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Orthodox Semirings with Additive Idempotents Satisfying Permutation Identities

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Abstract: This paper deals with orthodox semirings whose additive idempotents satisfy permutation identities. A structure theorem for such semirings is established.

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1. Introduction and Preliminaries

Throughout this paper, we use the terminologies and notions given in [3]. A semiring S is an algebraic structure $(S,+,\bullet)$ consisting of a non-empty set S together with two binary operations $+$ and $\bullet$ such that $(S,+)$ and $(S,\bullet)$ are semigroups, connected by ring-like distributivity(that is, $x(y+z)=xy+xz, (y+z)x=yx+zx$, for all x,y and z in S). Usually, we write $(S,+,\bullet)$ simply as S, and for any $x,y\in S$, write $x\bullet y$ simply as xy.

An element a of a semiring S is called an idempotent if it satisfies $a+a=aa=a$. A semiring S is an idempotent semiring if all of its elements are idempotents. An idempotent semiring S is called a band semiring[2], if it satisfies the following conditions

$$a+ab+a=a,\quad a+ba+a=a \tag{1.1}$$

for any $a,b\in S$. A T band semiring S is a band semiring such that $(S,+)$ is a T band[2]. In [6], the authors proved that band semirings are always regular band semirings.

Let D be a distributive lattice. For each $\alpha\in D$, let S_α be a semiring and assume that $S_\alpha\cap S_\beta=\emptyset$ if $\alpha\neq\beta$. For each pair $\alpha,\beta\in D$ such that $\alpha\leq\beta$, let $\varphi_{\alpha,\beta}: S_\alpha\longrightarrow S_\beta$ be a semiring homomorphism such that

(1) $\varphi_{\alpha,\alpha}=1_{S_\alpha}$;

(2) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma}=\varphi_{\alpha,\gamma}$, if $\alpha\leq\beta\leq\gamma$;

(3) $\varphi_{\alpha,\beta}$ is injective, if $\alpha\leq\beta$;

(4) $S_\alpha\varphi_{\alpha,\gamma}S_\beta\varphi_{\beta,\gamma}\subseteq S_{\alpha\beta}\varphi_{\alpha\beta,\gamma}$, if $\alpha+\beta\leq\gamma$.

On $S=\cup_{\alpha\in D}S_\alpha$, $+$ and $\bullet$ are defined as follows: For $a\in S_\alpha$ and $b\in S_\beta$,

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(5) $a+b=a\varphi_{\alpha,\alpha+\beta}+b\varphi_{\beta,\alpha+\beta}$;

(6) $ab=(a\varphi_{\alpha,\alpha+\beta}b\varphi_{\beta,\alpha+\beta})\varphi^{-1}_{\alpha\beta,\alpha+\beta}$.

With the above operations, S is a semiring, and each S_α is a subsemiring of S. Write S as $[D;S_\alpha,\varphi_{\alpha,\beta}]$, and call it a strong distributive lattice D of semirings S_α[1].

Let S be a semiring, and A a subset of S. Then A is said to satisfy the permutation identity if

$$(\forall x_1,x_2,\cdots,x_n\in A)\ \ x_1+x_2\cdots+x_n=x_{p_1}+x_{p_2}+\cdots+x_{p_n},$$

where $(p_1p_2\cdots p_n)$ is a nontrivial permutation of $(12\cdots n)$. Yamada[7] investigated the regular semigroups whose idempotents satisfy permutation identities and discussed the structure of such semigroups.

If U is a subsemiring of S, the restriction of the relation $\mathcal{R}$ on S to U will be denoted by $\mathcal{R}_U$. Also, we denote the set of all additive idempotents (if there exist) of a semiring S by E. We remark that E is an ideal of the multiplicative reduct $(S,\bullet)$.

We first recall some results about band semirings.

Theorem 1.1[5] *A semiring S is a rectangular band semiring if and only if S is isomorphic to the direct product of a left zero band semiring and a right zero band semiring.*

Theorem 1.2[5] *A semiring S is a normal band semiring if and only if S is a strong distributive lattice of rectangular band semirings.*

The following result will be used in the sequel.

Theorem 1.3[7] *Let S be a regular semigroup. Then the following statements are equivalent:*

(1) $E(S)$ satisfies a permutation identity;

(2) $E(S)$ is a normal band.

2. Orthodox semirings

A semiring S is *additively regular* if for each element a in S there exists an element a' such that $a=a+a'+a$. If, in addition, the element a' is unique, and satisfies $a'=a'+a+a'$, then S is an additively inverse semiring. Usually, we denote the unique additive inverse of a by a^{-1}. Additively inverse semirings were first studied by Karvellas[4] in 1974, and he proved the following theorem:

Theorem 2.1[4] *Let S be an additively inverse semiring. Then for x,y in S,*

$$(xy)^{-1}=x^{-1}y=xy^{-1},xy=x^{-1}y^{-1}.$$

Definition 2.2 *An additively regular semiring S is an orthodox semiring if E is a band semiring.*

Definition 2.3 *An additively regular semiring S is called a d-inverse semiring if E is a distributive lattice.*

Suppose S is an orthodox semiring. Then we define a relation σ on S as follows:

$$a\sigma b \quad \text{if and only if} \quad V^+(a)=V^+(b),$$

where $V^+(x)$ is the set of all additive inverses of x.

Proposition 2.4 *If S is an orthodox semiring, then σ is a congruence on S and S/σ is a d-inverse semiring.*

Proof Since σ is an inverse semigroup congruence on $(S,+)$, we just need to prove σ is compatible with multiplication. Let $a\sigma b$ and $c\in S$. Then by distributive laws, we can prove easily that $ca'\in V^+(ca)\cap V^+(cb), a'c\in V^+(ac)\cap V^+(bc)$ where a' is an additive inverse of a. It follows that $V^+(ac)=V^+(bc)$ and $V^+(ca)=V^+(cb)$ which mean that σ is a congruence on S. Now we prove S/σ is a d-inverse semiring. Clearly, S/σ is also an orthodox semiring and $E^+(S/\sigma)=\{e\sigma|e\in E^+(S)\}$. Since

$$ef+efe+ef=ef(f+fe+f)=ef,$$

$$efe+ef+efe=ef(e+ef+e)=efe,$$

we have $ef\sigma efe$. Similarly, we can prove $fe\sigma efe$. So $ef\sigma fe$. Also, $(e+ef)\sigma(e+e+ef)\sigma(e+ef+e)=e$. Therefore, $E^+(S/\sigma)$ is a distributive lattice.

3. The quasi-spined product structure

First, we introduce the definition of quasi-spined product.

Let T be a d-inverse semiring whose distributive lattice of additive idempotents is D, $L=[D;L_\alpha,\varphi_{\alpha,\beta}]$ a strong distributive lattice of left zero band semirings L_α, and $R=[D;R_\alpha,\psi_{\alpha,\beta}]$ a strong distributive lattice of right zero band semirings R_α. Let

$$M=\{(e,\xi,f)\in L\times T\times R:\xi\in T, e\in L_{\xi+\xi^{-1}}, f\in R_{\xi^{-1}+\xi}\}.$$

We define addition "+" and multiplication "·" as follows:

$$(e,\xi,f)+(g,\eta,h)=(e+u,\xi+\eta,v+h),$$

$$(e,\xi,f)\cdot(g,\eta,h)=(eg,\xi\eta,fh),$$

where $u\in L_{\xi+\eta+(\xi+\eta)^{-1}}$, $v\in R_{(\xi+\eta)^{-1}+\xi+\eta}$. It is easy to see that the addition and the multiplication above are well defined respectively.

Using Theorem 2.1, we can easily prove the following lemma by simple calculation:

Lemma 3.1 *$(M,+,\cdot)$ is a semiring.*

We call $(M,+,\cdot)$ the quasi-spined product of the left normal band semiring L, the right normal band semiring R and the d-inverse semiring T on the distributive lattice D in this paper. We denote it by $QS(L,R,T;D)$.

Lemma 3.2 *Let $(e,\xi,f),(g,\eta,h)\in QS(L,R,T;D)$. Then (e,ξ,f) is an idempotent if and only* *if ξ is an idempotent of T.*

Theorem 3.3 $S \stackrel{d}{=} QS(L,R,T;D)$ *is an orthodox semirings whose additive idempotents satisfy a permutation and* $S/\sigma \cong T$.

Proof Let $(e,\xi,f),(g,\eta,h) \in E^+(S)$. Then

$$\begin{aligned}(e,\xi,f)+(e,\xi,f)(g,\eta,h)+(e,\xi,f) &= (e,\xi,f)+(eg,\xi\eta,fh)+(e,\xi,f)\\ &= (e+u,\xi+\xi\eta,v+fh)+(e,\xi,f)\\ &= (e,\xi,v+fh)+(e,\xi,f)\\ &= (e,\xi,f).\end{aligned}$$

Similarly, we can prove $(e,\xi,f)+(g,\eta,h)(e,\xi,f)+(e,\xi,f)=(e,\xi,f)$. Also,

$$\begin{aligned}(e,\xi,f)+(g,\eta,h)+(i,\tau,j)+(e,\xi,f) &= (e+u,\xi+\eta+\tau+\xi,v+f)\\ &= (e,\xi,f)+(i,\tau,j)+(g,\eta,h)+(e,\xi,f).\end{aligned}$$

Therefore, S is an orthodox semiring whose additive idempotents satisfy a permutation. Now, we define a mapping $\varphi: S\to T$ by $(e,\xi,f)\varphi=\xi$. Easily, we can prove φ is a surjective semiring homomorphism and $\ker\varphi=\sigma$. That is, $S/\sigma\cong T$.

Theorem 3.4 *If S is an orthodox semiring whose additive idempotents satisfy a permutation identity, then S is isomorphic to $QS(L,R,T;D)$ where D is a distributive lattice, $L=\cup_{\alpha\in D}L_\alpha, R=\cup_{\alpha\in D}R_\alpha$ are left and right normal band semiring respectively, and T is a d-inverse semiring.*

Proof Suppose that S is an orthodox semiring whose additive idempotents satisfy a permutation identity. Then E is a normal band semiring. So from Theorem 1.2, E is a strong distributive lattice $[E/\stackrel{+}{\mathcal{D}}_E;E_\alpha,\theta_{\alpha,\beta}]$ of rectangular band semirings E_α. Let E_α be the direct product $L_\alpha\times R_\alpha$ of a left zero band semiring L_α and a right zero band semiring R_α. Denote $\cup_{\alpha\in D}L_\alpha,\cup_{\alpha\in D}R_\alpha$ by L and R respectively. For any $\alpha,\beta\in D$ with $\alpha\le\beta$, by Corollary IV 3.6 in [3], the additive semigroup homomorphism $\theta_{\alpha,\beta}:E_\alpha\to E_\beta$ determines additive semigroup homomorphisms $\varphi_{\alpha,\beta}:L_\alpha\to L_\beta$ and $\psi_{\alpha,\beta}:R_\alpha\to R_\beta$ such that

$$(l_\alpha,r_\alpha)\theta_{\alpha,\beta}=(l_\alpha\varphi_{\alpha,\beta},r_\alpha\psi_{\alpha,\beta})$$

for all (l_α,r_α) in E_α. Easily, we can prove that $\varphi_{\alpha,\beta}$ is a semiring homomorphism. Furthermore, since $\theta_{\alpha,\beta}$ satisfies (1)–(4), we have $\varphi_{\alpha,\beta}$ satisfies (1)–(4) accordingly. That is, $L=\cup_{\alpha\in D}L_\alpha$ is a strong distributive lattice $[E/\stackrel{+}{\mathcal{D}}_E;L_\alpha,\varphi_{\alpha,\beta}]$ of left zero band semirings L_α. Similarly, $R=\cup_{\alpha\in D}R_\alpha$ is a strong distributive lattice $[E/\stackrel{+}{\mathcal{D}}_E;R_\alpha,\psi_{\alpha,\beta}]$ of right zero band semirings R_α. Obviously, $E/\stackrel{+}{\mathcal{R}}_E$ and $E/\stackrel{+}{\mathcal{L}}_E$ are isomorphic to L and R. Note that $\sigma_E=\stackrel{+}{\mathcal{D}}_E$. So, $E^+(S/\sigma)$ is isomorphic to D.

For the sake of simplicity, we identity $E/\overset{+}{\mathcal{R}}_E$, $E/\overset{+}{\mathcal{L}}_E$ and $E/\overset{+}{\mathcal{D}}_E$ with L, R and D respectively. Now, define a mapping $\chi : S \to QS(L,R,T;D)$ as follows:

$$a\chi = (\widetilde{a+a'}, \overline{a}, \widehat{a'+a}),$$

where a' is an inverse of a. By the properties of L and R, χ is well defined and is an additive semigroup isomorphism. Let $a, b \in S$. Then,

$$\begin{aligned} a\chi b\chi &= (\widetilde{a+a'}, \overline{a}, \widehat{a'+a})(\widetilde{b+b'}, \overline{b}, \widehat{b'+b}) \\ &= (\widetilde{(a+a')(b+b')}, \overline{ab}, \widehat{(a'+a)(b'+b)}) \\ &= (\widetilde{ab+ab'}, \overline{ab}, \widehat{ab'+ab}) \\ &= (\widetilde{ab+(ab)'}, \overline{ab}, \widehat{(ab)'+ab}) \\ &= (ab)\chi. \end{aligned}$$

Consequently, we have proved that χ is an isomorphism of S onto $QS(L,R,T;D)$.

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加法幂等元满足置换等式的纯整半环

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摘要: 本文主要研究加法幂等元满足置换等式的纯整半环. 对于这类半环，建立了一个结构定理.

关键词: 纯整半环; d- 逆半环; 带半环.