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On Weakly Reducible SD-Splittings of Inner Genus 1

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Abstract: Let $(M; H_1, H_2; F_0)$ be a SD-splitting for bordered 3-manifold M. The splitting is reducible (weakly reducible, respectively) if there exist essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ such that $\partial D_1, \partial D_2 \subset F_0$ and $\partial D_1 = \partial D_2$ ($\partial D_1 \cap \partial D_2 = \emptyset$, respectively). A SD-splitting $(M; H_1, H_2; F_0)$ for bordered 3-manifold M is of inner genus 1 if F_0 is a punctured torus. In the present paper, we show that a weakly reducible SD-splitting of inner genus 1 is either reducible or bilongitudional.

Key words: SD-splitting; reducibility; inner genus
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1. Introduction

It is a well known fact that any closed orientable connected 3-manifold M admits a Heegaard splitting $V \cup_F W$, where V, W are handlebodies with the same genus, $V \cup_F W = M$, $V \cap W = \partial V = \partial W = F$. For a bordered compact orientable 3-manifold M, there is a natural way, still called Heegaard splitting of M, to generalize the above splitting, namely $M = V \cup_F W$, $V \cap W = \partial_+ V = \partial_+ W = F$, where V, W are compression bodies.

There is another way, called SD-splitting, to generalize Heegaard splittings for bordered 3-manifold as follows: Any bordered compact orientable 3-manifold M adimits a splitting as $M = V \cup_F W$, where V, W are homeomorphic handlebodies, and the union is made along a connected surface $F \subset \partial V$, ∂W, and ∂F cuts each component of ∂M into two homeomorphic planar surfaces, and some additional condition is satisfied (see section 2 for the definition).

The existence of SD-splittings for a bordered 3-manifold was first proved by Downing[2] in 1970, and Roeling[3] further discussed some properties of genus of such splittings for bordered 3-manifolds with connected boundary. Suzuki[4] reported some results on SD-splittings for bordered 3-manifolds in slightly modified and generalized forms.

Heegaard splittings have been extensively studied so far. However, quite few is known about SD-splittings. In [4] the bordered 3-manifolds with genus 1 D-splittings and SD-splittings are characterized.

In this paper, we show that a weakly reducible SD-splitting of inner genus 1 is either reducible or bilongitudional.

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Some preliminaries are included in Section 2, and the main result and its proof, together with some corollaries, are included in Section 3.

2. Preliminaries

Throughout this paper, we work in the piecewise-linear category. All 3-manifolds are assumed to be compact, connected, orientable and with nonempty boundary. For related manifolds, we assume that they are in general position. For the definitions of incompressible surface, ∂-incompressible surface, irreducible 3-manifold, the connected sum of 3-manifolds, the boundary connected sum of 3-manifolds, etc., we refer to [5].

The following definition is due to Suzuki[3], which is a modified and generalized version of Downing's definition[1]:

Definition 2.1 *For every bordered 3-manifold M with m boundary components B_1, B_2, $\cdots$, B_m, there exist handlebodies H_1 and H_2 in M which satisfy the followings:*

(0) $H_1 \cong H_2$,

(1) $M = H_1 \cup H_2$,

(2) $H_1 \cap H_2 = \partial H_1 \cap \partial H_2 = F_0$ *is a connected surface,*

(3) $H_j \cap B_i = \partial H_j \cap B_i = F_{ji}$ *is a disk with* $g(B_i)$ *holes, and hence,* $F_{1i} \cong F_{2i}$ ($j = 1, 2$; $i = 1, 2, \cdots, m$),

(4) F_{ji} *is incompressible in* H_j ($j = 1, 2; i = 1, 2, \cdots, m$).

We call $M = H_1 \cup_{F_0} H_2$ *a* Downing splitting *(or simply, D-splitting) of genus* $n = g(H_1) = g(H_2)$ *for* M, *and denote it by* $(M; H_1, H_2; F_0)$. *Call* F_0 *a D-splitting surface in* M. *Call the minimum genus of such splittings for* M *the* D-genus *of* M *and denote it by* $D_g(M)$.

We call such a D-splitting for M *a Special Downing splitting (or simply, SD-splitting) of genus* g *for* M, *if the D-splitting* $H_1 \cup_{F_0} H_2$ *satisfies the following additional condition:*

(5) there exists a complete system of disks $\mathcal{D}_| = \{\mathcal{D}_{|\infty}, \cdots, \mathcal{D}_{|\backslash}\}$ *for* H_j *with the property:* $\partial D_{jk} \cap (F_{j1} \cup \cdots \cup F_{jm})$ *consists of at most one simple arc* ($j = 1, 2; k = 1, 2, \cdots, n$), *and the resulting surface obtained by cutting each* F_{ji} *along all these arcs is a disk.*

Call F_0 *a SD-splitting surface in* M. *Call the minimum genus of SD-splittings for* M *the SD-genus of* M *and denote it by* $SD_g(M)$.

Definition 2.2 *Let* $H_1 \cup_{F_0} H_2$ *be a SD-splitting for a bordered 3-manifold* M. $H_1 \cup_{F_0} H_2$ *is weakly reducible if there exist essential disks* $D_1 \subset H_1$ *and* $D_2 \subset H_2$ *with* $\partial D_1, \partial D_2 \subset F_0$ *such that* $\partial D_1 \cap \partial D_2 = \emptyset$ *is reducible if there exist such disks with* $\partial D_1 = \partial D_2$.

Clearly, a stabilized SD-splitting is reducible, and a reducible SD-splitting is weakly reducible.

Definition 2.3 *A SD-splitting* $(M; H_1, H_2; F_0)$ *for bordered 3-manifold* M *is of inner genus 1 if* F_0 *is a punctured torus.*

Definition 2.4 *A SD-splitting* $(M; H_1, H_2; F_0)$ *for bordered 3-manifold* M *is bilongitudinal if* *there exist essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ with $\partial D_1, \partial D_2 \subset F_0$ and a simple closed curve $C \subset F_0$ such that C intersects ∂D_i in one point for each $i = 1, 2$.*

3. Weakly reducible SD-splitting of inner genus 1

Here is the main result of the paper:

Theorem 3.1 *Let $V \cup_{F_0} W$ be a SD-splitting of inner genus 1 for a bordered 3-manifold M. Suppose $V \cup_{F_0} W$ is weakly reducible. Then either $V \cup_{F_0} W$ is reducible or it is bilongitudional.*

Proof With no loss we assume that the boundary of M is connected. (The case that the boundary of M is not connected can be proved similarly.) Suppose the genus of $V \cup_{F_0} W$ (therefore V, W) is $g+1$. Then F_0 is a $(g+1)$-punctured torus. Since $V \cup_{F_0} W$ is weakly reducible, there exist essential disks $(D, \partial D) \subset (V, F_0), (E, \partial E) \subset (W, F_0)$ such that $\partial D \cap \partial E = \emptyset$.

Let $B_1, \cdots, B_{g+1}$ be the $g+1$ boundary components of F_0. Then, by definition, there exists a complete disk system $\mathcal{D} = \{D_1, \cdots, D_g, D_{g+1}\}$ of V, such that $D_i \cap F_0 = a_i$ is a proper arc in F_0, $1 \le i \le g$, and the surface obtained by cutting F_0 open along $\{a_i, 1 \le i \le g\}$ is a once-punctured torus in which D_{g+1} lies. Thus, we may assume, without loss of generality, that a_i connects B_i to B_{i+1}, $1 \le i \le g$. See Figure 1 below.

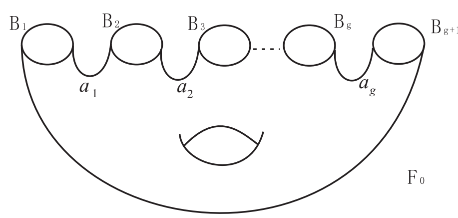

Figure 1

We divide it into two cases to discuss:

Case 1. One of ∂D and ∂E, say ∂D, is separating in F_0.

In the case, we claim that ∂D cuts F_0 into a once punctured torus and a $(g+3)$-punctured sphere.

Suppose ∂D cuts F_0 into a punctured torus T_0 and a punctured sphere S_0, and there is a partition $\{B_{i_1}, \cdots, B_{i_k}\} \cup \{B_{i_{k+1}}, \cdots, B_{i_{g+1}}\}$ of ∂F_0 such that $B_{i_1}, \cdots, B_{i_k} \subset \partial T_0$ and $B_{i_{k+1}}, \cdots, B_{i_{g+1}} \subset \partial S_0$, $1 \le k \le g+1$. By our choice, there must be some a_j $(1 \le j \le g)$ which connects some B_{i_p} lying in ∂T_0 to some B_{i_q} lying in ∂S_0 on F_0. Since ∂D is separating in F_0, the number of the intersection points of a_j and ∂D is odd. Denote $\{D_1, \cdots, D_g\}$ by $\mathcal{D}'$. On the other hand, we may assume that D and $\mathcal{D}'$ are in general position, thus the number of $\partial D \cap \partial D_j = \partial D \cap a_j$, which are the end points of a collection of pairwise disjoint simple arcs on D, is even. The contradiction shows that the claim holds.

If ∂E is separating in F_0, by a similar argument as above, we can show that ∂D cuts F_0 into a once punctured torus and a $(g+3)$-punctured sphere. Since $\partial D \cap \partial E = \emptyset$, it is easy to see that ∂E is parallel to ∂D in F_0. Thus $(V, W; F_0)$ is reducible.

If ∂E is nonseparating in F_0, then ∂E is a non-separating curve on T_0. Since E is an essential disk in W, we can see ∂D bounds a disk in W. Thus $(V, W; F_0)$ is again reducible.

Case 2. Both ∂D and ∂E are nonseparating in F_0.

There are two subcases:

Subcase 1. ∂D and ∂E are parallel on F_0. In this subcase, $(V, W; F_0)$ is reducible.

Subcase 2. ∂D and ∂E are not parallel on F_0. If this subcase happens, the surface obtained by cutting F_0 open along ∂D is a $(g+3)$-punctured sphere, and the surface obtained by cutting F_0 open along $\partial D \cup \partial E$ are two connected punctured spheres S_1 and S_2. Moreover, there is a partition $\{B_{i_1}, \cdots, B_{i_k}\} \cup \{B_{i_{k+1}}, \cdots, B_{i_{g+1}}\}$ of ∂F_0 such that $B_{i_1}, \cdots, B_{i_k} \subset \partial S_1$ and $B_{i_{k+1}}, \cdots, B_{i_{g+1}} \subset \partial S_2$, $1 \leq k \leq g+1$. See Figure 2 below:

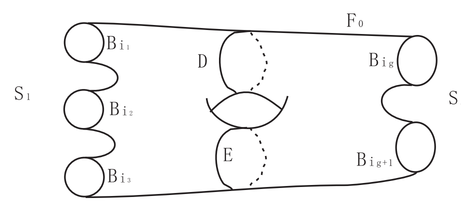

Figure 2

Clearly, there exists a simple closed curve C on F_0 such that C intersects both ∂D and ∂E in a single point. Thus the SD-splitting $(V, W; F_0)$ is bilongitudional.

This finishes the proof of Theorem 3.1.

Next we give an application of Theorem 3.1. First, we prove the following lemma.

Lemma 3.1 *Let M be a bordered 3-manifold with connected boundary S of genus g. Then M has a SD-splitting of genus g if and only if M is a handlebody of genus g.*

Proof The proof is done by induction on genus g. If $g = 0$, then it is clear that M is a 3-ball. If $g = 1$, by Theorem 2.1, M is a solid torus.

Now we consider the case $g > 1$. Let $(V, W; F_0)$ be the SD-splitting of genus g in M. Then F_0 is a $(g+1)$-punctured 2-sphere. Let $V \bigcup_{\partial V \setminus F_0} V = M'$. By $g(S) = g$ and the condition (5) in Definition 2.1, there exists a complete system of meridian-disks $\mathcal{D} = \{D_1, \cdots, D_g\}$ of V such that $\partial D_i \cap (\partial V \setminus F_0) = \alpha_i$ is a simple arc, $i = 1, 2, \cdots, g$, and $(\partial V \setminus F_0) \setminus (\alpha_1 \cup \cdots \cup \alpha_g)$ is a disk. Since $\partial V \setminus F_0$ is incompressible in V(or M'), it is ∂-compressible in M'. Let D_i be a ∂-compression disk of $\partial V \setminus F_0$ in M', $\beta_i = D_i \cap F_0$, $\alpha_i = D_i \cap (\partial V \setminus F_0)$, and $\alpha_i \cup \beta_i = \partial D_i$, $i = 1, 2, \cdots, g$. Note that α_i connects the distinct boundary components of $\partial V \setminus F_0$. ∂-compress $\partial V \setminus F_0$ along $D_i (i \leq i \leq g)$ in M' to get a disk B^2 and compress V along $\mathcal{D}$ to get a 3-ball B^3. The resulting 3-manifold is $B^3 \#_{B^2} B^3$, where # denotes the boundary connected sum. It is clear that B^2 is ∂-parallel in $B^3 \#_{B^2} B^3$. Then $\partial V \setminus F_0$ is ∂-parallel in M', which implies that $\partial V \setminus F_0$ is parallel to F_0. Similarly, F_0 is parallel to $\partial W \setminus F_0$. Then $V \cong F_0 \times I \cong W$. So $V \bigcup_{F_0} W = M$ is homeomorphic to a handlebody of genus g.

The other direction is obvious. This finishes the proof of Lemma 3.1.

Now we come to

Corollary 3.1 *Let M be a bordered 3-manifold with connected boundary S of genus g. Suppose that M admits a weakly reducible SD-splitting of inner genus 1 which are not bilongitudinal. Then M is either H_g, or $L(p,q)\#H_g$, or $S^2\times S^1\#H_g$, where H_g is a handlebody of genus g.*

Proof Let $(V,W;F_0)$ be a SD-splitting of inner genus 1 for the bordered 3-manifold M. By assumption, $(V,W;F_0)$ is not bilongitudional, so from Theorem 3.1 we know that $(V,W;F_0)$ is reducible. Thus, there exist essential disks $D\subset V$ and $E\subset W$ with $\partial D=\partial E$. Let $S=D\cup E$. Then S is an essential 2-sphere in M. If S separates M, then $M=M_1\#_S M_2$, i.e. M is a connected sum of M_1 and M_2, where M_1 is a closed 3-manifold with a genus 1 Heegaard splitting, and M_2 is a bordered 3-manifold with connected boundary of genus g which admits a SD-splitting of genus g. Thus M_1 is either S^3, or $L(p,q)$, or $S^2\times S^1$, and by Lemma 3.1, M_2 is a handlebody H_g of genus g. So M is either H_g, or $L(p,q)\#H_g$, or $S^2\times S^1\#H_g$.

If S is nonseparating in M, we can choose an essential separating 2-sphere in M with similar property as S. By applying the above argument, the conclusion follows.

This completes the proof of Corollary 3.1.

References:

[1] DOWNING J S. *Decomposing compact 3-manifolds into homeomorphic handlebodies* [J]. Proc. Amer. Math. Soc., 1970, **24**: 241–244.

[2] ROELING L G. *The genus of an orientable 3-manifold with connected boundary* [J]. Illinois J. Math., 1973, **17**: 558–562.

[3] SUZUKI S. *Handlebody splittings of compact 3-manifolds with boundary* [C]. Proceedings of Knot Theory 1999 in Toronto.

[4] LEI Feng-chun, YANG Li. *SD-splittings for some bordered 3-manifolds* [J]. Acta Math. Sinica,

[5] JACO W. *Lectures on Three-Manifold Topology* [M]. CBMS Regional Conference Series in Mathematics, 43. American Mathematical Society, Providence, R.I., 1980.

[6] CASSON A J, GORDON C M. *Reducing Heegaard splitting* [J]. Topology Appl., 1987, **27**: 275–283.

关于内亏格 1 的弱可约 SD- 分解

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摘要: 设 $(M;H_1,H_2;F_0)$ 为带边 3- 流形 M 的一个 SD- 分解. 称该分解为可约的(或弱可约的) 若存在本质圆片 $D_1\subset H_1$，$D_2\subset H_2$ 使得 $\partial D_1,\partial D_2\subset F_0$ 并且 $\partial D_1=\partial D_2$ (或 $\partial D_1\cap\partial D_2=\emptyset$). 称 $(M;H_1,H_2;F_0)$ 为内亏格 1 若 F_0 为穿孔环面. 本文主要结果：一个弱可约的内亏格 1 的 SD-分解或是可约的或是双经的.

关键词: SD- 分解；可约性；内亏格.