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Asymptotic Behavior of Solutions to Equations Modelling Non-Newtonian Flows

YIN Gu-liang[1], *DONG Bo-qing*[2]

(1. College Mathematical and Information Sciences, Wenzhou University, Zhejiang 325000, China;
2. School of Mathematical Sciences, Nankai University, Tianjin 300071, China)
(E-mail: ygl1818@163.com)

Abstract: This paper is concerned with the system of equations that model incompressible non-Newtonian fluid motion with p-growth dissipative potential $1+\frac{2n}{n+2} \le p < 3$ in R^n ($n=2,3$). Using the improved Fourier splitting method, we prove that a weak solution decays in L^2 norm at the same rate as $(1+t)^{-n/4}$ as the time t approaches infinity.

Key words: asymptotic behavior; non-Newtonian flows; Fourier splitting.
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1. Introduction

In this paper we consider the L^2 time decay for weak solutions to the incompressible non-Newtonian fluids governed by the following momentum and continuity equations

$$u_t+(u\cdot\nabla)u-\nabla\cdot\tau^v+\nabla\pi=0 \ \text{ in } \ \mathbf{R}^n\times(0,\infty), \tag{1.1}$$

$$\nabla\cdot u=0 \ \text{ in } \ \mathbf{R}^n\times(0,\infty), \tag{1.2}$$

$$u(x,0)=u_0 \ \text{ in } \ \mathbf{R}^n, \tag{1.3}$$

where $x=(x_1,x_2\cdots x_n)$ is the spatial variables with $n=2,3$, the gradient $\nabla=(\partial_{x_1},\cdots,\partial_{x_n})$, and the unknown $u=(u_1,\cdots,u_n)$ and π denote the velocity and pressure of the fluid motion, respectively. The stress tensor $\tau=(\tau_{ij})$ is specified in the form

$$\tau_{ij}^v=2(\mu_0+\mu_1|e(u)|^{p-2})e_{ij}(u), \tag{1.4}$$

where the viscous coefficients μ_0,μ_1 are positive constants. And the component of the symmetric deformations velocity tensor is given by

$$e(u)_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \qquad |e(u)|=(e_{ij}(u)e_{ij}(u))^{\frac{1}{2}}.$$

When $p=2$, the system (1.1)-(1.4) turns out to be the famous Navier-Stokes equations[9,13].

There is an extensive literature on the solutions of the incompressible non-Newtonian fluids. Ladyzhenskaya[8] and Lions[10] first discussed the existence and uniqueness of weak solutions of

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the sort model, and more recently, Du and Gunzguiger[6] studied the somewhat more general existence and uniqueness results in bounded domains. Pokorny[11] also investigated the Cauchy problem of (1.1)–(1.4) in whole spaces. Beirão da Veiga[1] and Guo and Zhu[7] examined the regularity of weak solutions of (1.1)–(1.4). One may also consult with Bellout et al.[2] and Dong[4,5] for a study on the other nonlinear viscous fluids. Moreover, based on the Fourier splitting method[12,14], the decay rates of the non-Newtonian fluid flows in R^n with high dissipative potential ($p \geq 3$) were recently derived by Guo and Zhu[7] as

$$\|u(t)\|_{L^2} \leq ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}, \quad n \geq 2,$$
$$\|u(t) - e^{t\Delta}u_0\|_{L^2} \to 0, \quad t \to \infty.$$

By developing the Fourier splitting method, the optimal algebraic decay rate in $\mathbf{R}^2$ is derived by Dong and Li[3] as

$$\|u(t)\|_{L^2} \leq c(1+t)^{-\frac{1}{2}}, \quad \|u(t) - e^{t\Delta}u_0\|_{L^2} \leq c(1+t)^{-\frac{3}{4}}, \quad \forall\, t > 0.$$

It should be mentioned that in [3,7], the bounds of p-power ($p \geq 3$) potential is obtained by the energy inequality, whereas the bounds are not applicable for the case $1 + \frac{2n}{n+2} \leq p < 3$. The aim of this paper is to investigate the decay problem of this sort fluids with p-growth dissipative potential for the case $1 + \frac{2n}{n+2} \leq p < 3$. With the aid of the improved Fourier splitting method and some inequalities, we prove that a weak solution decays in L^2 norm at $(1+t)^{-n/4}$ as the time t approaches infinity, which is the same as the decay rates of the solution to the heat equation.

The remains of this paper are organized as follows: In Section 2, we introduce the mathematical preliminaries and state the main results; We present some auxiliary lemmas in Section 3, and prove the main results in the Section 4.

2. Preliminaries and main results

Let $\|\cdot\|_q = \|\cdot\|_{L^q}$ ($\|\cdot\| = \|\cdot\|_2$) be the norm of the Lebesgue space $L^q(\mathbf{R}^n)$ and $\|\cdot\|_{m,q} = \|\cdot\|_{W^{m,q}}$ be the norm of the Sobolev space $W^{m,q}(\mathbf{R}^n)$. The space $L^q_\sigma(\mathbf{R}^n)^n$ denotes the L^q-closure of $C^\infty_{0,\sigma}(\mathbf{R}^n)^n$, which is the set of smooth divergence-free vector fields with compact supports in $\mathbf{R}^n$. The space $W^{1,q}_{0,\sigma}(\mathbf{R}^n)^n$ denotes the closure of $C^\infty_{0,\sigma}(\mathbf{R}^n)^n$ in $W^{1,q}(\mathbf{R}^n)^n$. C or $c > 0$, independent of the quantities t, x, u and v, is a generic constant, which may depend on the initial data u_0. The Fourier transformation of a function f is denoted by$\hat{f}$ or $F[f]$.

Without loss of generality, we assume that $\mu_0 = \mu_1 = 1$ in (1.5). Substitution of (1.4) into (1.1) produces

$$u_t - \triangle u + (u \cdot \nabla)u - \nabla \cdot (|e(u)|^{p-2}e(u)) + \nabla \pi = 0. \tag{2.1}$$

By a weak solution of the initial value problem (1.2–1.4, 2.1), we mean a function $u(x,t)$ which is as follows.

Definition 2.1[10] *Let $u_0 \in W^{1,2} \cap L^2_\sigma$ and $p \geq 1 + \frac{2n}{n+2}$. Then a function $u(x,t)$ where*

$$u \in L^p((0,T); W^{1,p}) \cap C([0,T); L^2_\sigma) \cap L^2((0,T); W^{1,2})$$

$$\frac{\partial u}{\partial t} \in L^2((0,T);L^2_\sigma) \quad \textit{for} \;\; \forall\, T>0$$

is called a weak solution of the problem (1.2–1.4, 2.1) if

$$\int_{\mathbf{R}^n} u(t)\cdot\varphi(t)\,\mathrm{d}x - \int_0^t\int_{\mathbf{R}^n} u\cdot\frac{\partial\varphi}{\partial s}\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\mathbf{R}^n} u_j\frac{\partial u_i}{\partial x_j}\varphi_i\,\mathrm{d}x\mathrm{d}s +$$
$$\int_0^t\int_{\mathbf{R}^n}(1+|e(u)|^{p-2})e_{ij}(u)\cdot e_{ij}(\varphi)\mathrm{d}x\mathrm{d}s - \int_{\mathbf{R}^n} u_0\cdot\varphi(0)\,\mathrm{d}x = 0 \tag{2.2}$$

for every $\varphi\in C^\infty([0,\infty);C^\infty_{0,\sigma}(\mathbf{R}^n)^n)$ vanishing near $t=\infty$. Moreover, the weak solution satisfies the following energy inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}|u|^2\mathrm{d}x + \int_{\mathbf{R}^n}|\nabla u|^2 + \int_{\mathbf{R}^n}|\nabla u|^p\mathrm{d}x \le 0. \tag{2.3}$$

Theorem 2.1[11] *Let $u_0\in W^{1,2}\cap L^2_\sigma$ and $p\ge 1+\frac{2n}{n+2}$, $n=2,3$. Then there exists a weak solution of the system (1.2–1.4, 2.1) in the sense of Definition 2.1. Moreover, the solution is regular, i.e.*

$$u\in L^\infty((0,T);W^{1,2})\cap L^2((0,T);W^{2,2})\cap L^p((0,T);W^{1,3p}). \tag{2.4}$$

Theorem 2.2 (Main Theorem) *Assume $u_0\in W^{1,2}\cap L^2_\sigma\cap L^1$ $(n=2,3)$ and let u be a weak solution of the problem (1.2–1.4, 2.1), then*

$$\|u(t)\|\le C(1+t)^{-\frac{1}{2}} \quad \text{if} \;\; n=2, 2<p<3; \tag{2.5}$$

$$\|u(t)\|\le C(1+t)^{-\frac{3}{4}} \quad \text{if} \;\; n=3, \frac{11}{5}\le p<3. \tag{2.6}$$

3. Some auxiliary lemmas

In this section, we recall and prove some lemmas, which will be employed in the proof of the theorems.

Lemma 3.1 (Gagliardo-Nirenberg Inequality) *For all $1\le p,\ q,\ r\le\infty$ and for all integer $n\ge 1, m>k\ge 0$, there exist two constants $0\le\alpha\le 1, C>1$, such that for all $u\in C^\infty_0(\mathbf{R}^n)$*

$$\|\nabla^k u\|_p \le C\|\nabla^m u\|_r^\alpha\|u\|_q^{1-\alpha}, \quad \textit{for} \;\; \frac{1}{p}-\frac{k}{n}=\alpha(\frac{1}{r}-\frac{m}{n})+\frac{1}{q}(1-\alpha). \tag{3.1}$$

The only exception is that $\alpha\ne 1$, if $m-\frac{n}{r}=k, 1<r<\infty$.

Lemma 3.2 (Gronwall's Inequality) *Let $f(t), g(t), h(t)$ be nonnegative continuous functions and satisfy the inequality*

$$g(t)\le f(t)+\int_0^t g(s)h(s)\mathrm{d}s, \quad \forall\, t>0,$$

where $f'(t)\ge 0$. Then

$$g(t)\le f(t)\exp\left(\int_0^t h(s)\mathrm{d}s\right), \quad \forall\, t>0. \tag{3.2}$$

Lemma 3.3 *Assume $u_0 \in W^{1,2} \cap L^2_\sigma \cap L^1$, and let u be a weak solution of (1.2–1.4, 2.1). Then we have*

(I) $\sup_{0\le t\le\infty} \|u(t)\| \le \|u_0\|$; (3.3)

(II) For $n=2, 2<p<3$,

$$|\hat{u}(\xi,t)| \le \|u_0\|_{L^1} + C|\xi|\int_0^t \|u(s)\|^2 \mathrm{d}s + C|\xi|\left(\int_0^t \|u(s)\|^{\frac{2}{4p-2}}\mathrm{d}s\right)^{\frac{4-p}{2}} \tag{3.4}$$

$$\le C + C|\xi|t + C|\xi|t^{\frac{4-p}{2}}; \tag{3.5}$$

(III) For $n=3, \frac{11}{5}\le p<3$,

$$|\hat{u}(\xi,t)| \le C + C|\xi|\int_0^t \|u(s)\|^2 \mathrm{d}s + C|\xi|\left(\int_0^t \|u(s)\|^{\frac{14-2p}{19-5p}}\mathrm{d}s\right)^{\frac{19-5p}{8}}. \tag{3.6}$$

Proof From the energy Inequality (2.3), it is easy to get (3.3). To obtain (3.4)–(3.6), applying the Fourier transformation of (2.1) yields

$$\hat{u}_t + |\xi|^2\hat{u} = F[\nabla\cdot(|e(u)|^{p-2}e(u)) - (u\cdot\nabla)u - \nabla\pi] =: G(\xi,t). \tag{3.7}$$

Now we need to estimate $G(\xi,t)$. First

$$\begin{aligned}|F[(u\cdot\nabla)u]| = |F[\mathrm{div}(u\otimes u)]| &\le \sum_{i,j}\int_{\mathbf{R}^n} |u_iu_j||\xi_j|\mathrm{d}x\\ &\le \sum_{i,j}|\xi_j|\|u_i\|\|u_j\| \le |\xi|\|u\|^2.\end{aligned} \tag{3.8}$$

According to Lemma 3.1, when $n=2$,

$$\begin{aligned}|F[\nabla\cdot(|e(u)|^{p-2}e(u))]| \le |\xi||F[|e(u)|^{p-2}e(u)]| &\le |\xi|\int_{\mathbf{R}^n}|\nabla u|^{p-1}\mathrm{d}x\\ &\le C|\xi|\|u\|\|\nabla^2 u\|^{p-2},\end{aligned} \tag{3.9}$$

and when $n=3$,

$$\begin{aligned}|F[\nabla\cdot(|\nabla u|^{p-2}Du)]| \le |\xi||F[|e(u)|^{p-2}e(u)]| &\le |\xi|\int_{\mathbf{R}^n}|\nabla u|^{p-1}\mathrm{d}x\\ &\le C|\xi|\|u\|^{\frac{7-p}{4}}\|\nabla^2 u\|^{\frac{5p-11}{4}}.\end{aligned} \tag{3.10}$$

Take divergence in (2.1) to get

$$\Delta\pi = \sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}[-u_iu_j + |e(u)|^{p-2}e_{ij}(u)],$$

and the Fourier transformation to set

$$|\xi|^2F[\pi] = \sum_{i,j}\xi_i\xi_jF[-u_iu_j + |e(u)|^{p-2}e_{ij}(u)].$$

Thus we have

$$|F[\nabla\pi]| \le |F[\nabla\cdot(|\nabla u|^{p-2}Du)]| + |F[(u\cdot\nabla)u]|. \tag{3.11}$$

So combining (3.7)–(3.11) shows that

$$|G(\xi,t)| \le C|\xi|\|u\|^2 + C|\xi|\|u\|\|\nabla^2 u\|^{p-2}, \quad \text{for} \quad n=2, \tag{3.12}$$

$$|G(\xi,t)| \le C|\xi|\|u\|^2 + C|\xi|\|u\|^{\frac{7-p}{4}}\|\nabla^2 u\|_p^{\frac{5p-11}{4}}, \quad \text{for} \quad n=3. \tag{3.13}$$

Moreover, from (3.7), it follows easily that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{u}e^{-|\xi|^2 t}\right) \le G(\xi,t)e^{-|\xi|^2 t}.$$

Integrating in time gives

$$\hat{u}(\xi,t) \le e^{-|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t G(\xi,t)e^{-|\xi|^2(t-s)}\mathrm{d}s.$$

According to Theorem 2.1, we may as well assume that $\int_0^\infty \|\nabla^2 u(t)\|^2\mathrm{d}t < C$, and C is independence of time. Therefore from (3.12), (3.13) we have

(I) when $n=2, 2<p<3$,

$$\begin{aligned}
|\hat{u}(\xi,t)| &\le |\hat{u}_0(\xi)| + C\int_0^t |\xi|\|u(s)\|^2\mathrm{d}s + C\int_0^t |\xi|\|u(s)\|\|\nabla^2 u(s)\|^{p-2}\mathrm{d}s\\
&\le \|u_0\|_{L^1} + C|\xi|\int_0^t \|u(s)\|^2\mathrm{d}s + C|\xi|\left(\int_0^t \|u(s)\|^{\frac{2}{4-p}}\mathrm{d}s\right)^{\frac{4-p}{2}}\left(\int_0^t \|\nabla^2 u(s)\|^2\mathrm{d}s\right)^{\frac{p-2}{2}}\\
&\le \|u_0\|_{L^1} + C|\xi|\int_0^t \|u(s)\|^2\mathrm{d}s + C|\xi|\left(\int_0^t \|u(s)\|^{\frac{2}{4-p}}\mathrm{d}s\right)^{\frac{4-p}{2}}\\
&\le C + C|\xi|t + C|\xi|t^{\frac{4-p}{2}}.
\end{aligned}$$

(II) When $n=3, \frac{11}{5} \le p < 3$,

$$\begin{aligned}
|\hat{u}(\xi,t)| &\le |\hat{u}_0(\xi)| + C\int_0^t |\xi|\|u(s)\|^2\mathrm{d}s + C\int_0^t |\xi|\|u(s)\|^{\frac{7-p}{4}}\|\nabla^2 u(s)\|^{\frac{5p-11}{4}}\mathrm{d}s\\
&\le \|u_0\|_{L^1} + C|\xi|\int_0^t \|u(s)\|^2\mathrm{d}s + C|\xi|\left(\int_0^\infty \|\nabla^2 u(t)\|^2\mathrm{d}t\right)^{\frac{5p-11}{8}}\left(\int_0^t \|u(s)\|^{\frac{14-2p}{19-5p}}\mathrm{d}s\right)^{\frac{19-5p}{8}}\\
&\le \|u_0\|_{L^1} + C|\xi|\int_0^t \|u(s)\|^2\mathrm{d}s + C|\xi|\left(\int_0^t \|u(s)\|^{\frac{14-2p}{19-5p}}\mathrm{d}s\right)^{\frac{19-5p}{8}}.
\end{aligned}$$

Thus the proof of this lemma is completed.

4. Proof of the main theorem

From the energy inequality (2.3), it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}|u|^2\mathrm{d}x + \int_{\mathbf{R}^n}|\nabla u|^2 \le 0. \tag{4.1}$$

Applying Plancherel's theorem to (4.1) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}|\hat{u}(\xi,t)|^2\mathrm{d}\xi+\int_{\mathbf{R}^n}|\xi|^2|\hat{u}(\xi,t)|^2d\xi\le 0. \tag{4.2}$$

Let $f(t)$ be a continuous function of t with $f(0)=1, f(t)>0$ and $f'(t)>0$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)\int_{\mathbf{R}^n}|\hat{u}(\xi,t)|^2\mathrm{d}\xi\right)+2f(t)\int_{\mathbf{R}^n}|\xi|^2|\hat{u}(\xi,t)|^2\mathrm{d}\xi\le f'(t)\int_{\mathbf{R}^n}|\hat{u}(\xi,t)|^2\mathrm{d}\xi.$$

Let $B(t)=\{\xi\in\mathbf{R}^n: 2f(t)|\xi|^2\le f'(t)\}$, then

$$\begin{aligned}2f(t)\int_{\mathbf{R}^n}|\xi|^2|\hat{u}(\xi,t)|^2\mathrm{d}\xi&=2f(t)\int_{B(t)}|\xi|^2|\hat{u}(\xi,t)|^2\mathrm{d}\xi+2f(t)\int_{B(t)^c}|\xi|^2|\hat{u}(\xi,t)|^2\mathrm{d}\xi\\&\ge 2f(t)\int_{B(t)^c}|\xi|^2|\hat{u}(\xi,t)|^2\mathrm{d}\xi\ge f'(t)\int_{\mathbf{R}^n}|\hat{u}(\xi,t)|^2\mathrm{d}\xi-f'(t)\int_{B(t)}|\hat{u}(\xi,t)|^2\mathrm{d}\xi.\end{aligned}$$

Therefore we have the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)\int_{\mathbf{R}^n}|\hat{u}(\xi,t)|^2\mathrm{d}\xi\right)\le f'(t)\int_{B(t)}|\hat{u}(\xi,t)|^2\mathrm{d}\xi.$$

Integrating in time yields

$$f(t)\int_{\mathbf{R}^n}|\hat{u}(\xi,t)|^2\mathrm{d}\xi\le\int_{\mathbf{R}^n}|\hat{u}_0|^2\mathrm{d}\xi+C\int_0^t f'(s)\int_{B(s)}|\hat{u}(\xi,s)|^2\mathrm{d}\xi\mathrm{d}s. \tag{4.3}$$

(I) The Case $n=2, 2<p<3$.

According to (3.4) and (4.3), let $A^2=\frac{f'(t)}{2f(t)}$, and ω_n be area of unit ball in $\mathbf{R}^2$, then we have

$$\begin{aligned}&f(t)\int_{\mathbf{R}^2}|\hat{u}(\xi,t)|^2\mathrm{d}\xi\le\int_{\mathbf{R}^2}|\hat{u}_0|^2\mathrm{d}\xi+\\&\quad C\omega_n\int_0^t f'(s)\int_0^A\left(\|u_0\|+r\int_0^s\|u(\tau)\|^2\mathrm{d}\tau+r\left(\int_0^s\|u(\tau)\|^{\frac{2}{4-p}}\mathrm{d}\tau\right)^{\frac{4-p}{2}}\right)^2 r\mathrm{d}r\mathrm{d}s\\&\quad\le C+C\int_0^t f'(s)\left(\frac{f'(s)}{2f(s)}+\left(\frac{f'(s)}{2f(s)}\right)^2\left(\int_0^s\|u(\tau)\|^2\mathrm{d}\tau\right)^2\right)\mathrm{d}s+\\&\quad C\int_0^t f'(s)\left(\left(\frac{f'(s)}{2f(s)}\right)^2\left(\int_0^s\|u(\tau)\|^{\frac{2}{4-p}}\mathrm{d}\tau\right)^{4-p}\right)\mathrm{d}s.\end{aligned} \tag{4.4}$$

Applying (3.5) to (4.4) yields

$$\begin{aligned}f(t)\int_{\mathbf{R}^n}|\hat{u}(\xi,t)|^2\mathrm{d}\xi&\le\int_{\mathbf{R}^n}|\hat{u}_0|\mathrm{d}\xi+C\omega_n\int_0^t f'(s)\int_0^A\left(r+r^3s^2+r^3s^{4-p}\right)\mathrm{d}r\mathrm{d}s\\&\le C+C\int_0^t f'(s)\left(\frac{f'(s)}{2f(s)}+\left(\frac{f'(s)}{2f(s)}\right)^2(s^2+s^{4-p})\right)\mathrm{d}s.\end{aligned}$$

Let $f(t)=(\ln(e+t))^3$, then $f'(t)=\frac{3(\ln(e+t))^2}{e+t}, \frac{f'(t)}{2f(t)}=\frac{3}{(e+t)\ln(e+t)}$. By Plancherel's theorem, and simple directly calculating, we have

$$\|u(t)\|\le C(\ln(e+t))^{-1}. \tag{4.5}$$

Note that $\int_0^t(\ln(e+s))^{-m}\mathrm{d}s \le C_m(t+e)\ln(e+t))^{-m}$. Therefore from (4.4) and (4.5), one shows that

$$\|u(x,t)\| \le C(\ln(e+t))^{-2}.$$

By iteration step by step,

$$\|u(t)\| \le C(\ln(e+t))^{-m}, \quad \text{for } \ \forall\, m \in \mathbf{N}. \tag{4.6}$$

Applying (4.4) and Hölder Inequality yields

$$\begin{aligned} f(t)\int_{\mathbf{R}^2}|\hat{u}(\xi,t)|^2\mathrm{d}\xi \le & C + C\int_0^t f'(s)\frac{f'(s)}{2f(s)}\mathrm{d}s + \int_0^t sf'(s)\left(\frac{f'(s)}{2f(s)}\right)^2\int_0^t\|u(s)\|^4\mathrm{d}s + \\ & C\int_0^t s^{\frac{4-p}{2}}f'(s)\left(\frac{f'(s)}{2f(s)}\right)^2\left(\int_0^t\|u(s)\|^{\frac{4}{4-p}}\mathrm{d}s\right)^{\frac{4-p}{2}}\mathrm{d}s. \end{aligned}$$

Let $f(t) = (1+t)^2$, then

$$\begin{aligned} (1+t)^2\int_{\mathbf{R}^2}|\hat{u}(\xi,t)|^2\mathrm{d}\xi \le & C(1+t) + C(1+t)\int_0^t\|u(s)\|^4\mathrm{d}s + \\ & C(1+t)^{\frac{4-p}{2}}\left(\int_0^t\|u(s)\|^{\frac{4}{4-p}}\mathrm{d}s\right)^{\frac{4-p}{2}}. \end{aligned}$$

Noting $\frac{1}{2} < \frac{4-p}{2} < 1$ and using the Young Inequality to the last term yield

$$\left(\int_0^t\|u(s)\|^{\frac{4}{4-p}}\mathrm{d}s\right)^{\frac{4-p}{2}} \le C\int_0^t\|u(s)\|^{\frac{4}{4-p}}\mathrm{d}s + C.$$

Thus from (4.4), we get the following inequality

$$\begin{aligned} & (1+t)\int_{\mathbf{R}^2}|\hat{u}(\xi,t)|^2\mathrm{d}\xi \le C + \\ & C\int_0^t\|u(s)\|^2(1+s)\left((1+s)^{-1}(\ln(e+s))^{-m} + (1+s)^{-1}(\ln(e+s))^{-\frac{m(2p-4)}{4-p}}\right)\mathrm{d}s. \end{aligned}$$

Let

$$g(t) = (1+t)\int_{\mathbf{R}^2}|\hat{u}(\xi,t)|^2\mathrm{d}\xi = (1+t)\int_{\mathbf{R}^n}|u(x,t)|^2,$$

$$h(t) = (1+t)^{-1}(\ln(e+t))^{-m} + (1+t)^{-1}(\ln(e+t))^{-\frac{m(2p-4)}{4-p}}.$$

When m is suitable large, it is simple to deduce that $h(t)$ satisfies $\int_0^\infty h(t)\mathrm{d}t < \infty$, so by Lemma 3.2, we have

$$g(t) \le C\exp\left(\int_0^\infty h(t)\mathrm{d}t\right) \le C.$$

So

$$\|u(t)\| \le C(1+t)^{-\frac{1}{2}}.$$

(II) The Case $n = 3, \frac{11}{5} \le p < 3$.

From (3.6) and (4.3), similarly, by Hölder Inequality and Young Inequality we have

$$
\begin{aligned}
f(t)\int_{\mathbf{R}^3}|\hat{u}(\xi,t)|^2\mathrm{d}\xi \le \int_{\mathbf{R}^3}|\hat{u}_0|\mathrm{d}\xi+ \\
C\omega_n\int_0^t f'(s)\int_0^A\left(\|u_0\|+r\int_0^s\|u(\tau)\|^2\mathrm{d}\tau+r\left(\int_0^s\|u(\tau)\|^{\frac{14-2p}{19-5p}}\mathrm{d}\tau\right)^{\frac{19-5p}{8}}\right)^2 r^2\mathrm{d}r\mathrm{d}s \\
\le C+C\int_0^t f'(s)\left(\left(\frac{f'(s)}{2f(s)}\right)^{\frac{3}{2}}+\left(\frac{f'(s)}{2f(s)}\right)^{\frac{5}{2}}s\int_0^t\|u(s)\|^4\mathrm{d}s\right)\mathrm{d}s+ \\
C\int_0^t f'(s)\left(\frac{f'(s)}{2f(s)}\right)^{\frac{5}{2}}s^{\frac{19-5p}{8}}\left(\int_0^t\|u(s)\|^{\frac{28-4p}{19-5p}}\mathrm{d}s\right)^{\frac{19-5p}{8}}\mathrm{d}s \\
\le C+C\int_0^t f'(s)\left(\left(\frac{f'(s)}{2f(s)}\right)^{\frac{3}{2}}+\left(\frac{f'(s)}{2f(s)}\right)^{\frac{5}{2}}s\int_0^t\|u(s)\|^4\mathrm{d}s\right)\mathrm{d}s+ \\
C\int_0^t f'(s)\left(\frac{f'(s)}{2f(s)}\right)^{\frac{5}{2}}s^{\frac{19-5p}{8}}\left(\int_0^t\|u(s)\|^{\frac{28-4p}{19-5p}}\mathrm{d}s+C\right)\mathrm{d}s.
\end{aligned}
$$

Note $\frac{1}{2}<\frac{19-5p}{8}\le 1$, $\frac{28-4p}{19-5p}>2$, and let $f(t)=(1+t)^2$, then from (3.3) we have

$$
\begin{aligned}
&(1+t)^2\int_{\mathbf{R}^3}|\hat{u}(\xi,t)|^2\mathrm{d}\xi \\
&\le C(1+t)^{\frac{1}{2}}+C(1+t)^{\frac{1}{2}}\int_0^t\|u(s)\|^4\mathrm{d}s+C(1+t)^{\frac{1}{2}}\int_0^t\|u(s)\|^{\frac{28-4p}{19-5p}}\mathrm{d}s \\
&\le C(1+t)^{\frac{1}{2}}+C(1+t)^{\frac{1}{2}}\int_0^t\|u(s)\|^2\mathrm{d}s.
\end{aligned}
$$

So

$$
(1+t)^{\frac{3}{2}}\int_{\mathbf{R}^3}|\hat{u}(\xi,t)|^2\mathrm{d}\xi\le C+C\int_0^t\{(1+s)^{\frac{3}{2}}\|u(s)\|^2\}(1+s)^{-\frac{3}{2}}\mathrm{d}s.
$$

According to Lemma 3.2, it is easy to deduce

$$
\|u(t)\|\le C(1+t)^{-\frac{3}{4}}.
$$

Hence the proof of theorem is completed.

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一类非牛顿流体模型解的渐近性态

殷谷良[1]，董柏青[2]
(1. 温州大学数学与信息科学学院，浙江 温州 325000; 2. 南开大学数学科学学院，天津 300071)

摘要: 本文主要讨论一类带 $p\ (1+\frac{2n}{n+2} \leq p < 3)$ 幂增长耗散位势的非牛顿流体模型解的渐近性态，利用改进的 Fourier 分解方法，证明了其解在 L^2 范数下衰减率为 $(1+t)^{-\frac{n}{4}}$.

关键词: 渐进性态；非牛顿流体； Fourier 分解.