

Boundedness of Commutators of Generalized Calderón-Zygmund Operators

ZHAO Kai¹, MA Li-min², ZHOU Shu-juan¹

(1. Department of Mathematics., Qingdao University, Shandong 266071, China;

2. Department of Mathematics., Laiyang Agriculture College, Shandong 266109, China)

(E-mail: zkzc@yahoo.com.cn)

Abstract: $[b, T]$ denotes the commutator of generalized Calderón-Zygmund operators T with Lipschitz function b . This paper deals with that $[b, T]$ is bounded in classical Hardy spaces and Herz type Hardy spaces, and proves that $[b, T]$ is bounded from Hardy spaces to weak Lebesgue spaces and from Herz type Hardy spaces to weak Herz spaces on critical point.

Key words: commutator; Lipschitz function; Hardy space; Herz space; boundedness .

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1. Introduction

The singular integral operators and their commutators have been extensively studied in recent years, and the results are plentiful and substantial. Especially, their boundedness on some spaces have been resolved with the establishment of the decomposition theory on these spaces. But the commutators of $\theta(t)$ -type Calderón-Zygmund operators which were introduced in [18] by Yabuta and in [15] by Lizhong Peng, have not been discussed extensively. However, the introduction of this kind of operators has profound background of partial differential equation. By the studies of $\theta(t)$ -type Calderón-Zygmund operators, we know that the studies of this kind of operators and commutators are more complicated and difficult than those of standard Calderón-Zygmund operators and commutators, see [1], [2], [3], [18] and [19] etc for details. In 2002, Shanzhen Lu, Qiang Wu and Dachun Yang studied the boundedness of commutators generated by standard Calderón-Zygmund operators and Lipschitz functions on the Hardy type spaces in [14]. Inspired by the results in [14] and other papers, we studied the commutators generated by generalized Calderón-Zygmund operators and Lipschitz functions. By the Minkowski integral inequality and the Jensen inequality to control some inequalities, in this paper, we obtain the boundedness of the commutator $[b, T]$ generated by $\theta(t)$ -type Calderón-Zygmund operator T and Lipschitz function b on the Hardy spaces and Herz type Hardy spaces. On critical point, we prove that this commutator is bounded from Hardy spaces to weak Lebesgue spaces and from Herz type Hardy spaces to weak Herz spaces.

First of all, let us introduce the definitions of $\theta(t)$ -type Calderón-Zygmund operator, cf. [18], [15] and [19] etc for details.

Definition 1.1 Suppose that T is a bounded operator from Schwartz class $S(\mathbb{R}^n)$ to its dual $S'(\mathbb{R}^n)$, satisfying the following conditions:

- (i) There exists $C > 0$, such that for any $f \in C_0^\infty(\mathbb{R}^n)$, $\|Tf\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}$.
- (ii) There exists a continuous function $k(x, y)$ defined on $\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ and $C > 0$ such that

- a) $|k(x, y)| \leq C|x - y|^{-n}$ for all $(x, y) \in \Omega$;
- b) for all $x, x_0, y \in \mathbb{R}^n$ with $2|x - x_0| < |y - x_0|$,

$$|k(x, y) - k(x_0, y)| + |k(y, x) - k(y, x_0)| \leq C\theta\left(\frac{|x_0 - x|}{|x_0 - y|}\right)|x_0 - y|^{-n},$$

where $\theta(t)$ is a nonnegative nondecreasing function on $[0, +\infty)$ with $\int_0^1 \frac{\theta(t)}{t} dt < +\infty$ and $\theta(0) = 0$, $\theta(2t) \leq C\theta(t)$;

- c) $Tf(x) = \int k(x, y)f(y)dy$, a.e. $x \notin \text{supp} f$. Then T is said to be a $\theta(t)$ -type Calderón-Zygmund operator.

For some properties of $\theta(t)$ -type Calderón-Zygmund operator defined above, especially the boundedness on some spaces, see [15], [18] and [19] etc for details.

Similar to the definitions of other commutators, we introduce the definition of commutator generated by $\theta(t)$ -type Calderón-Zygmund operator and Lipschitz function as follows.

Definition 1.2 Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$, and T be a $\theta(t)$ -type Calderón-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x) = \int k(x, y)(b(x) - b(y))f(y)dy. \quad (1.1)$$

For $\beta > 0$, Lipschitz space $\text{Lip}_\beta(\mathbb{R}^n)$ is defined by

$$\text{Lip}_\beta(\mathbb{R}^n) = \{f : \|f\|_{\text{Lip}_\beta(\mathbb{R}^n)} = \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty\}. \quad (1.2)$$

Obviously, if $\beta > 1$, then $\text{Lip}_\beta(\mathbb{R}^n)$ only includes constant, and $[b, T] \equiv 0$ is trivial. So we restrict $0 < \beta \leq 1$.

Secondly, let us introduce Riesz integral I_α and its boundedness on Lebesgue spaces which are useful to the proofs of theorems in the following.

Definition 1.3^[16] For $0 < \alpha < n$, Riesz integral I_α is defined by

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy.$$

Lemma 1.1^[16] Let $0 < \alpha < n$, $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$. Then there exists $C > 0$, such that

$$\|I_\alpha(f)\|_q \leq C\|f\|_p.$$

The boundedness of commutator $[b, T]$ on the Lebesgue spaces is as follows

Proposition 1.1 *Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$ ($\beta = n(\frac{1}{p} - \frac{1}{q})$), $1 < p < q < \infty$. Then commutator $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Proof Let $f \in L^p(\mathbb{R}^n)$. Then

$$\begin{aligned} |[b, T]f| &= \left| \int k(x, y)(b(x) - b(y))f(y)dy \right| \\ &\leq \int |k(x, y)| \cdot |b(x) - b(y)| \cdot |f(y)|dy \\ &\leq C\|b\|_{\text{Lip}_\beta(\mathbb{R}^n)} \int |x - y|^{\beta-n} |f(y)|dy \\ &= C\|b\|_{\text{Lip}_\beta(\mathbb{R}^n)} I_\alpha(|f|)(x). \end{aligned}$$

So

$$\|[b, T]f\|_q \leq C\|b\|_{\text{Lip}_\beta(\mathbb{R}^n)} \|I_\alpha(|f|)\|_q.$$

Applying Lemma 1.1 for I_α , we have

$$\|[b, T]f\|_q \leq C\|b\|_{\text{Lip}_\beta(\mathbb{R}^n)} \|f\|_p.$$

This completes the proof of Proposition 1.1. \square

Here, we write the Minkowski integral inequality, which plays an important role in this paper, as the Lemma 1.2 below^[6].

Lemma 1.2 *Let μ and ν be σ -finite measures, and $1 \leq p < \infty$. If $f(x, y)$ is measurable on $\nu \times \mu$, then*

$$\left\| \int f(x, y) d\nu(x) \right\|_{L^p(\mu)} \leq \int \|f(y)\|_{L^p(\mu)} d\nu(y). \quad (1.3)$$

Obviously, when $p = 1$, Lemma 1.2 is the Fubini theorem.

We will study the boundedness of the commutator on Hardy spaces and Herz type Hardy spaces in Sections 2 and 3, respectively.

2. Boundedness on Hardy spaces

In this section, we discuss the boundedness of commutator $[b, T]$ generated by $\theta(t)$ -type Calderón-Zygmund operator T and Lipschitz function b on the classical Hardy spaces. And the boundedness of commutator $[b, T]$ from the Hardy spaces to weak Lebesgue spaces is obtained.

First, let us review the atomic decomposition of the Hardy spaces, which is important to study the boundedness of operators. For the following definition and lemma, please see [4], [11] and [17] for details.

Definition 2.1 *Let $0 < p \leq 1$. A function $a(x)$ on $L^2(\mathbb{R}^n)$ is said to be a $(p, 2)$ -atom, if it satisfies the following conditions.*

- 1) *There exist $x_0 \in \mathbb{R}^n$ and $r > 0$, such that $\text{supp } a \subset B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$;*
- 2) *$\|a\|_2 \leq |B(x_0, r)|^{1/2-1/p}$;*

3) $\int_{\mathbb{R}^n} a(x)x^\gamma dx = 0$ for $|\gamma| \leq [n(1/p - 1)]$,

where $[s]$ is the largest integer not bigger than s .

Lemma 2.1 Let $0 < p \leq 1$. A distribution f on \mathbb{R}^n is said to belong to $H^p(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ in the sense of distribution, where a_j is $(p, 2)$ -atom, and $\{\lambda_j\}$ is a sequence of numbers with $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. Moreover

$$\|f\|_{H^p(\mathbb{R}^n)} \approx \inf \left\{ \sum_{j=-\infty}^{\infty} |\lambda_j|^p \right\}^{1/p}, \quad (2.1)$$

where the infimum is taken over all the atomic decompositions of f .

Here we obtain the following results.

Theorem 2.1 Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$ ($0 < \beta \leq 1$), $n/(n + \beta) < p \leq 1$ and $1/q = 1/p - \beta/n$. If $\int_0^1 \frac{\theta(t)}{t^{1+\beta}} dt < +\infty$, then $[b, T]$ is bounded from $H^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

Proof By Lemma 2.1, we only need to prove that there exists a constant $C > 0$, for any $(p, 2)$ -atom a , such that $\|[b, T]a\|_q \leq C\|b\|_{\text{Lip}_\beta}$ (C being independent of a).

Suppose $\text{supp } a \subset B = B(x_0, r)$. It is easy to see that

$$\begin{aligned} \|[b, T]a\|_q &\leq \left(\int_{|x-x_0| \leq 2r} |[b, T]a(x)|^q dx \right)^{1/q} + \left(\int_{|x-x_0| > 2r} |[b, T]a(x)|^q dx \right)^{1/q} \\ &:= \text{I} + \text{II}. \end{aligned} \quad (2.2)$$

Pick real numbers p_1 and q_1 satisfying $1 < p_1 < \min(2, n/\beta)$, $1/q_1 = 1/p_1 - \beta/n$. Using the Hölder inequality and applying the (L^{p_1}, L^{q_1}) boundedness of $[b, T]$ in the Proposition 1.1 and the size condition of a , we obtain

$$\text{I} \leq C\|b\|_{\text{Lip}_\beta} \|a\|_{p_1} |B|^{1/q_1 - 1/q_1} \leq C\|b\|_{\text{Lip}_\beta} \|a\|_2 |B|^{1/p - 1/2} \leq C\|b\|_{\text{Lip}_\beta}. \quad (2.3)$$

For II, the Minkowski inequality says that

$$\begin{aligned} \text{II} &= \left(\int_{|x-x_0| > 2r} \left| \int_B k(x, y)(b(x) - b(y))a(y) dy \right|^q dx \right)^{1/q} \\ &\leq \left(\int_{|x-x_0| > 2r} \left| (b(x) - b(x_0)) \int_B k(x, y)a(y) dy \right|^q dx \right)^{1/q} + \\ &\quad \left(\int_{|x-x_0| > 2r} \left| \int_B k(x, y)(b(y) - b(x_0))a(y) dy \right|^q dx \right)^{1/q} \\ &:= \text{II}_1 + \text{II}_2. \end{aligned} \quad (2.4)$$

If $|x - x_0| > 2r$, then

$$\begin{aligned} \left| \int_B k(x, y)(b(y) - b(x_0))a(y) dy \right| &\leq C\|b\|_{\text{Lip}_\beta} \int_B \frac{|y - x_0|^\beta}{|x - y|^n} |a(y)| dy \\ &\leq C\|b\|_{\text{Lip}_\beta} |x - x_0|^{-n} \int_B |y - x_0|^\beta |a(y)| dy \\ &\leq C\|b\|_{\text{Lip}_\beta} |x - x_0|^{-n} r^{\beta + n(1-1/p)}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\Pi_2 &\leq C\|b\|_{\text{Lip}_\beta} r^{\beta+n(1-1/p)} \left(\int_{|x-x_0|>2r} |x-x_0|^{-nq} dx \right)^{1/q} \\
&\leq C\|b\|_{\text{Lip}_\beta} r^{\beta+n(1-1/p)} \left(\int_{|t|>2r} t^{-nq+n-1} dt \right)^{1/q} \\
&\leq C\|b\|_{\text{Lip}_\beta}.
\end{aligned} \tag{2.5}$$

For Π_1 , by the vanishing condition of a , using the Lemma 1.2 and the Hölder inequality, we have

$$\begin{aligned}
\Pi_1 &= \left(\int_{|x-x_0|>2r} \left| (b(x) - b(x_0)) \int_B (k(x, y) - k(x, x_0)) a(y) dy \right|^q dx \right)^{1/q} \\
&\leq C\|b\|_{\text{Lip}_\beta} \left(\int_{|x-x_0|>2r} \left(|x-x_0|^\beta \int_B \theta \left(\frac{|y-x_0|}{|x-x_0|} \right) |x-x_0|^{-n} |a(y)| dy \right)^q dx \right)^{1/q} \\
&\leq C\|b\|_{\text{Lip}_\beta} \int_B |a(y)| \left(\int_{|x-x_0|>2r} |x-x_0|^{\beta q - nq} \theta^q \left(\frac{|y-x_0|}{|x-x_0|} \right) dx \right)^{1/q} dy \\
&\leq C\|b\|_{\text{Lip}_\beta} \int_B |a(y)| \cdot |y-x_0|^{\beta-n+n/q} \left(\int_{t<1/2} \frac{\theta^q(t)}{t^{q(\beta-n)+1+n}} dt \right)^{1/q} dy.
\end{aligned}$$

And the Jensen inequality tells us

$$\begin{aligned}
\left(\int_0^1 \frac{\theta^q(t)}{t^{q(\beta-n)+1+n}} dt \right)^{1/q} &\leq \left(\sum_{k=1}^{\infty} \int_{2^{-k}}^{2^{-k+1}} \frac{\theta^q(t)}{t^{q(\beta-n)+1+n}} dt \right)^{1/q} \\
&\leq C \left(\sum_{k=1}^{\infty} \theta^q(2^{-k+1}) \cdot 2^{k(q(\beta-n)+n)} \right)^{1/q} \\
&\leq C \left(\sum_{k=1}^{\infty} (\theta(2^{-k+1}) \cdot 2^{k(\beta-n+n/q)})^{1/2} \right)^2 \\
&\leq C \left(\int_0^1 \left(\frac{\theta(t)}{t^{\beta-n+2+n/q}} \right)^{1/2} dt \right)^2 \\
&\leq C \left(\int_0^1 \frac{\theta(t)}{t^{1+\beta}} dt \right) \cdot \left(\int_0^1 \frac{1}{t^{1-n+n/q}} dt \right) \\
&\leq C.
\end{aligned}$$

Thus

$$\Pi_1 \leq C\|b\|_{\text{Lip}_\beta} \int_B |a(y)| \cdot |y-x_0|^{\beta-n+n/q} dy \leq C\|b\|_{\text{Lip}_\beta}. \tag{2.6}$$

So, by the estimates (2.3), (2.4), (2.5) and (2.6), we get that $\|[b, T]a\|_q \leq C\|b\|_{\text{Lip}_\beta}$. This completes the proof of Theorem 2.1. \square

Noticing the fact that $\text{BMO}(\mathbb{R}^n)$ is the dual space of $H^1(\mathbb{R}^n)$, we obtain a corollary of Theorem 2.1 as follows.

Corollary 2.1 *Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$ ($0 < \beta \leq 1$), and $\int_0^1 \frac{\theta(t)}{t^{1+\beta}} dt < +\infty$. Then $[b, T]$ is bounded from $L^{n/\beta}(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R}^n)$.*

Generally, $[b, T]$ is not bounded from $H^{n/(n+\beta)}$ to L^1 except for the trivial case, but it has the following weak type estimate.

Theorem 2.2 Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$ ($0 < \beta \leq 1$), and $\int_0^1 \frac{\theta(t)}{t^{1+\beta}} dt < +\infty$. Then $[b, T]$ is bounded from $H^{n/(n+\beta)}(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$.

Proof By the Lemma 2.1, we know that there exist $(n/(n+\beta), 2)$ -atom a_j , $\text{supp } a_j \subset B_j = B(x_j, r_j)$, and number λ_j , $j \in \mathbb{Z}$, such that

$$f = \sum_{j=-\infty}^{\infty} \lambda_j a_j, \quad \text{and} \quad \sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty.$$

It is easy to see that

$$\begin{aligned} [b, T]f(x) &= \sum_{j=-\infty}^{\infty} \lambda_j (b(x) - b(x_j)) T a_j(x) \chi_{2B_j}(x) + \\ &\quad \sum_{j=-\infty}^{\infty} \lambda_j (b(x) - b(x_j)) T a_j(x) \chi_{(2B_j)^c}(x) - \\ &\quad T \left(\sum_{j=-\infty}^{\infty} \lambda_j (b - b(x_j)) a_j \right)(x) \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{2.7}$$

By the Hölder inequality and the boundedness of T on L^2 , we have

$$\begin{aligned} \|(b - b(x_j)) T a_j(x) \chi_{2B_j}\|_1 &= \int_{2B_j} |b(x) - b(x_j)| \cdot |T a_j| dx \\ &\leq C \|b\|_{\text{Lip}_\beta} \int_{2B_j} |x - x_j|^\beta |T a_j| dx \\ &\leq C \|b\|_{\text{Lip}_\beta} \left(\int_{2B_j} |x - x_j|^{2\beta} dx \right) \|T a_j\|_2 \\ &\leq C \|b\|_{\text{Lip}_\beta} r_j^{\beta+n/2} \|a_j\|_2 \\ &\leq C \|b\|_{\text{Lip}_\beta}. \end{aligned}$$

And the vanishing condition of atom a_j shows that

$$\begin{aligned} \|(b - b(x_j)) T a_j(x) \chi_{(2B_j)^c}\|_1 &= \int_{(2B_j)^c} \left| (b(x) - b(x_j)) \int_{B_j} k(x, y) a_j(y) dy \right| dx \\ &\leq C \|b\|_{\text{Lip}_\beta} \int_{(2B_j)^c} |x - x_j|^\beta \left| \int_{B_j} (k(x, y) - k(x, x_j)) a_j(y) dy \right| dx \\ &\leq C \|b\|_{\text{Lip}_\beta} \int_{(2B_j)^c} |x - x_j|^\beta \int_{B_j} \theta \left(\frac{|y - x_j|}{|x - x_j|} \right) |x - x_j|^{-n} |a_j(y)| dy dx \\ &\leq C \|b\|_{\text{Lip}_\beta} \int_{B_j} |a_j(y)| \cdot |y - x_j|^\beta \int_0^1 \frac{\theta(t)}{t^{1+\beta}} dt dy \\ &\leq C \|b\|_{\text{Lip}_\beta}. \end{aligned}$$

Therefore,

$$|\{x \in \mathbb{R}^n : |I_i| > \lambda/3\}| \leq C\lambda^{-1} \|I_i\|_1 \leq C\lambda^{-1} \sum_{j=-\infty}^{\infty} |\lambda_j|, \quad i = 1, 2. \quad (2.8)$$

By

$$\|(b - b(x_j))a_j\|_1 \leq C\|b\|_{\text{Lip}_\beta} \int_{B_j} r^\beta |a(y)| dy \leq C,$$

and the weak type (1.1) property of T , we have

$$|\{x \in \mathbb{R}^n : |I_3| > \lambda/3\}| \leq C\lambda^{-1} \left\| \sum_{j=-\infty}^{\infty} \lambda_j (b - b(x_j))a_j \right\|_1 \leq C\lambda^{-1} \sum_{j=-\infty}^{\infty} |\lambda_j|. \quad (2.9)$$

Noticing the fact that $n/(n + \beta) < 1$ and using the Jensen inequality, we get

$$\begin{aligned} |\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}| &\leq \sum_{i=1}^3 |\{x \in \mathbb{R}^n : |I_i(x)| > \lambda/3\}| \\ &\leq C\lambda^{-1} \sum_{j=-\infty}^{\infty} |\lambda_j| \\ &\leq C\lambda^{-1} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{n/(n+\beta)} \right)^{(n+\beta)/n}. \end{aligned} \quad (2.10)$$

Taking infimum over all the central atomic decomposition of f finishes the proof of Theorem 2.2. \square

3. Boundedness on Herz type Hardy spaces

In this section, we discuss the boundedness of the commutator $[b, T]$ generated by $\theta(t)$ -type Calderón-Zygmund operator T and Lipschitz function b on the Herz type Hardy spaces.

First, we introduce the definition of Herz type Hardy space and its atomic decomposition characterization as follows^[8,12].

Definition 3.1 Let $S'(\mathbb{R}^n)$ be the distribution function space on \mathbb{R}^n , and $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$. For $k \in \mathbb{Z}$, suppose that $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $E_k = B_k \setminus B^{k-1}$, and $\chi_k = \chi_{E_k}$ is the characteristic function of E_k .

(i) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f : f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}), \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p}.$$

(ii) The nonhomogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f : f \in L_{loc}^q(\mathbb{R}^n), \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = (\|f\chi_{B_0}\|_q^p + \sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L_q}^p)^{1/p}.$$

For all $0 < p < \infty$ and $\alpha \in \mathbb{R}$, it is easy to check that

$$\dot{K}_p^{0,p}(\mathbb{R}^n) = K_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n), \quad \text{and} \quad \dot{K}_p^{\alpha/p,p}(\mathbb{R}^n) = L_{|x|^\alpha}^p(\mathbb{R}^n).$$

So, the Herz spaces are the generalization of the Lebesgue spaces, and the homogeneous Herz spaces contain the Lebesgue spaces with power weights.

Noticing the fact that $[b, T]f(x) \leq C\|b\|_{\text{Lip}_\beta} I_\beta(|f|)(x)$, by the Theorems 2.3 and 2.4 of [10], we can obtain the boundedness of commutator $[b, T]$ on the Herz spaces as follows.

Proposition 3.1 *Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$ ($0 < \beta \leq 1$). If $0 < p \leq \infty$, $1 < q_1, q_2 < \infty$, and $1/q_2 = 1/q_1 - \beta/n$, $-n/q_2 < \alpha < n(1 - 1/q_2)$, then $[b, T]$ is bounded from $\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)$ to $\dot{K}_{q_2}^{\alpha,p}(\mathbb{R}^n)$ and from $K_{q_1}^{\alpha,p}(\mathbb{R}^n)$ to $K_{q_2}^{\alpha,p}(\mathbb{R}^n)$.*

Definition 3.2 *Let $S'(\mathbb{R}^n)$ be the distribution function space on \mathbb{R}^n . Suppose that $G(f)$ is the grand maximal function of f [12, 17], and $\alpha \in \mathbb{R}$, $0 < p, q < \infty$.*

(i) *The homogeneous Herz type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by*

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : G(f) \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|G(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

(ii) *The nonhomogeneous Herz type Hardy space $HK_q^{\alpha,p}(\mathbb{R}^n)$ is defined by*

$$HK_q^{\alpha,p}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : G(f) \in K_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define

$$\|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)} = \|G(f)\|_{K_q^{\alpha,p}(\mathbb{R}^n)}.$$

For $0 < p < \infty$, it is easy to check that

$$H\dot{K}_p^{0,p}(\mathbb{R}^n) = HK_p^{0,p}(\mathbb{R}^n) = H^p(\mathbb{R}^n), \quad \text{and} \quad H\dot{K}_p^{\alpha/p,p}(\mathbb{R}^n) = H_{|x|^\alpha}^p(\mathbb{R}^n).$$

This means that the Herz type Hardy space is the generalization of the classical Hardy spaces, and the homogeneous Herz type Hardy space contains the classical Hardy spaces with power weights.

For $1 < q < +\infty$, it can be proved that if $-n/q < \alpha < n(1 - 1/q)$, then

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n), \quad \text{and} \quad HK_q^{\alpha,p}(\mathbb{R}^n) = K_q^{\alpha,p}(\mathbb{R}^n).$$

And if $\alpha \geq n(1 - 1/q)$, then one can prove that

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \neq \dot{K}_q^{\alpha,p}(\mathbb{R}^n), \quad \text{and} \quad HK_q^{\alpha,p}(\mathbb{R}^n) \subset K_q^{\alpha,p}(\mathbb{R}^n)^{[10,12]}.$$

Now, we introduce the atomic decomposition of the Herz type Hardy spaces, which makes the study of the boundedness of operators on these spaces convenient.

Definition 3.3 Let $\alpha \in \mathbb{R}$, $1 < q < \infty$. A function $a(x)$ on \mathbb{R}^n is called a central (α, q) -atom, if it satisfies the following conditions.

- 1) There exists $r > 0$, such that $\text{supp } a \subset B(0, r)$;
- 2) $\|a\|_q \leq |B(0, r)|^{-\alpha/n}$;
- 3) $\int_{\mathbb{R}^n} a(x) x^\gamma dx = 0$, for $|\gamma| \leq [\alpha - n(1 - 1/q)]$.

Lemma 3.1 Let $0 < p < \infty$, $1 < q < \infty$, and $\alpha \geq n(1 - 1/q)$. Then a distribution f on \mathbb{R}^n is said to belong to $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$ in the sense of distribution, where a_k is central (α, q) -atom with support B_k , and $\{\lambda_k\}$ is a sequence of number with $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. Furthermore

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \approx \inf \left\{ \sum_{k=-\infty}^{\infty} |\lambda_k|^p \right\}^{1/p}, \quad (3.1)$$

where the infimum is taken over all atomic decompositions of f [12].

Here, we obtain that the commutator $[b, T]$ generated by $\theta(t)$ -type Calderón-Zygmund operator T and Lipschitz function b is bounded from the Herz type Hardy spaces to the Herz spaces. That is the following theorem.

Theorem 3.1 Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$ ($0 < \beta \leq 1$). For $0 < p < +\infty$, $1 < q_1, q_2 < +\infty$, $1/q_2 = 1/q_1 - \beta/n$, and $n(1 - 1/q_1) \leq \alpha < n(1 - 1/q_1) + \beta$. Suppose $\int_0^1 \frac{\theta(t)}{t^{1+\beta}} dt < \infty$. Then $[b, T]$ is bounded from $H\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)$ to $\dot{K}_{q_2}^{\alpha,p}(\mathbb{R}^n)$.

Proof By Lemma 3.1, we know that $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$, where a_k is a central (α, q_1) -atom with support B_k , and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. Write

$$\begin{aligned} \|[b, T]f\|_{\dot{K}_{q_2}^{\alpha,p}}^p &= \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|[b, T]f\chi_j\|_{q_2}^p \\ &\leq C \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \left(\sum_{k=-\infty}^{j-2} |\lambda_k| \cdot \|([b, T]a_k)\chi_j\|_{q_2} \right)^p + \\ &\quad C \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \left(\sum_{k=j-1}^{\infty} |\lambda_k| \cdot \|([b, T]a_k)\chi_j\|_{q_2} \right)^p \\ &:= \text{I} + \text{II}. \end{aligned} \quad (3.2)$$

Noticing the fact that $\|a_k\|_q \leq 2^{-\alpha k}$, and using the (L^{q_1}, L^{q_2}) boundedness of $[b, T]$ in the Proposition 1.1, the Abel lemma and the Hölder inequality, one can obtain that

$$\text{II} = C \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \left(\sum_{k=j-1}^{\infty} |\lambda_k| \cdot \|([b, T]a_k)\chi_j\|_{q_2} \right)^p$$

$$\begin{aligned}
&\leq C \|b\|_{\text{Lip}_\beta}^p \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \left(\sum_{k=j-1}^{\infty} |\lambda_k| \cdot \|a_k\|_{q_1} \right)^p \\
&\leq C \|b\|_{\text{Lip}_\beta}^p \begin{cases} \sum_{k=-\infty}^{\infty} |\lambda_k|^p \sum_{j=-\infty}^{k+1} 2^{(j-k)\alpha p}, & \text{if } p \leq 1 \\ \sum_{j=-\infty}^{\infty} \left(\sum_{k=j-1}^{\infty} |\lambda_k|^p 2^{(j-k)\alpha p/2} \right) \left(\sum_{k=j-1}^{\infty} 2^{(j-k)\alpha p'/2} \right)^{p/p'}, & \text{if } p > 1 \end{cases} \\
&\leq C \|b\|_{\text{Lip}_\beta}^p \begin{cases} \sum_{k=-\infty}^{\infty} |\lambda_k|^p \sum_{j=-\infty}^{k+1} 2^{(j-k)\alpha p}, & \text{if } p \leq 1 \\ \sum_{k=-\infty}^{\infty} |\lambda_k|^p \sum_{j=-\infty}^{k+1} 2^{(j-k)\alpha p/2}, & \text{if } p > 1 \end{cases} \\
&\leq C \|b\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} |\lambda_k|^p. \tag{3.3}
\end{aligned}$$

For I, we have

$$\begin{aligned}
\|([b, T]a_k)\chi_j\|_{q_2} &= \left(\int_{B_j \setminus B_{j-1}} \left| \int_{B_k} (b(x) - b(y))k(x, y)a_k(y)dy \right|^{q_2} dx \right)^{1/q_2} \\
&\leq \left(\int_{B_j \setminus B_{j-1}} \left| (b(x) - b(0)) \int_{B_k} k(x, y)a_k(y)dy \right|^{q_2} dx \right)^{1/q_2} + \\
&\quad \left(\int_{B_j \setminus B_{j-1}} \left| \int_{B_k} (b(y) - b(0))k(x, y)a_k(y)dy \right|^{q_2} dx \right)^{1/q_2} \\
&:= I_1 + I_2. \tag{3.4}
\end{aligned}$$

In addition

$$\begin{aligned}
\left| \int_{B_k} (b(y) - b(0))k(x, y)a_k(y)dy \right| &\leq C \|b\|_{\text{Lip}_\beta} \int_{B_k} \frac{|y|^\beta}{|x - y|^n} |a_k(y)| dy \\
&\leq C \|b\|_{\text{Lip}_\beta} |x|^{-n} 2^{k(\beta + n(1-1/q_1) - \alpha)}.
\end{aligned}$$

Therefore,

$$I_2 \leq C \|b\|_{\text{Lip}_\beta} 2^{-j\alpha} 2^{(j-k)(\alpha - n(1-1/q_1) - \beta)} := C \|b\|_{\text{Lip}_\beta} 2^{-j\alpha} W(j, k), \tag{3.5}$$

where

$$W(j, k) = 2^{(j-k)(\alpha - n(1-1/q_1) - \beta)}.$$

For I_1 , noticing that $j \geq k + 2$, by the vanishing condition of a_k and the Lemma 1.2, we get

$$\begin{aligned}
I_1 &= \left(\int_{B_j \setminus B_{j-1}} \left| (b(x) - b(0)) \int_{B_k} (k(x, y) - k(x, 0))a_k(y)dy \right|^{q_2} dx \right)^{1/q_2} \\
&\leq C \|b\|_{\text{Lip}_\beta} \left(\int_{B_j \setminus B_{j-1}} \left(|x|^\beta \int_{B_k} \theta\left(\frac{|y|}{|x|}\right) |x|^{-n} |a_k(y)| dy \right)^{q_2} dx \right)^{1/q_2}
\end{aligned}$$

$$\begin{aligned}
&\leq C\|b\|_{\text{Lip}_\beta} \int_{B_k} |a_k(y)| \left(\int_{B_j \setminus B_{j-1}} \left(\theta \left(\frac{|y|}{|x|} \right) |x|^{\beta-n} \right)^{q_2} dx \right)^{1/q_2} dy \\
&\leq C\|b\|_{\text{Lip}_\beta} \int_{B_k} |a_k(y)| \cdot |y|^\beta 2^{-j(n(1-1/q_1)+\beta)} \left(\int_0^1 \frac{\theta^{q_2}(t)}{t^{1+\beta q_2}} dt \right)^{1/q_2} dy. \tag{3.6}
\end{aligned}$$

In addition the Jensen inequality, one shows that

$$\begin{aligned}
\left(\int_0^1 \frac{\theta^{q_2}(t)}{t^{1+\beta q_2}} dt \right)^{1/q_2} &\leq \left(\sum_{k=1}^{\infty} \int_{2^{-k}}^{2^{-k+1}} \frac{\theta^{q_2}(t)}{t^{1+\beta q_2}} dt \right)^{1/q_2} \leq C \left(\sum_{k=1}^{\infty} (\theta(2^{-k+1}) \cdot 2^{k\beta})^{q_2} \right)^{1/q_2} \\
&\leq C \sum_{k=1}^{\infty} \theta(2^{-k+1}) \cdot 2^{k\beta} \leq C \int_0^1 \frac{\theta(t)}{t^{1+\beta}} dt \leq C.
\end{aligned}$$

Thus

$$\begin{aligned}
I_1 &\leq C\|b\|_{\text{Lip}_\beta} 2^{-k(\alpha-n(1-1/q_1)-\beta)} 2^{-j(n(1-1/q_1)+\beta)} \\
&= C\|b\|_{\text{Lip}_\beta} 2^{-j\alpha} 2^{(j-k)(\alpha-n(1-1/q_1)-\beta)} \\
&:= C\|b\|_{\text{Lip}_\beta} 2^{-j\alpha} W(j, k). \tag{3.7}
\end{aligned}$$

Similar to (3.3), we have that

$$\begin{aligned}
I &\leq C\|b\|_{\text{Lip}_\beta}^p \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{j-2} |\lambda_k| W(j, k) \right)^p \\
&\leq C\|b\|_{\text{Lip}_\beta}^p \begin{cases} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{j-2} |\lambda_k|^p W(j, k)^p, & \text{if } p \leq 1 \\ \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{j-2} |\lambda_k|^p W(j, k) \right) \left(\sum_{k=-\infty}^{j-2} W(j, k) \right)^{p/p'}, & \text{if } p > 1 \end{cases} \tag{3.8} \\
&\leq C\|b\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} |\lambda_k|^p.
\end{aligned}$$

Colligating the above estimates of (3.3) and (3.8), we have

$$\|[b, T]f\|_{\dot{K}_{q_2}^{\alpha, p}} \leq C\|b\|_{\text{Lip}_\beta} \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p}. \tag{3.9}$$

Taking the infimum over all the central atomic decomposition of f , we finish the proof of Theorem 3.1. \square

Similarly, the results related to the nonhomogeneous Herz spaces also hold. And the proof is similar to Theorem 3.1. Here, we omit the details.

On critical point, the commutator $[b, T]$ generated by $\theta(t)$ -type Calderón-Zygmund operator T and Lipschitz function b has weak type estimate similar to Theorem 2.2. So, we introduce the definition of the weak Herz spaces, which was introduced in [1] initially.

Definition 3.4 Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q < \infty$ (B_k and E_k are the same as Definition 3.1).

(i) A measurable function f on \mathbb{R}^n is said to belong to the homogeneous weak Herz spaces $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, if $\|f\|_{W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} |\{x \in E_k : |f(x)| > \lambda\}|^{p/q} \}^{1/p} < \infty$.

(ii) A measurable function f on \mathbb{R}^n is said to belong to the nonhomogeneous weak Herz spaces $WK_q^{\alpha,p}(\mathbb{R}^n)$, if

$$\|f\|_{WK_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \left\{ |\{x \in B_0 : |f(x)| > \lambda\}|^{p/q} + \sum_{k=1}^{\infty} 2^{k\alpha p} |\{x \in E_k : |f(x)| > \lambda\}|^{p/q} \right\}^{1/p} < \infty.$$

For all $0 < p < \infty$ and $\alpha \in \mathbb{R}$, one can get

$$W\dot{K}_p^{0,p}(\mathbb{R}^n) = WK_p^{0,p}(\mathbb{R}^n) = WL^p(\mathbb{R}^n), \quad \text{and} \quad W\dot{K}_p^{\alpha/p,p}(\mathbb{R}^n) = WL_{|x|^\alpha}^p(\mathbb{R}^n).$$

This shows that the weak Herz space contains weak $L^p(\mathbb{R}^n)$ space, and the homogeneous weak Herz space contains weak $L^p(\mathbb{R}^n)$ spaces with power weights.

Theorem 3.2 Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$ ($0 < \beta \leq 1$). Suppose that $0 < p \leq 1$, $1 < q_1, q_2 < +\infty$, $1/q_2 = 1/q_1 - \beta/n$, and $\int_0^1 \frac{\theta(t)}{t^{1+\beta}} dt < \infty$. Then $[b, T]$ is bounded from $H\dot{K}_{q_1}^{n(1-1/q_1)+\beta,p}(\mathbb{R}^n)$ to $W\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}(\mathbb{R}^n)$.

Proof Let $f \in H\dot{K}_{q_1}^{n(1-1/q_1)+\beta,p}(\mathbb{R}^n)$. By Lemma 3.1, one has

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

where a_k is a central $(n(1 - \frac{1}{q_1}) + \beta, q_1)$ -atom with support B_k , and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. Write

$$\begin{aligned} \| [b, T]f \|_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}}^p &= \sup_{\lambda>0} \lambda^p \sum_{j=-\infty}^{\infty} 2^{j(n(1-1/q_1)+\beta)p} |\{x \in E_j : |[b, T]f| > \lambda\}|^{p/q_2} \\ &\leq C \sup_{\lambda>0} \lambda^p \sum_{j=-\infty}^{\infty} 2^{j(n(1-1/q_1)+\beta)p} \left| \left\{ x \in E_j : \left| \sum_{k=-\infty}^{j-2} \lambda_k (b(x) - b(0)) T a_k(x) \right| > \lambda/3 \right\} \right|^{p/q_2} + \\ &\quad C \sup_{\lambda>0} \lambda^p \sum_{j=-\infty}^{\infty} 2^{j(n(1-1/q_1)+\beta)p} \left| \left\{ x \in E_j : \left| \sum_{k=-\infty}^{j-2} \lambda_k T((b - b(0)) a_k)(x) \right| > \lambda/3 \right\} \right|^{p/q_2} + \\ &\quad C \sup_{\lambda>0} \lambda^p \sum_{j=-\infty}^{\infty} 2^{j(n(1-1/q_1)+\beta)p} \left| \left\{ x \in E_j : \left| \sum_{k=j-1}^{\infty} \lambda_k [b, T] a_k(x) \right| > \lambda/3 \right\} \right|^{p/q_2} \\ &:= M_1 + M_2 + M_3. \end{aligned} \tag{3.10}$$

Similar to II in Theorem 3.1, it is easy to know that

$$M_3 \leq C \sum_{j=-\infty}^{\infty} 2^{j(n(1-1/q_1)+\beta)p} \left\| \sum_{k=j-1}^{\infty} \lambda_k [b, T] a_k(x) \chi_j \right\|_{q_2}^p \leq C \|b\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} |\lambda_k|^p. \tag{3.11}$$

Similar to I_1 in Theorem 3.1, we deduce that

$$M_1 \leq C \sum_{j=-\infty}^{\infty} 2^{j(n(1-1/q_1)+\beta)p} \left\| \sum_{k=-\infty}^{j-2} \lambda_k (b(x) - b(0)) T a_k(x) \chi_j \right\|_{q_2}^p \leq C \|b\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} |\lambda_k|^p. \quad (3.12)$$

Noticing the fact that $x \in E_j$, we have

$$\begin{aligned} \left| \sum_{k=-\infty}^{j-2} \lambda_k T((b - b(0))a_k)(x) \right| &= \left| \sum_{k=-\infty}^{j-2} \lambda_k \int_{B_k} k(x, y) (b(y) - b(0)) a_k(y) dy \right| \\ &\leq C \|b\|_{\text{Lip}_\beta} \left| \sum_{k=-\infty}^{j-2} \lambda_k \int_{B_k} \frac{|y|^\beta}{|x - y|^n} a_k(y) dy \right| \leq C 2^{-jn} \|b\|_{\text{Lip}_\beta} \sum_{k=-\infty}^{j-2} |\lambda_k| \\ &\leq C 2^{-jn} \|b\|_{\text{Lip}_\beta} \sum_{k=-\infty}^{\infty} |\lambda_k|. \end{aligned} \quad (3.13)$$

Choosing $j_0 \in \mathbb{Z}$ such that

$$2^{j_0 n} \leq 3C\lambda^{-1} \|b\|_{\text{Lip}_\beta} \sum_{k=-\infty}^{\infty} |\lambda_k| < 2^{(j_0+1)n}. \quad (3.14)$$

For $j \geq j_0 + 1$, it is easy to see that

$$\left\{ x \in E_j : \left| \sum_{k=-\infty}^{j-2} \lambda_k T((b - b(0))a_k)(x) \right| > \lambda/3 \right\}$$

is an empty set. Therefore,

$$\begin{aligned} M_2 &\leq C \sup_{\lambda > 0} \lambda^p \sum_{j=-\infty}^{j_0} 2^{j(n(1-1/q_1)+\beta)p} |E_j|^{p/q_2} \leq C \|b\|_{\text{Lip}_\beta}^p \left(\sum_{k=-\infty}^{\infty} |\lambda_k| \right)^p \\ &\leq C \|b\|_{\text{Lip}_\beta}^p \sum_{k=-\infty}^{\infty} |\lambda_k|^p. \end{aligned} \quad (3.15)$$

For any central atomic decomposition of f , colligating the estimates of M_1, M_2, M_3 above, we have

$$\|[b, T]f\|_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta, p}} \leq C \|b\|_{\text{Lip}_\beta} \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p}. \quad (3.16)$$

Taking the infimum, we complete the proof of Theorem 3.2. \square

The similar results related to the nonhomogeneous weak Herz spaces also hold. The details are omitted to save the space.

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广义 Calderón-Zygmund 算子交换子在 Hardy 型空间上的有界性

赵 凯¹, 马丽敏², 周淑娟¹

(1. 青岛大学数学系, 山东 青岛 266071; 2. 莱阳农业学院数学系, 山东 莱阳 266109)

摘要: $[b, T]$ 表示由 Lipschitz 函数 b 与广义 Calderón-Zygmund 算子 T 生成的交换子. 本文研究了 $[b, T]$ 在经典 Hardy 空间和 Herz 型 Hardy 空间上的有界性, 并且在临界点情形证明了该交换子是从 Hardy 空间到弱 Lebesgue 空间以及 Herz 型 Hardy 到弱 Herz 空间有界的.

关键词: 交换子; Lipschitz 函数; Hardy 空间; Herz 空间; 有界性.