On Signed Edge Domination of Graphs

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Abstract: Let $\gamma'_s(G)$ and $\gamma'_l(G)$ be the numbers of the signed edge and local signed edge domination of a graph $G$ [2], respectively. In this paper we prove mainly that $\gamma'_s(G) \leq \frac{1}{2}n - 1$ and $\gamma'_l(G) \leq 2n - 4$ hold for any graph $G$ of order $n(n \geq 4)$, and pose several open problems and conjectures.

Key words: local signed edge domination function; local signed edge domination number; signed edge domination function; signed edge domination number.

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1. Introduction

We use Bondy and Murty[1] and Xu[2] for terminology and notation not defined here and consider simple graphs only.

Let $G = (V, E)$ be a graph. If $e = uv \in E$, then $N_G[e] = \{u'v' \in E | u' = u \ or \ v' = v\}$ is called the closed edge-neighbourhood of $e$ in $G$, and $N_G(e) = N_G[e]\{e\}$ is the open one. If $v \in V$, then $E_G(v) = \{uv \in E | u \in V\}$. For simplicity, sometimes, $N_G[e]$ and $E_G(v)$ are denoted by $N[e]$ and $E(v)$, respectively. In [2] we introduced the signed edge domination of graphs as follows:

Definition 1[2] Let $G = (V, E)$ be a nonempty graph. A function $f : E \rightarrow \{+1, -1\}$ is called the signed edge domination function (SEDF) of $G$ if $\sum_{e' \in N[e]} f(e') \geq 1$ for every $e \in E(G)$. The signed edge domination number of $G$ is defined as $\gamma'_s(G) = \min\{\sum_{e \in E} f(e) | f \text{ is an SEDF of } G\}$.

And define $\gamma'_s(\overline{K}_n) = 0$ for all totally disconnected graphs $\overline{K}_n$.

Next we introduce a new concept of edge domination in graphs:

Definition 2 Let $G = (V, E)$ be a graph without isolated vertices. A function $f : E \rightarrow \{+1, -1\}$ is called the local signed edge domination function (LSEDF) of $G$ if $\sum_{e \in E(v)} f(e) \geq 1$ for every $v \in V(G)$. The local signed edge domination number of $G$ is defined as $\gamma'_l(G) = \min\{\sum_{e \in E} f(e) | f \text{ is an LSEDF of } G\}$. Obviously, $|\gamma'_l(G)| \leq |E(G)|$. It seems natural to define $\gamma'_l(\overline{K}_n) = 0$ for all totally disconnected graphs $\overline{K}_n$.

Clearly, $\gamma'_l(G_1 \cup G_2) = \gamma'_l(G_1) + \gamma'_l(G_2)$ and $\gamma'_s(G_1 \cup G_2) = \gamma'_s(G_1) + \gamma'_s(G_2)$ for any two disjoint graphs $G_1$ and $G_2$. In comparison with the above two definitions, we see that each
LSEDF of $G$ is an SEDF of $G$, and hence we have

**Lemma 1** For all graphs $G$, $\gamma'_l(G) \leq \gamma'_i(G)$.

By Definition 2, we have

**Lemma 2** For all graphs $G$, $v \in V(G)$, then $\gamma'_l(G) \leq \gamma'_i(G - v) + d_G(v)$.

In recent years, some kinds of domination in graphs have been investigated. Most of those belong to the vertex domination of graphs, such as signed domination\cite{3,4}, minus domination\cite{5}, majority domination\cite{6}, domination\cite{7}, etc. A few of results have been obtained about the edge domination of graphs\cite{2}. In this paper we discuss mainly the upper bounds for (local) signed domination numbers of graphs, and pose several open problems and conjectures.

A graph $G$ is said to be a $\theta$-graph if $G$ is a connected graph with degree sequence $d = (2, 2, \cdots, 2, 3, 3)$. That is, a $\theta$-graph consists of a cycle and a path such that two end-vertices of the path are on the cycle.

**Lemma 3** Any $\theta$-graph contains a cycle of even length (even cycle).

**Proof** It is obvious.

**Lemma 4** For any graph $G$, if $\delta(G) \geq 3$, then $G$ contains a $\theta$-graph as subgraph, and hence $G$ contains an even cycle.

**Proof** Without loss of generality, we may suppose that $G$ is a connected graph. Let $T$ be a spanning tree of $G$, and $v$ a pendant-vertex of $T$. That is, $d_T(v) = 1$. Since $\delta(G) \geq 3$, there exist at least two vertices $u$ and $w$ such that $uv, wv \in E(G) \setminus E(T)$. Define $H = T + \{uv, wv\}$. Then obviously, $H$ contains a $\theta$-graph as subgraph, which is the maximum 2-connected subgraph of $H$. In view of $H \subseteq G$ and Lemma 3, we have completed the proof of Lemma 4.

For a graph $G$, if there exist some subgraphs $G_i (i = 1, 2, \cdots, q)$ of $G$ such that $E(G) = \bigcup_{i=1}^{q} E(G_i)$ and $E(G_i) \cap E(G_j) = \emptyset (1 \leq i \neq j \leq q)$, then we say that $G$ can be decomposed into $G_1, G_2, \cdots, G_q$.

**Lemma 5** Any forest $F$ can be decomposed into some paths $P_{m_i} (i = 1, 2, \cdots, q; m_i \geq 2)$ such that all end-vertices of all these paths are pairwise distinct.

**Proof** We use the induction on $m = |E(F)|$.

It is trivial for $m = 0$. Suppose that the lemma is true for all forests of size $k \leq m - 1$. Now we consider a forest $F$ of size $m$ ($m \geq 1$). In $F$ we choose a path $P_t$ ($t \geq 2$) whose end-vertices are two pendant-vertices of $F$.

Let $F_1 = F - E(P_t)$. Clearly, $F_1$ is a forest of size at most $m - 1$. By the induction hypothesis, $F_1$ can be decomposed into some paths $P_{m_i} (i = 1, 2, \cdots, q; m_i \geq 2)$ such that all end-vertices of all these paths are pairwise distinct. Thus, $F$ can be decomposed into the paths $P_{m_i} (i = 1, 2, \cdots, q)$ and $P_t$, all end-vertices of the $q + 1$ paths are pairwise distinct. So, the lemma is true for all forests $F$ of size $m$. We have completed the proof of Lemma 5. \qed
For cycles $C_n (n \geq 3)$ and complete graphs $K_n (n \geq 1)$, we have

**Lemma 6** \( \gamma'_s (C_n) = n - 2 \left\lceil \frac{n}{3} \right\rceil \) and \( \gamma'_s (K_n) = \left\lceil \frac{n-1}{2} \right\rceil \).

### 2. Main results

We first give an upper bound of \( \gamma'_s (G) \) for all graphs \( G \).

**Theorem 1** For any graph \( G \) of order \( n \) \((n \geq 4)\), \( \gamma'_s (G) \leq 2n - 4 \), and this bound is sharp.

**Proof** We use the induction on \( m = |E(G)| \). The result is clearly true for \( m \leq 3 \) (note that \( n \geq 4 \)).

Suppose that the theorem is true for all graphs of size \( k \) \((k \leq m - 1)\). Now we consider a graph \( G \) with \( |E(G)| = m \). By Lemma 2, we may suppose \( \delta (G) \geq 1 \).

**Case 1.** \( \delta (G) \leq 2 \)

There exists a vertex \( v \in V(G) \) such that \( d_G (v) = \delta (G) \leq 2 \). Note that \( |E(G - v)| \leq m - 1 \).

By the induction hypothesis, we have \( \gamma'_s (G - v) \leq 2(n - 1) - 4 = 2n - 6 \). We see from Lemma 2 that \( \gamma'_s (G) \leq \gamma'_s (G - v) + d_G (v) \leq 2n - 6 + 2 = 2n - 4 \).

**Case 2.** \( \delta (G) \geq 3 \)

We see from Lemma 4 that \( G \) contains an even cycle \( C \). Let \( H = G - E(C) \). By the induction hypothesis, \( H \) has an LSEDF \( f \) with \( \sum_{e \in E(H)} f (e) \leq 2n - 4 \). Extending \( f \) from \( H \) by signing +1 and −1 alternatively along \( C \), we obtain an LSEDF for \( G \), and hence \( \gamma'_s (G) \leq 2n - 4 \).

Since \( \gamma'_s (K_{2,n-2}) = 2n - 4 (n \geq 4) \), the upper bound given in Theorem 1 is sharp. We have completed the proof of Theorem 1.

For signed edge domination number, by Theorem 1 and Lemma 1, we have

**Corollary 1** For all graphs \( G \) of order \( n (n \geq 3) \), \( \gamma'_s (G) \leq 2n - 4 \).

For the lower bound of \( \gamma'_s (G) \), we have

**Corollary 2** For all graphs \( G \) of order \( n \), if \( \delta (G) \geq 1 \), then \( \gamma'_s (G) \geq \left\lceil \frac{n}{4} \right\rceil \).

**Proof** Let \( f \) be an LSEDF of \( G \) such that \( \gamma'_s (G) = \sum_{e \in E(G)} f (e) \). For every edge \( e = uv \in E(G) \), \( e \in E(u) \) and \( e \in E(v) \). Thus, we have

\[
\gamma'_s (G) = \sum_{e \in E(G)} f (e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \in E(v)} f (e) \geq \frac{1}{2} \sum_{v \in V(G)} 1 = \frac{n}{2}
\]

Note that \( \gamma'_s (G) \) is an integer. The proof is complete.

We know from Definition 2 that the inequality \( \gamma'_s (G) \leq |E(G)| \) holds for all graphs \( G \).

This equality holds for some graphs only.

**Theorem 2** Let \( G \) be a graph, \( D_3 (G) = \{v \in V(G) | d_G (v) \geq 3\} \). Then \( \gamma'_s (G) = |E(G)| \) if and only if either \( D_3 (G) = \phi \) or \( D_3 (G) \) is an independent set of \( G \).

**Proof** It is not difficult to check that the following four statements are equivalent:
(1) $\gamma_0'(G) = |E(G)|$;
(2) For any LSEDF $f$ of $G$ satisfying $\gamma_0'(G) = \sum_{e \in E(G)} f(e)$ and every edge $e \in E(G)$, $f(e) = 1$;
(3) For any two vertices $u$ and $v$ of degree at least 3, $uv \notin E(G)$;
(4) $D_3(G) = \phi$ or $D_3(G)$ is an independent set of $G$.

We have completed the proof of Theorem 2. \qed

Next we give an upper bound of $\gamma_0'(G)$ for general graphs $G$.

**Theorem 3** For any graph $G$ of order $n$, $\gamma_0'(G) \leq \lfloor \frac{14}{3} n - 1 \rfloor$.

**Proof** Without loss of generality, we may suppose that $G$ is a connected graph and $n \geq 4$.

When $G$ contains a Hamilton cycle $C_n$, let $T = C_n$.

When $G$ has no Hamilton cycle, we choose a spanning tree $T$ of $G$ such that $|\{v \in V(T) | d_T(v) = 1\}|$ is as small as possible (taken over all spanning tree of $G$). It is easy to see that any two pendant-vertices of $T$ are not adjacent in $G$. (Otherwise, there exists a spanning tree $T'$ of $G$ such that $T'$ contains less pendant-vertices than $T$, which contradicts the choice of $T$ in $G$.)

Thus, $n - 1 \leq |E(T)| \leq n$.

For every edge $e \in E(T)$, define $f(e) = +1$.

Let $A = \{v \in V(T) | d_T(v) = 1\}$, note that $A = \phi$ when $T = C_n$.

$$T_0 = T \setminus A, A_0 = \{u \in V(T_0) | d_{T_0}(u) = 1\} \quad \text{(it is possible that } A_0 = \phi).$$

For each vertex $u_0 \in A_0$, we choose exactly one edge $e_0 \in E(u_0) \setminus E(T)$ when $E(u_0) \setminus E(T) \neq \phi$, where $E(u_0) \setminus E(T) = \{u_0u \in E(G) | u \in V(G)\}$. Let $M$ be the set of all edges chosen. Clearly, $|M| \leq |A_0| \leq |A|$ and $A \cap A_0 = \phi$, thus $|M| \leq \lfloor \frac{1}{3} n \rfloor$.

For every edge $e \in M$, we define $f(e) = +1$.

It is easy to check the following statements:

For every nonpendant-edge $e$ of $T$, $N_G[e]$ contains at least three edges of $T$. For any pendant-edge $e$ of $T$, $e = uv \in E(T)$ with $d_T(u) = 1$, when $d_G(v) \geq 3$; $N_G[e]$ has at least three edges in $E(T) \cup M$, when $d_G(v) = 2$ (note that $d_G(v) \neq 1$); $N_G[e]$ contains two edges of $T$. For every edge $e \in E(G) \setminus E(T)$, since any two vertices of $A$ are not adjacent in $G$, $N_G[e]$ contains at least three edges of $T$.

Write $G_0 = G - (E(T) \cup M)$.

If there exist even circuits in $G_0$, then we choose some pairwise edge-disjoint even circuits, denoted by $H_i \ (1 \leq i \leq t)$, so that the graph $G_1 = G_0 - \bigcup_{i=1}^t E(H_i)$ contains no even circuit. If there is no even circuits in $G_0$, then $G_1 = G_0$.

For each even circuit $H_i$, we define $f$ by signing $+1$ and $-1$ alternatively along $H_i (1 \leq i \leq t)$.

Since $G_1$ does not contain any even circuit, any two odd cycles in $G_1$ are vertex-disjoint.

(Otherwise, there exists an even circuit in $G_1$, which is impossible.)

Let $C_r (1 \leq i \leq s)$ be all odd cycles of $G_1$, where $r_i \geq 3$ is odd for each $i$. Noting that $V(C_{r_i}) \cap V(C_{r_j}) = \phi (1 \leq i \neq j \leq s)$, we have $s \leq \lfloor \frac{2}{3} \rfloor$. 


For every $C_{r_i}$, let $M_i$ be a maximum matching of $C_{r_i}$, and define $f$ as follows:

$$f(e) = \begin{cases} 
-1, & \text{when } e \in M_i \\
+1, & \text{when } e \in E(C_{r_i}) \setminus M_i
\end{cases}$$

Clearly, $\sum_{e \in E(C_{r_i})} f(e) = 1$ for each $i$ ($1 \leq i \leq s$).

Let $F = G_1 - \bigcup_{i=1}^{s} E(C_{r_i})$. Obviously, $F$ is a forest. By Lemma 5, $F$ can be decomposed into some paths such that all end-vertices of these paths are pairwise distinct. These paths are written as $P_m$ ($m_i \geq 2, 1 \leq i \leq q$), namely, $E(F) = \bigcup_{i=1}^{q} E(P_m)$ and $E(P_m) \cap E(P_{m_j}) = \phi$ ($1 \leq i \neq j \leq q$).

For every path $P_m$ ($1 \leq i \leq q$), $m_i \geq 2$, let $N_i$ be a maximum matching of $P_{m_i}$. When $e \in N_i$, define $f(e) = -1$; when $e \in E(P_{m_i}) \setminus N_i$, define $f(e) = +1$. Note that $|N_i| = \lceil \frac{m_i}{2} \rceil \geq |E(P_{m_i})|$. When $e \in E(P_{m_i}) \setminus N_i$, we have $-1 \leq \sum_{e \in E(P_{m_i})} f(e) \leq 0, i = 1, 2, \cdots, q$.

We have completed the definition of $f$ on $E(G)$.

Next we check that $f$ is an SEDF of $G$.

1. For any edge $e = uv \in E(G) \setminus E(T)$.

Since any two vertices of $A$ are not adjacent in $G$, thus, $N_G[e]$ contains at least three edges of $T$. Note that $u$ (also, $v$) is an end-vertex of at most one path defined before, thus $N_G[e]$ contains at most two pendant-edges of all paths $P_{m_i} (1 \leq i \leq q)$. So, we have $\sum_{e' \in N_G[e]} f(e') \geq 1$.

2. For any edge $e = uv \in E(T)$.

When $e$ is not any pendant-edge of $T$, obviously, $N_G[e]$ contains at least three edges of $T$. Similarly to (1), we have $\sum_{e' \in N_G[e]} f(e') \geq 1$.

When $e = uv$ is a pendant-edge of $T$, where $u \in A$ and $v \in A_0$. If $d_G(v) \geq 3$, then $N_G[e]$ contains at least three edges in $E(T) \cup M$. Similarly to (1), we have $\sum_{e' \in N_G[e]} f(e') \geq 1$. If $d_G(v) = 2$ (note that $d_G(v) \neq 1$), $N_G[e]$ contains two edges of $T$, and $v$ is not end-vertex of any path $P_{m_i}$ ($1 \leq i \leq q$). Thus $N_G[e]$ contains at most one pendant-edge in $\bigcup_{i=1}^{q} E(P_{m_i})$, and we have $\sum_{e' \in N_G[e]} f(e') \geq 1$.

So, $f$ is an SEDF of $G$. Note $n - 1 \leq |E(T)| \leq n$. When $T = C_n, A_0 = \phi$ and hence $M = \phi$; when $T$ is a spanning tree of $G$, $|M| \leq \lceil \frac{q}{2} \rceil$. These imply $|E(T)| + |M| \leq n - 1 + \lceil \frac{q}{2} \rceil$.

Note that $s \leq \lceil \frac{q}{2} \rceil$, we have

$$\sum_{e \in E(G)} f(e) = |E(T)| + |M| + \sum_{i=1}^{t} \sum_{e \in E(H_i)} f(e) + \sum_{i=1}^{s} \sum_{e \in E(C_{r_i})} f(e) + \sum_{i=1}^{q} \sum_{e \in E(P_{m_i})} f(e) \leq n - 1 + \lfloor \frac{n}{2} \rfloor + 0 + s + 0 \leq \left\lceil \frac{11}{6} n - 1 \right\rceil.$$  

Therefore, $\gamma_s'(G) \leq \sum_{e \in E(G)} f(e) \leq \left\lceil \frac{11}{6} n - 1 \right\rceil$. We have completed the proof of Theorem 3. \hfill \Box

In particular, if $G$ is a bipartite graph, then in the proof of Theorem 3, $s = 0$. So we have

**Corollary 3** For any bipartite graph $G$ of order $n$, $\gamma_s'(G) \leq \left\lceil \frac{3}{2} n - 1 \right\rceil$.

If a graph $G$ has a 2-regular spanning subgraph $H$, then in the proof of Theorem 3, let $T = H$, and hence $M = \phi$. Analogously, we have $\gamma_s'(G) \leq \sum_{e \in E(G)} f(e) \leq |E(H)| + s \leq n + \left\lceil \frac{q}{2} \right\rceil$, where $n = |V(G)|$. Namely, we have
Corollary 4  Let $G$ be a graph of order $n$. If $G$ has a 2-regular spanning subgraph, then

$$\gamma'_{s}(G) \leq \lfloor \frac{4}{3} n \rfloor.$$ 

3. Some open problems and conjectures

We know from Lemma 1 that $\gamma'_{s}(G) \leq \gamma'_{L}(G)$. A natural problem is

Problem 1  Characterize the graphs which satisfy the equality $\gamma'_{s}(G) = \gamma'_{L}(G)$.

Although in [2] we have determined the exact value of $\psi(m) = \min\{\gamma'_{s}(G)|G\text{ is a graph of size } m\}$ for all positive integers $m$, it seems more difficult to solve the following

Problem 2  Determine the exact value of $g(n) = \min\{\gamma'_{s}(G)|G\text{ is a graph of order } n\}$ for every positive integer $n$.

Conjecture 1  For any graph $G$ of order $n(n \geq 1)$, $\gamma'_{s}(G) \leq n - 1$.

If it is true, the super bound is the best possible for odd $n$. For example, let $G$ be the subdivision of the star $K_{1,\frac{n-1}{2}}$, then clearly, $\gamma'_{s}(G) = |E(G)| = n - 1$. (The subdivision of a graph $G$ is the graph obtained from $G$ by subdividing each edge of $G$ exactly once.)

References:


关于图符号的边控制

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摘要: 设 $\gamma'_{s}(G)$ 和 $\gamma'_{l}(G)$ 分别表示图 $G$ 的符号边和局部符号边控制数，本文主要证明了：对任何 $n$ 阶图 $G(n \geq 4)$，均有 $\gamma'_{s}(G) \leq \lfloor \frac{11}{6} n - 1 \rfloor$ 和 $\gamma'_{l}(G) \leq 2n - 4$ 成立，并提出了若干问题和猜想。

关键词: 局部符号边控制数；局部符号边控制数；符号边控制数；符号边控制数.