian), if M satisfies ACC on non-essential (resp.uniform) submodules. A ring R is said to be right Ne-Noetherian (resp.right U-Noetherian), if R_R is a Ne-Noetherian (resp. U-Noetherian) module. A ring R is said to be Ne-Noetherian (resp. U-Noetherian), if R is not only a left Ne-Noetherian (resp. U-Noetherian) ring but also a right Ne-Noetherian (resp. U-Noetherian) ring. Clearly Noetherian ring is both Ne-Noetherian and U-Noetherian. Smith and Vedadi in [2] have investigated Ne-Noetherian ring and given some basic properties and many examples of Ne-Noetherian ring. Motivated by Bass-Papp Theorem, in this paper, we will give some characterizations of Ne-Noetherian rings and U-Noetherian rings by Ne-injective modules and U-injective modules, and discuss Ne-Noetherian property and U-Noetherian property of the subcategory of $MOD-R$.

2. Main results

An R-module Q is called Ne-injective (resp. U-injective), if for each non-essential (resp. uniform) right ideal I of R, every R-homomorphism $f : I \to Q$ extends to R. Equivalently if f is a left multiplication by some element of R. A ring R is called right Ne-injective (resp. right

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U-injective), if R_R is a Ne-injective (resp. U-injective) module. Left Ne-injective (resp. left U-injective) rings are defined similarly.

Lemma 1[2] *The following statements are equivalent for a module M:*

(1) M is Ne-Noetherian.

(2) Every non-essential submodule of M is Noetherian.

(3) Every non-essential submodule of M is finitely generated.

Remark Any direct sum of Ne-Noetherian modules need not be Ne-Noetherian. For example, let M be Ne-Noetherian module but not Noetherian. Then $M \oplus M$ is not Ne-Noetherian. Because $M \oplus 0$ is non-essential submodule of $M \oplus M$, however which is not Noetherian.

We say a module M has property $(*)$, if for any ascending chain $M_1 \leq M_2 \leq \cdots$ of non-essential submodules of $M, N = \cup_{i=1}^{\infty} M_i$ is also non-essential submodule of M.

Theorem 2 *Suppose that R_R has property $(*)$. Then R is a right Ne-Noetherian ring if and only if every direct sum of Ne-injective R-modules is Ne-injective.*

Proof $(\Rightarrow)$ Let $\{E_\lambda\}_{\lambda\in\Lambda}$ be a family of Ne-injective modules. Set $E = \oplus_{\lambda\in\Lambda} E_\lambda$. Let I be a non-essential right ideal of R and $f : I \to R$ any R-homomorphism. Since R is right Ne-Noetherian, hence I is finitely generated, and consequently $f(I)$ is finitely generated. Let $f(I) = \sum_{i=1}^{n} x_i R$, $x_i \in E$. Since $x_1, x_2, \cdots, x_n$ are contained in a finite direct sum of E_λ, i.e. there is a finite subset $\Lambda_0 \subseteq \Lambda$ such that $f(I) \subseteq \oplus_{\lambda\in\Lambda_0} E_\lambda$. Since every finite direct sum of Ne-injective modules is Ne-injective, there exists R-homomorphism $g : R \to \oplus_{\lambda\in\Lambda_0} E_\lambda$ such that $g|_I = f$. Let $\tau : \oplus_{\lambda\in\Lambda_0} E_\lambda \to E$ be injection with $h = \tau g$. Then $h|_I = f$. This means that E is Ne-injective.

$(\Longleftarrow)$ Let $L_1 \leq L_2 \leq \cdots$ be any ascending chain of non-essential right ideals of R. Since R_R has property $(*)$, it follows that $L = \cup_{i=1}^{\infty} L_i$ is a non-essential right ideal of R. Let $E = \oplus_{i=1}^{\infty} E(R/L_i)$ and $f : L \to E$ be the mapping defined by $f(x) = (x + L_i)_{i\geq 1}$, $x \in L$, where $x + L_i \in R/L_i \leq E(R/L_i)$. Clearly, f is an R-homomorphism. By hypothesis E is Ne-injective and thus R-homomorphism f can be expressed in the form $f(x) = ex$ for all $x \in L$, where $e = (e_i)_{i\geq 1}$ is a suitable element in E. Now for sufficiently large i, we have $e_i = 0$. So we also have $0 = f(x)_i = x + L_i$ for every $x \in L$. This means that for sufficiently large $i, L = L_i$, i.e. R is right Ne-Noetherian.

Lemma 3 *The following properties are equivalent for a module M:*

(1) M is U-Noetherian.

(2) Every uniform submodule of M is Noetherian.

(3) Every uniform submodule of M is finitely generated.

Proof It follows from the similar method in the proof of [2, Theorem 1.4].

Remark From [2, Proposition 1.9], if a module M is Ne-Noetherian, then so is the module M/K, where K is a closed submodule of M. For Noetherian module M, the homomorphic image of M is U-Noetherian. However it is not true in general. For example, let M be U-Noetherian module but not uniform and Noetherian. Let K be a submodule of M but not uniform. Hence there is $0 \neq K_1 \leq K$, $0 \neq K_2 \leq K$, such that $K_1 \cap K_2 = 0$. Let N/K be a uniform submodule of M/K. Then $K_1 \leq N$, $K_2 \leq N$, consequently N is not uniform submodule of M. It follows that N/K need not be Noetherian, that is M/K need not be U-Noetherian.

Proposition 4 *Let I be any index set $\pi_i : M \rightarrow M_i$ natural projection. If each uniform submodule of M is invariant under π_i for all $i \in I$. Then $M = \oplus_{i\in I} M_i$ is U-Noetherian module if and only if M_i is U-Noetherian module for all $i \in I$.*

Proof Let K be a uniform submodule of M. If $K \cap M_i \neq 0$ for some $i \in I$, thus $K \cap M_i$ is uniform submodule of M_i. $K \cap M_i$ is summand of K, since K is invariant π_j for all $j \in I$ consequently $K = K \cap M_i$. By Lemma 3, K is finitely generated. Hence M is U-Noetherian. Conversely is obvious.

Lemma 5 *Let M be a module and $M_1 \leq M_2 \leq \cdots$ any ascending chain of uniform submodules of M, Then $N = \cup_{i=1}^{\infty} M_i$ is also uniform submodule of M.*

Proof Let $0 \neq K \leq N$, $0 \neq H \leq N$. Then there is $i, j \in \{1, 2, \cdots\}$ such that $K \cap M_i \neq 0$, $H \cap M_j \neq 0$. Let $l = \max\{i, j\}$. Thus $0 \neq K \cap M_l \leq M_l$, $0 \neq H \cap M_l \leq M_l$. Since M_l is uniform submodule of M, therefore $K \cap M_l \cap H \neq 0$ and consequently $K \cap H \neq 0$. We conclude that N is uniform submodule of M.

By the similar method of Theorem 2, we get the following result.

Theorem 6 *Let R be a ring, Then R is a right U-Noetherian ring if and only if every direct sum of U-injective R-modules is U-injective.*

Let M be an R-module. We say that an R-module N is subgenerated by M, or that M is subgenerator for N, if N is isomorphic to a submodule of an M-generated module. Following [1], we denote by $\sigma[M]$ the full subcategory of $MOD - R$, whose objects are all R-module subgenerated by M. $N \in \sigma[M]$, the injective hull of N in $\sigma[M]$ is also called an M-injective hull of N and is usually denoted by $I(N)$. An R-module U is said to be Ne-M-injective, if for any R-module N and every R-monomorphism $h : N \rightarrow M$ with $Im(h)$ is non-essential submodule of M, any R-homomorphism $f : N \rightarrow U$ can be extended to an R-homomorphism $g : M \rightarrow U$, i.e. $gh = f$. An R-module U is called weakly Ne-M-injective, if for any finitely generated non-essential submodule K of M, each R-homomorphism $f : K \rightarrow U$ extends to M. We say that M is locally Ne-Noetherian, if M satisfies ACC on finitely generated non-essential submodules. Call M is Ne-V-module, if every simple module is Ne-M-injective.

Lemma 7 *U is a Ne-M-injective module if and only if for every finitely generated submodule N of M, U is Ne-N-injective.*

Proof It follows from the similar method in the proof of [1,16.3].

Remark By the similar method of [1,16.1], we can prove that the direct product of weakly Ne-M-injective modules is weakly Ne-M-injective. Therefore, the finite direct sum of weakly Ne-M-injective modules is weakly Ne-M-injective.

Lemma 8 *The direct sum of Ne-M-injective modules is weakly Ne-M-injective.*

Proof It follows from the similar method in the proof of [1,16.10].

Theorem 9 *Suppose that M is a module with any finitely generated submodule has property $(*)$, Then the following assertions are equivalent:*

(a) M is locally Ne-Noetherian.

(b) Every weakly Ne-M-injective module is Ne-M-injective.

(c) Every direct sum of Ne-M-injective modules is Ne-M-injective.

(d) Every countable direct sum of M-injective hulls of simple modules in $\sigma[M]$ is Ne-M-injective.

Proof (a)⇒(b) Let N be a finitely generated submodule of M and U a weakly Ne-M-injective module. Let K be a non-essential submodule of N. Then K is finitely generated with K is non-essential in M by (a) and Lemma 1. Therefore every R-homomorphism $f : K \to U$ can be extended to an R-homomorphism $g : M \to U$. Let $h = g|_N$. Thus $h : N \to U$ extends f. Hence U is Ne-N-injective for every finitely generated submodule $N \subseteq M$ and by Lemma 7, U is Ne-M-injective.

(b)⇒(c) It follows from Lemma 8.

(c)⇒(d) is trivial.

(d)⇒(a) Let K be a finitely generated submodule of M and $U_0 \le U_1 \le U_2 \le \cdots$ any ascending chain of non-essential submodules of K. For every $U_i, i \in \{1, 2, \cdots\}$, we choose a maximal submodule $V_i \le U_i$ with $U_{i-1} \le V_i$. So we obtain the ascending chain $U_0 \le V_1 \le U_1 \le V_2 \le U_2 \le \cdots$, where the factors $E_i = U_i/V_i \ne 0$ are simple modules as long as $U_{i-1} \ne U_i$ with the M-injective hulls $I(E_i)$ of E_i for all $i \in \{1, 2, \cdots\}$. Let $U = \bigcup_{i=0}^{\infty} U_i$. Then U is a non-essential submodule of K by the condition. Then for every R-homomorphism $h_i : U_i/V_i \to I(E_i)$, there exists R-homomorphism $g_i : U/V_i \to I(E_i)$. Hence a family of mappings $f_i : U \to^{p_i} U/V_i \to^{g_i} I(E_i), i \in \{1, 2, \cdots\}$, yielding a map into the product $f : U \to \prod_{i=1}^{\infty} I(E_i)$. Now any $u \in U$ is not contained in at most finitely many $V_i^{'}s$. Hence $\pi_i f(u) - f_i(u) \ne 0$ only for finitely many $i \in \{1, 2, \cdots\}$, where $\pi_i : \prod_{i=1}^{\infty} I(E_i) \to I(E_i)$ is projective,which means $\mathrm{Im}(f) \subseteq \oplus_{i=1}^{\infty} I(E_i)$. By assumption (d), this sum is Ne-M-injective, and hence f can be extended to an R-homomorphism $g : K \to \oplus_{i=1}^{\infty} I(E_i)$, Since K is finitely generated, so $Im(g)$ is contained in a finite partial sum, i.e. $f(U) \subseteq g(K) \subseteq I(E_1) \oplus \cdots \oplus I(E_r)$ for some $r \in \{1, 2, \cdots\}$. Then for $k \ge r$, we must get $0 = f_k(U) = g_k p_k(U)$ and $0 = f_k(U_k) = g_k(U_k/V_k) = U_k/V_k$, implying $U_k = V_k$. Hence the sequence considerated terminates at r and K is Ne-Noetherian.

Corollary 10 *Let M is a finitely generated module. Then the following statements are equivalent:*

(i) M is Ne-Noetherian, Ne-V-module.

(ii) Every semisimple module is Ne-M-injective.

(iii) Every countable generated semisimple module is Ne-M-injective.

Let M be an R-module. An R-module U is said to be U-M-injective, if for any R-module N and every R-monomorphism $h: N \to M$ with $Im(h)$ is uniform submodule of M, any R-homomorphism $f: N \to U$ can be extended to an R-homomorphism $g: M \to U$. An R-module U is called weakly U-M-injective, if for any finitely generated uniform submodule K of M, every R-homomorphism $f: K \to U$ extends to M. We call that M is locally U-Noetherian, if M satisfies ACC on finitely generated uniform submodules.

By the similar methed of Theorem 9, we get the following result.

Theorem 11 *Let M be a module. Then the following statements are equivalent:*

(a) M is locally U-Noetherian.

(b) Every weakly U-M-injective module is U-M-injective.

(c) Every direct sum of U-M-injective modules is U-M-injective.

(d) Every countable direct sum of $M-$injective hulls of simple modules in $\sigma[M]$ is U-M-injective.

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环的广义 Noether 性

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摘要: 众所周知，环 R 是右 Noether 的当且仅当任意内射右 R- 模的直和是内射的. 本文我们将用 Ne- 内射模和 U- 内射模来刻画 Ne-Noether 环和 U-Noether 环.

关键词: Ne- 内射模; Ne-Noether 环; U- 内射模; U-Noether 环.