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On Soluble Block-Transitive 2-$(5^6, 7, 1)$ Designs

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Abstract: Let G be a soluble block-transitive automorphism group of 2-$(5^6, 7, 1)$ design $\mathbf{D}$. Then G is flag-transitive or $G \leq A\Gamma L(1, 5^6)$.

Key words: design; block-transitive; automorphism; soluble group.
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1. Introduction

A 2-$(v, k, 1)$ design is a pair $(\mathbf{P}, \mathbf{L})$, where $\mathbf{P}$ is a point set of v points, and $\mathbf{L}$ is the set of some k-subsets of $\mathbf{P}$, such that for any 2 points of $\mathbf{P}$, there is exactly one element of $\mathbf{L}$ containing them. The elements of $\mathbf{P}$ are called points and the elements of $\mathbf{L}$ are called blocks. We denote by $\mathbf{D}$ the 2-$(v, k, 1)$ design, by b the number of blocks in $\mathbf{D}$, by r the number of blocks containing a fixed point of $\mathbf{P}$. For a point-block pair (α, L), $\alpha \in L$, we call it the flag of $\mathbf{D}$. We say G is flag-transitive if G is transitive on the set of flags of $\mathbf{D}$.

The notations and terminology above can be found in [1] and [8].

Denote by π the permutations of the point set $\mathbf{P}$. We say π is an automorphism of $\mathbf{D}$ if it transforms the blocks of $\mathbf{D}$ into themselves. All the automorphisms of $\mathbf{D}$ consist of a group, denote by $\mathrm{Aut}(\mathbf{D})$. Put $G \leq \mathrm{Aut}(\mathbf{D})$. We call G is block-transitive (point-transitive) if G acts transitively on the block set (point set). The results in [2] show that if G is block-transitive, then G is point-transitive. We say G is block-primitive (point-primitive) if G acts primitively on the block set (point set). Few years ago, Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl classified the flag-transitive designs. Recently, A.R. Camina[3] provides a scheme to classify the block-transitive designs. It says that if G is block-transitive and point-primitive, then the socle of G is either elementary abelian groups or non-abelian simple groups. So we can discuss the structure of G by using the classification of finite simple groups theorem.

In the beginning of 1980's, people have completely finished the classification of block-transitive 2-$(v, 3, 1)$ designs (to see [7,9,12,14]). In [6], the classification of soluble block-transitive 2-$(v, 4, 1)$ designs is obtained by A. Camina and J. Siemons. In [15], H. L. Li classified the non-soluble block-transitive 2-$(v, 4, 1)$ designs. In [20], H. L. Li and W. W. Tong classified the soluble block-transitive 2-$(v, 5, 1)$ designs. In [17], W. J. Liu, H. L. Li and C. G. Ma classified the soluble

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block-transitive $2-(v,6,1)$ designs. In [18], the authors got the following theorem:

Theorem 1.1 *Let G be a soluble block-transitive automorphism group of $2-(v,7,1)$ design. Then G is point-primitive and one of the following statements holds:*

(1) $v=7^n$ *and G is flag-transitive;*

(2) $v=5^6$ *and* $G=Z_{5^6}:H$, *where H is a soluble and irreducible subgroup of* $\mathrm{GL}(6,5)$;

(3) $v=p^n$ *and* $G\leq A\Gamma L(1,p^n)$.

In particular, $p\neq 2$ *and* $p^n\equiv 1 \pmod{42}$.

In this paper, we discuss the case (2) of the above theorem, and set the following theorem:

Main Theorem *Let G be a soluble block-transitive automorphism group of 2-$(5^6,7,1)$ design. Then G is flag-transitive and* $G\leq A\Gamma L(1,5^6)$.

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2. Preparation

Definition 2.1 *Let p be a prime and n, t be positive integers. We say that t is a p-primitive divisor of p^n-1, if $t>0$ and $(t,p^m-1)=1$ for all m with $0<m<n$. Call t the maximal p-primitive divisor of p^n-1 if t is a primitive divisor of p^n-1 and $s|t$ for all p-primitive divisors s of p^n-1.*

By [21], we know that there always exists a p-primitive divisor of p^n-1 that is not equal to 1 except that $n=1$ and $p=2$, $n=2, n=6$ and $p=2$.

For a 2-$(v,k,1)$ design, there is a well known result as below.

Lemma 2.1 *Let $\mathbf{D}$ be a 2-$(v,k,1)$ design. Denote by b the number of blocks of $\mathbf{D}$. Then*

(i) $bk(k-1)=v(v-1)$;

(ii) $b\geq v$;

(iii) either $v=k^2-k+1$ *or* $v\geq k^2$.

Let $G\leq \mathrm{Aut}(\mathbf{D})$ and B be a block of $\mathbf{D}$. Denote by G_B the subgroup of G stabilizing B (as a set).

Lemma 2.2 (Lemma 2 of [6]) *Let G act as a block-transitive automorphism group of a linear space $\mathbf{D}$. Let B be a block and H a subgroup of G_B. Assume that H satisfies the following two conditions:*

(i) $|\mathrm{Fix}(H)\cap B|\geq 2$ *and*

(ii) if $K\leq G_L$ and $|\mathrm{Fix}(K)\cap B|\geq 2$ and K is conjugate to H in G then H is conjugate to K in G_B.

Then either (a) $\mathrm{Fix}(H)\subseteq B$ or (b) the induced structure on $\mathrm{Fix}(H)$ is also a regular linear space with parameters (b_0,v_0,r_0,k_0), where $v_0=|\mathrm{Fix}(H)|$, $k_0=|\mathrm{Fix}(H)\cap B|$. Furthermore, $N_G(H)$ acts as a line-transitive group on this linear space.

Lemma 2.3 (Lemma 4 of [6]) *Let G act block-transitively on a 2-$(v,k,1)$ design. If s is an* *involution in G which fixes no points, then k divides v.*

Recall [4], we have that G is flag-transitive.

Lemma 2.4 (Lemma 5 of [6]) *Let G act block-transitively on a 2-$(v, k, 1)$ design. Assume that G contains a regular normal subgroup V whose elements are identified with $\mathbf{P}$. Suppose that some element in G maps every element of $\mathbf{P}$ onto its inverse. If $k > 2$ then any block containing 1 is a subgroup of V so that k divides v.*

The following Lemma is a generalization of Lemma 2.2 of [5].

Lemma 2.5 *Let G be a group acting block-transitively on a 2-$(v, k, 1)$ design $\mathcal{D}$. Let g be an element of order s of G_B, where s is a prime and B is a block of $\mathcal{D}$. Assume that there is a normal subgroup N of G with $|G : N| = s$, such that $g \notin N$. Then N also acts block-transitively.*

Proof Since $N \trianglelefteq G$, $N \cap G_B = N_B \trianglelefteq G_B$. By $g \in G_B$ and $g \notin N$, we get $N < NG_B \leq G$. Because $|G : N| = s$ is a prime, $G = NG_B$. Hence

$$G_B/N_B = G_B/(N \cap G_B) \cong NG_B/N = G/N,$$

and so $|G/G_B| = |N/N_B|$. It follows that N is block-transitive.

Lemma 2.6[19] *The maximal irreducible soluble subgroup of linear group* $\mathrm{GL}(6, 5)$ *is isomorphic to the following three subgroups:*

(1) $(Z_{124} : Z_3)wrS_2$, its order is $2^5 \cdot 31^2$;

(2) $(Z_{31} \times NS) : Z_3$, where NS is a group of order 96;

(3) $Z_{5^6-1} : Z_6$, that is, $\Gamma L(1, 5^6)$.

Lemma 2.7 (Theorem 3.2 of [10]) *Let G be a primitive group of degree n, and N a minimal normal subgroup. If N is soluble, then*

(a) N is regular and elementary abelian, and the degree n is a prime power p^n;

(b) $G = NG_1$ and $N \cap G_1 = E$, where G_1 denotes the stabilizer in G of the element 1;

(c) $C_G(N) = N$;

(d) N is unique minimal normal subgroup.

Lemma 2.8 (Theorem 7.3 of [11]) *Any a soluble 2-transitive group with degrees p^n is isomorphic to a subgroup of* $\mathrm{A\Gamma L}(1, \mathrm{p^n})$, *unless $p^n = 3^2, 5^2, 7^2, 11^2, 23^2$ or 3^4.*

Lemma 2.9[16] *Let G be a soluble block-transitive automorphism group of a 2-$(v, k, 1)$ design. If G is point -primitive, then*

(i) there exists a prime number p and a positive integer n such that $v = p^n$, and

(ii) if there exists a p-primitive prime divisor r of $p^n - 1$, such that $r||G|$, then either $G \leq A\Gamma L(1, p^n)$ or $k|v$.

3. Proof of the main theorem

Let $\mathbf{D}$ be a 2-($5^6, 7, 1$) design, and denote by $\mathbf{P}$ the point set of $\mathbf{D}$, and $G \leq \mathrm{Aut}(\mathbf{D})$. We have $v=5^6$ and $b=2^2\cdot 3\cdot 5^6\cdot 31$. Hence $v>(7(7-1)/2-1)^2=400$. Since $\mathbf{D}$ is block-transitive, by [8], G acts primitively on the point set of $\mathbf{D}$. It follows that G is a soluble primitive group. By Lemma 2.7, G contains an abelian minimal normal subgroup V. By identifying the point set $\mathbf{P}$ of $\mathbf{D}$ with V, we can regard $\mathbf{P}$ as an n-dimensional vector space over the field $\mathrm{GF}(p)$, and G_0, the stabilizer in G of the zero vector, is a subgroup of $\mathrm{GL}(6,5)$ and irreducible. We know that

$$\mathrm{Fix}(\langle g\rangle)=\mathrm{C_V(g)}=\{\,\mathrm{v}\in\mathrm{V}\mid \mathrm{v^g}=\mathrm{v}\,\},$$

where $g\in G_0$. If G is flag-transitive, then by [13] and Lemma 2.8, $G\le \mathrm{A\Gamma L}(1,5^6)$.

We assume that G is a group of least order which is not flag-transitive as below.

(i) $G\not\le \mathrm{A\Gamma L}(1,5^6)$.

Assume that $G\le \mathrm{A\Gamma L}(1,5^6)$. Then G_0 contains a subgroup of an even order of a Singer cycle. But then G_0 contains the involution s so that $v^s=-v$ for all $v\in V$. Then G would be flag-transitive by Lemma 2.4. Hence we can assume that $G\not\le \mathrm{A\Gamma L}(1,5^6)$.

(ii) 7 does not divide $|G_0|$.

Note that 7 is a 5-primitive divisor of 5^6-1. Hence if $7\mid |G_0|$, then, by Lemma 2.9 and [13], we have $G\le \mathrm{A\Gamma L}(1,5^6)$. This conflicts with (i).

(iii) There is no involution whose determinant is -1.

By second isomorphism theorem of groups, we have

$$G_0/(G_0\cap \mathrm{SL}(6,5))\cong G_0\mathrm{SL}(6,5)/\mathrm{SL}(6,5)\le \mathrm{GL}(6,5)/\mathrm{SL}(6,5)\le Z_4.$$

Since G_0 is soluble, we know that $\mathrm{SL}(6,5)\not\le G_0$. Therefore,

$$G_0/(G_0\cap \mathrm{SL}(6,5))\cong Z_2.$$

Thus if there is an involution whose determinant is -1, then we see that there is a normal subgroup $K=G_0\cap \mathrm{SL}(6,5)$ of index 2 in G_0 so that $G_0=K\langle s\rangle$. By Lemma 2.5, VK would be a group of smaller order which was block-transitive, a contradiction.

(iv) Each involution fixes 25 or 625 points.

Let s be an involution in G_0. If s fixes only one point then $v^s=-v$ for all v of V, which is false by Lemma 2.4 and as noted in (i). The eigenvalues of s are either $+1$ or -1 but by (iii) there must be an even number of -1's and this number is less than 6. For these involutions s, we have $|C_V(s)|=5^4$ or 5^2, and so (iv) is true.

(v) The only primes that divide the order of G_0 are 2, 3 and 31.

Note that $b=v(v-1)/(k(k-1))=2^2\cdot 3\cdot 5^6\cdot 31$ and G is block-transitive. Hence we have $2^2\cdot 3\cdot 31$ divides $|G_0|$. Since G_0 is a soluble irreducible subgroup of $\mathrm{GL}(6,5)$, by Lemma 2.6, we know that the conclusion is true.

(vi) We complete the analysis by showing that G_0 has no elementary abelian minimal normal subgroup, M.

(a) M is not a 31-group.

Assume M is a 31-group. Then by Lemma 2.6 $|M|=31^2$ or 31. If $|M|=31^2$, then there has to be an element of order 31 which fixes more than one point. In fact, if this is not true, then every element of M other than 1 fixes only one point. Thus this point is the vector 0. This implies that M acts semiregularly on $5^6 - 1$ points. Thus $|M|$ divides $5^6 - 1$, and so $|M| = 31$. Now let δ be an element of order 31 of M which fixes at least two points. Let x, y be any two points in $\mathrm{Fix}(\langle\delta\rangle)$, and let B be the block containing them. Since $31 > 7$, $\langle\delta\rangle$ fixes every point of B. Hence $|\mathrm{Fix}(\langle\delta\rangle)| = 5^e$, where $1 < e < 6$. Therefore, $\mathrm{Fix}(\langle\delta\rangle)$ is the point-set of a $2-(|\mathrm{Fix}(\langle\delta\rangle)|, 7, 1)$ design. By Lemma 2.1 there are no $2-(5^e, 7, 1)$ designs with $e < 6$, a contradiction. So M is a cyclic group of order 31. Let $M = \langle\delta\rangle$. Then

$$G_0/C_{G_0}(\delta) \leq Aut(M) = Z_{30}. \tag{1}$$

Note that

$$|G| = v|G_0| = b|G_B|,$$

that is,

$$|G_0| = 12 \cdot 31 \cdot |G_B|.$$

Hence 4 divides $|G_0|$, and hence by (1), 2 divides $|C_{G_0}(\delta)|$. Thus there is an involution s which centralizes M. This implies that $\mathrm{Fix}(s)$ is a fixed set of M. By the above argument M fixes only one point, that is, the zero vector of V. Hence M acts on the fixed points of s with only one fixed point. However 31 does not divide $25 - 1$ nor $625 - 1$. So M does not have order a power of 31.

(b) M is not a 3-group.

Let δ be an element of order 31. Then $|\mathrm{Fix}(\langle\delta\rangle)| = 1$ as the proof of (a). Suppose that δ can centralize an element η of order 3. Then $\mathrm{Fix}(\langle\eta\rangle)$ is a fixed set of $\langle\delta\rangle$, and so 31 divides $|\mathrm{Fix}(\langle\eta\rangle)| - 1$. Note that $|\mathrm{Fix}(\langle\eta\rangle)| = 5^e$, where $0 \leq e < 6$. Thus $e = 3$. But 3 divides $5^6 - |\mathrm{Fix}(\langle\eta\rangle)| = 5^6 - 5^3$, which is impossible. Thus δ can not centralize an element of order 3. It follows that 31 does divide $|C_{G_0}(M)|$. Let $|M| = 3^m$. Recall that

$$G_0/C_{G_0}(M) \leq \mathrm{Aut}(M) = \mathrm{GL}(m, 3).$$

Then $31 \mid |\mathrm{GL}(m, 3)|$, and so $|M| \geq 3^{30}$ as 3 is a primitive root modulo 31. This does not exist in $\mathrm{GL}(6, 5)$.

(c) M is not a 2-group.

Assume that there is an element of order 31 which centralizes an element of order 2. Then we can get a contradiction as in the proof of (a). Thus 31 does not divide $|C_{G_0}(M)|$. Let $M = Z_2^m$. Then

$$G_0/C_{G_0}(M) \leq \mathrm{Aut}(M) = \mathrm{GL}(m, 2).$$

Therefore 31 divides $|\mathrm{GL}(m, 2)|$. By Lemma 2.6, we get $m = 5$. Thus G_0 contains a subgroup of type $2^5 : 31$. Such a group is Frobenius and has no representations of degree 6.

Now we completed the proof of the main theorem.

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关于可解区组传递的 2-$(5^6,7,1)$ 设计

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摘要: 设 G 是设计 2-$(5^6,7,1)$ 的一个可解区传递自同构群，则 G 是旗传递的且 $G \leq A\Gamma L(1,5^6)$.

关键词: 设计；区组传递；自同构群；可解群.