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Meromorphic Solutions of a Type of Higher-Order Partial Differential Equations

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Abstract: Using the value distribution theory in several complex variables, we extend Malmquist type theorem of algebraic differential equation of Steinmetz to higher-order partial differential equations.

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1. Introduction and main result

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we define, for any $r \in \mathbb{R}^+$, $|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$, $\tau(z) = |z|^2$, $\mathbb{C}^\ltimes\langle r\rangle = \{z \in \mathbb{C}^n : |z| = r\}, \mathbb{C}^n(r) = \{z \in \mathbb{C}^n : |z| < r\}$. Let $\mathbb{C}^n[r] = \{z \in \mathbb{C}^n : |z| \le r\}, d = \partial + \overline{\partial}, d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$. We then write

$$\omega_n(z) = dd^c \log |z|^2, \sigma_n(z) = d^c \log |z|^2 \wedge \omega_n^{n-1}(z), z \in \mathbb{C}^n \setminus \{0\};$$

$$\nu_n(z) = dd^c|z|^2, \rho_n(z) = \nu_n^n(z), z \in \mathbb{C}^n.$$

Thus $\sigma_n(z)$ defines a positive measure on $\mathbb{C}^n\langle r\rangle$ with total measure one and ρ_n is a normalized Lebesgue measure on $\mathbb{C}^n$ such that $\mathbb{C}^n(r)$ has measure r^{2n}. Let $\mathbb{P}^1$ be the Riemann sphere, and f be a meromorphic function on $\mathbb{C}^n$, i.e., f can be written as a quotient of two holomorphic functions which are relatively prime. Thus f can be regarded as a meromorphic map f: $\mathbb{C}^n \to \mathbb{P}^1$ such that $f^{-1}(\infty) \neq \mathbb{C}^n$.

For $a, b \in \mathbb{P}^1$, the chordal distance from a to b is denoted by $\| a, b \|$,

$$\| a, \infty \| = \frac{1}{\sqrt{1 + |a|^2}}, \| a, b \| = \frac{|a - b|}{\sqrt{1 + |a|^2}\sqrt{1 + |b|^2}}, a, b \in \mathbb{C},$$

where $\| a, a \| = 0$ and $0 \le \| a, b \| = \| b, a \| \le 1$.

For $0 < s < r$, the characteristic of f is defined by

$$T(r, f) = \int_s^r \frac{1}{t^{2n-1}} \int_{\mathbb{C}^n[t]} f^*(\omega) \bigwedge \nu_n^{n-1} \mathrm{d}t = \int_s^r \frac{1}{t} \int_{\mathbb{C}^n[t]} f^*(\omega) \bigwedge \omega_n^{n-1} \mathrm{d}t.$$

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Let ν be a divisor on $\mathbb{C}^n$. We identify ν with its multiplicity function and define

$$\nu(r) = \{z \in \mathbb{C}^n : |z| < r\} \bigcap \operatorname{supp}\nu, r > 0.$$

The pre-counting function of ν is defined by

$$n(r,\nu) = \sum_{z\in\nu(r)} \nu(z), \quad \text{if} \quad n=1; n(r,\nu) = r^{2-2n}\int_{\nu(r)} \nu\nu_n^{n-1}, \quad \text{if} \quad n>1.$$

The counting function of ν is defined by

$$N(r,\nu) = \int_s^r n(t,\nu)\frac{\mathrm{d}t}{t}, r > s.$$

Let f be a meromorphic function on $\mathbb{C}^n$. If $a \in \mathbb{P}^1$ and $f^{-1}(a) \neq \mathbb{C}^n$, the a-divisor $\nu(f,a) \geq 0$ is defined, and its pre-counting function and counting function will be denoted by $n(r,f,a)$ and $N(r,f,a)$, respectively.

If $a \in \mathbb{P}^1$ and $f^{-1}(a) \neq \mathbb{C}^n$, then we define the proximity function as follows

$$m(r,f,a) = \int_{|z|=r} \log \frac{1}{\| a, f(z) \|}\sigma_n \geq 0, r > 0.$$

For a divisor ν on $\mathbb{C}^n$, let

$$\overline{n}(r,\nu) = \sum_{z\in\nu(r)} 1, \quad \text{if} \quad n=1; \ \overline{n}(r,\nu) = r^{2-2n}\int\limits_{\nu(r)} \nu_n^{n-1}, \quad \text{if} \quad n>1,$$

$$\overline{N}(r,\nu) = \int_s^r \overline{n}(t,\nu)\frac{\mathrm{d}t}{t}, \quad \overline{N}(r,f,a) = \overline{N}(r,\nu(f,a)).$$

The first main theorem states that

$$T(r,f) = N(r,f,a) + m(r,f,a) - m(s,f,a).$$

For a meromorphic function w on $\mathbb{C}^n$, let

$$\Omega_1(z,w,Dw,\ldots,D^n w) = \sum_{(i)\in I} a_{(i)}(z) w^{i_0}(Dw)^{i_1}\cdots(D^n w)^{i_n},$$

$$\Omega_2(z,w,Dw,\ldots,D^n w) = \sum_{(j)\in J} b_{(j)}(z) w^{j_0}(Dw)^{j_1}\cdots(D^n w)^{j_n},$$

where $D^k w = (\partial_1)^{k_1}\cdots(\partial_n)^{k_n} w$ is the partial derivative of w of order $k = k_1 + \cdots + k_n$, $\partial_j = \partial/\partial z_j$; $\{a_{(i)}(z)\}, \{b_{(j)}(z)\}$ are meromorphic functions on $\mathbb{C}^n$; I, J are two finite sets of multi-indices $(i) = (i_0, i_1, \ldots, i_n)$ and $(j) = (j_0, j_1, \ldots, j_n)$ respectively; and $i_0, i_1, \ldots, i_n, j_0, j_1, \ldots, j_n$ are non-negative integers.

For partial differential polynomials $\Omega_1(z,w,Dw,\ldots,D^n w), \Omega_2(z,w,Dw,\ldots,D^n w)$, we adopt the notation, respectively:

$$\lambda_1 = \max\{\sum_{l=0}^n i_l\}, \Delta_1 = \max\{\sum_{l=0}^n (l+1)i_l\}; \lambda_2 = \max\{\sum_{l=0}^n j_l\}, \Delta_2 = \max\{\sum_{l=0}^n (l+1)j_l\}.$$

In this paper we consider the following partial differential equation

$$\frac{\Omega_1(z,w,Dw,\ldots,D^n w)}{\Omega_2(z,w,Dw,\ldots,D^n w)} = H(z,w), \tag{1}$$

where $H(z,w)$ is a meromorphic function on $\mathbb{C}^{n+1}$ with $z \in \mathbb{C}^n$ and $w \in \mathbb{C}$.

In 1978, N. Steinmetz investigated the problem of the existence of admissible solutions of algebraic differential equation of the form

$$\Omega(z,w) = H(z,w), \tag{2}$$

where $\Omega(z,w) = \sum_{(i)} a_{(i)}(z) w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}$, and $H(z,w)$ is quotient of entire functions in variables z and w. They obtained

Theorem A[1] *If the differential equation (2) admits an admissible meromorphic solution $w(z)$, then (2) must be degenerate into a polynomial in w and*

$$\deg_w H(z,w) \le \Delta,$$

where $\Delta = \max\{i_0 + 2i_1 + \ldots + (n+1)i_n\}$.

Recently, the papers[2−4] have investigated the problem of some Malmquist-type theorems of partial differential equations on $\mathbb{C}^n$. In particular, [2] extends Theorem A to partial differential equations:

Theorem B[3] *Let $a_1,\ldots$ be a sequence of distinct complex numbers which tends to a finite limit value a, and set $H_j(z) = H(z,a_j)$. If the partial differential equation $\Omega(z,w,Dw,\ldots,D^n w) = H(z,w)$ admits a meromorphic solution $w(z)$ that satisfies the condition*

$$\sum_{(i)\in I} T(r,a_{(i)}) + T(r,H_j) = S(r,w), j = 1,2,\ldots,$$

then the equation is a polynomial in w and $\deg_w H(z,w) \le w(\Omega)$ (weight of Ω).

In [7] we considered the existence of admissible solution of general algebraic differential equations of the form

$$\frac{\Omega_1(z,w)}{\Omega_2(z,w)} = H(z,w), \tag{3}$$

where

$$\Omega_1(z,w) = \sum_{(i)} a_{(i)}(z) w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n},$$

$$\Omega_2(z,w) = \sum_{(j)} b_{(j)}(z) w^{j_0}(w')^{j_1}\cdots(w^{(n)})^{j_n}$$

are differential polynomials with meromorphic coefficients $\{a_{(i)}\}$ and $\{b_{(j)}\}$, respectively, (i) and (j) are two finite index sets, and $H(z,w)$ is a meromorphic function in z and w.

We obtained

Theorem C[7] *If $w(z)$ is an admissible meromorphic solutions of (3), then $H(z,w)$ must be rational function in w, and the degree of w satisfies*

$$\deg_w H(z,w) \le \lambda + (\Delta - \lambda)(1 - \theta(w,\infty)) \le \Delta,$$

where $\lambda = \max\{\lambda_1, \lambda_2\}$, $\Delta = \max\{\Delta_1, \Delta_2\}, \theta(w,\infty) = 1 - \limsup \frac{\overline{N}(r,w)}{T(r,w)}$.

For Equation (1), we will prove

Theorem 1 *Let $c_1, c_2, \ldots$ be a sequence of distinct complex numbers which tends to a finite limit value c. And set $H_j(z) = H(z, c_j)$. If Equation (1) admits a meromorphic solution $w(z)$ that satisfies the condition*

$$\sum_{(i)} T(r, a_{(i)}) + \sum_{(j)} T(r, b_{(j)}) + T(r, H_j) = S(r,w), j = 1, 2, \ldots,$$

then $H(z,w)$ must be rational function in w, and the degree of w satisfies

$$\deg_w H(z,w) \le \Delta,$$

where $\Delta = \max\{\Delta_1, \Delta_2\}$.

2. Some lemmas

Lemma 1[5] *Let $w(z)$ be a meromorphic function on $\mathbb{C}^n$. Then*

$$\int_{\mathbb{C}^n\langle r\rangle} \log^+(|D^k w(z)|/|w(z)|)\sigma_n \le 17(\log^+(rT(r,w))),$$

for all large r outside a set I with $\int_I \mathrm{d}\log r < \infty$, where $\log^+ x = \log x$, if $x \ge 1$; $\log^+ x = 0$, if $0 \le x < 1$.

Lemma 2 (The second main theorem)[3] *Let $f(z)$ be a meromorphic function on $\mathbb{C}^n$. If $a_1, \ldots, a_q \in \mathbb{P}^1$ are distinct constants, then*

$$(q-2)T(r,f) \le \sum_{i=1}^{q} \overline{N}(r,f,a_i) + S_1(r),$$

where $S_1(r) \le O\left(\log(rT(r,f))\right)$ for all large r outside a set I with $\int_I \mathrm{d}\log r < \infty$.

3. Proof of Theorem 1

Let $w(z)$ be an admissible meromorphic solutions of Equation (1). For $c_1 \in E$, set

$$\varphi_1(z;c_1) = \frac{\Omega_1}{H(z,c_1)(w-c_1)} - \frac{\Omega_2}{w-c_1} = \frac{\Omega_1 - \Omega_2 H(z,c_1)}{H(z,c_1)(w-c_1)}. \tag{4}$$

Because w is a meromorphic solutions of Equation (1), we know that

$$\mathrm{supp}\nu(w,c_1) \subseteq \mathrm{supp}\nu(\varphi_1(z;c_1),0).$$

Take $z \in \mathbb{C}^n$ with $\nu(w, c_1) > 0$ and let $\theta_{n,z}$ denote the ring of holomorphic functions defined in some neighborhood of $z \in \mathbb{C}^n$. If $w - c_1$ is irreducible in $\theta_{n,z}$, then $w - c_1$ devides $\Omega_1 - \Omega_2 H(z, c_1)$ in $\theta_{n,z}$ (Weak Nullstellensatz), which implies $\nu(\varphi_1(z; c_1), \infty) = 0$.

If $w - c_1$ is not irreducible, then there exists an irreducible $g \in \theta_{n,z}$ such that $g(z) = 0$ and g divides $w - c_1$ in $\theta_{n,z}$ because $\theta_{n,z}$ is a unique factorization domain. Then g divides $\Omega_1 - \Omega_2 H(z, c_1)$ in $\theta_{n,z}$. Consequently, we have

$$\nu(\varphi_1(z; c_1), \infty) \le \nu(w, c_1) - 1.$$

Now we take $c_1, c_2 \in E, c_1 \ne c_2$ and set

$$\varphi_2(z; c_1, c_2) = \frac{\Omega_1[H(z, c_2)(w - c_2) - H(z, c_1)(w - c_1)]}{H(z, c_2)(w - c_2)H(z, c_1)(w - c_1)} - \frac{(c_1 - c_2)\Omega_2 H(z, c_1)H(z, c_2)}{H(z, c_2)(w - c_2)H(z, c_1)(w - c_1)}.$$

If $\nu(w, c_j) > 0$ and $a_{(i)} \ne \infty, b_{(j)} \ne \infty, H(z, c_j) \ne 0, \infty$ $(j = 1, 2)$, we have

$$\begin{aligned}
&\Omega_1[H(z, c_2)(w - c_2) - H(z, c_1)(w - c_1)] - (c_1 - c_2)\Omega_2 H(z, c_1)H(z, c_2) \\
&= \Omega_1[H(z, c_2)(w - c_1 + c_1 - c_2) - H(z, c_1)(w - c_1)] - (c_1 - c_2)\Omega_2 H(z, c_1)H(z, c_2) \\
&= \Omega_1[H(z, c_2)(c_1 - c_2)] - (c_1 - c_2)\Omega_2 H(z, c_1)H(z, c_2) = 0.
\end{aligned}$$

It shows that $\nu(\varphi_2(z; c_1; c_2), \infty) \le \nu(w, c_j) - 1, j = 1, 2.$

In general, we take distinct $c_1, c_2, \ldots, c_k \in E$ and set

$$\begin{aligned}
\varphi_k(z; c_1, \ldots, c_k) =& \varphi_{k-1}(z; c_1, \ldots, c_{k-1}) - \varphi_{k-1}(z; c_1, \ldots, c_{k-2}, c_k) \\
=& (\Omega_1 Q_{k-1}(z, w) - \Omega_2 Q_{k-2}(z, w)) / \prod_{j=1}^{k} H(z, c_j)(w - c_j),
\end{aligned} \tag{5}$$

where $Q_k(z, w)$ is a polynomial of degree $k - 1$ in w, and its coefficients are combination with $H_j(z) (j = 1, 2, \ldots, k)$. By induction, from Equation (5), if $\nu(w, c_j) > 0$, and $a_{(i)} \ne \infty, b_{(j)} \ne \infty, H(z, c_j) \ne 0, \infty$ $(j = 1, 2, \ldots, k)$, we have

$$\nu(\varphi_k(z; c_1, c_2, \ldots, c_k), \infty) \le \nu(w, c_j) - 1.$$

Next we prove that $\varphi_{k+1} \equiv 0$ if $w(z)$ is a meromorphic solution of Equation (1).

Suppose $\deg_w H(z, w) = k = \Delta$ and $\varphi_{k+1} \not\equiv 0$. By the first main theorem, it follows that

$$\begin{aligned}
T(r, w) =& T(r, w - c_{k+1}) + O(1) = T(r, \prod_{j=1}^{k+1}(w - c_j) / \prod_{j=1}^{k}(w - c_j)) + o(1) \\
\le& T(r, \varphi_{k+1} / \prod_{j=1}^{k}(w - c_j)) + T(r, \varphi_{k+1} / \prod_{j=1}^{k+1}(w - c_j)) + O(1).
\end{aligned} \tag{6}$$

Now we estimate $T(r, \varphi_{k+1} / \prod_{j=1}^{k}(w - c_j))$ and $T(r, \varphi_{k+1} / \prod_{j=1}^{k+1}(w - c_j))$.

$$m(r, \frac{\varphi_{k+1}}{\prod\limits_{j=1}^{k}(w - c_j)}) = m(r, \frac{\Omega_1 Q_k(z, w) - \Omega_2 Q_{k-1}(z, w)}{\prod\limits_{j=1}^{k+1} H(z, c_j)(w - c_j) \prod\limits_{j=1}^{k}(w - c_j)})$$

$$\leq m(r, \frac{\Omega_1}{\prod\limits_{j=1}^{k+1}(w-c_j)}) + m(r, \frac{Q_k(z,w)}{\prod\limits_{j=1}^{k}(w-c_j)}) + m(r, \frac{\Omega_2}{\prod\limits_{j=1}^{k+1}(w-c_j)}) +$$

$$m(r, \frac{Q_{k-1}(z,w)}{\prod\limits_{j=1}^{k}(w-c_j)}) + 2\sum m(r, \frac{1}{H(z,c_j)}) + O(1).$$

We note that

$$|w/(w-c_j)| \leq 1 + |c_j|/|w-c_j| \leq (1+|c_j|)(1/|w-c_j|)^+ \leq c(1/|w-c_j|)^+, \tag{7}$$

where $|a|^+ = \max\{1, |a|\}$, $c = \max\{1+|c_j|\}$. Thus

$$|\Omega_1/\prod_{j=1}^{k+1}(w-c_j)| \leq c^{k+1}\sum |a_{(i)}(z)|(\prod_j |\frac{Dw}{(w-c_j)}|)\ldots(\prod_j |\frac{D^n w}{(w-c_j)}|)(\prod_j |\frac{1}{(w-c_j)}|^+),$$

$$|\Omega_2/\prod_{j=1}^{k+1}(w-c_j)| \leq c^{k+1}\sum |b_{(j)}(z)|(\prod_j |\frac{Dw}{(w-c_j)}|)\ldots(\prod_j |\frac{D^n w}{(w-c_j)}|)(\prod_j |\frac{1}{(w-c_j)}|^+),$$

where $\prod_j |\frac{D^\alpha w}{(w-c_j)}|$ is $i_{1\alpha}$-fold product, and $\prod_j (|\frac{1}{w-c_j}|)^+$ is $(k+1-\lambda_t-t_0)(t=i,j)$-fold product. So

$$m(r,\Omega_1/\prod_{j=1}^{k+1}(w-c_j)) \leq \sum_{j=1}^{k} m(r, \frac{1}{w-c_j}) + \sum_{(i)} m(r, a_{(i)}) + O\{\sum_{\alpha=1}^{n}\sum_{j=1}^{k+1} m(r, \frac{D^\alpha w}{w-c_j})\}. \tag{8}$$

$$m(r,\Omega_2/\prod_{j=1}^{k+1}(w-c_j)) \leq \sum_{j=1}^{k} m(r, \frac{1}{w-c_j}) + \sum_{(j)} m(r, b_{(j)}) + O\{\sum_{\alpha=1}^{n}\sum_{j=1}^{k+1} m(r, \frac{D^\alpha w}{w-c_j})\}. \tag{9}$$

$$m(r, Q_k(z,w)/\prod_{j=1}^{k}(w-c_j)) \leq \sum_{j=1}^{k} m(r, \frac{1}{w-c_j}) + \sum_{j=1}^{k} m(r, H_j) + O(1). \tag{10}$$

$$m(r, Q_{k-1}(z,w)/\prod_{j=1}^{k}(w-c_j)) \leq \sum_{j=1}^{k} m(r, \frac{1}{w-c_j}) + \sum_{j=1}^{k} m(r, H_j) + O(1). \tag{11}$$

By (8), (9), (10), (11) and Lemma 1, we have

$$\begin{aligned} m(r,\varphi_{k+1}/\prod_{j=1}^{k}(w-c_j)) \leq & 4\sum_{j=1}^{k} m(r, \frac{1}{w-c_j}) + \sum_{(i)} m(r, a_{(i)}) + \sum_{(j)} m(r, b_{(j)}) + \\ & 2\sum m(r, H_j) + S(r,w), \end{aligned} \tag{12}$$

where $S(r,w) = O\{\log(rT(r,w))\}$ for all large r outside a set I with $\int_I \mathrm{d}\log r < \infty$.

Similarly, we may deduce that

$$m(r,\varphi_{k+1}/\prod_{j=1}^{k+1}(w-c_j)) \leq 4\sum_{j=1}^{k+1} m(r, \frac{1}{w-c_j}) + \sum_{(i)} m(r, a_{(i)}) + \sum_{(j)} m(r, b_{(j)}) +$$

$$2\sum m(r,H_j)+S(r,w), \tag{13}$$

for all large r outside a set I with $\int_I \mathrm{d}\log r<\infty$.

Now we estimate $N(r,\varphi_{k+1}/\prod_{j=1}^{k}(w-c_j))$ and $N(r,\varphi_{k+1}/\prod_{j=1}^{k+1}(w-c_j))$. By

$$\varphi_{k+1}/\prod_{j=1}^{k}(w-c_j)=(\Omega_1 Q_k(z,w)-\Omega_2 Q_{k-1}(z,w))/\prod_{j=1}^{k+1}H(z,c_j)(w-c_j)\prod_{j=1}^{k}(w-c_j), \tag{14}$$

we know that the poles of $\varphi_{k+1}/\prod_{j=1}^{k}(w-c_j)$ may arise from one of the following cases:

(i). The poles of $\{a_{(i)}(z)\},\{b_{(j)}(z)\}$;

(ii). The poles and the zeros of $\{H_j(z)\}$;

(iii). The zeros of $w-c_j$ for which the cases (i) and (ii) are not satisfied;

(iv). The poles of $w(z)$.

Case (i). Its contribution to $N(r,\varphi_{k+1}/\prod_{j=1}^{k}(w-c_j))$ is $\sum N(r,\nu(a_{(i)},\infty))+\sum N(r,\nu(b_{(j)},\infty))$.

Case (ii). Its contribution to $N(r,\varphi_{k+1}/\prod_{j=1}^{k}(w-c_j))$ is $\sum N(r,\nu(H_j,\infty))+\sum N(r,\nu(H_j,0))$.

Case (iii). According to the above discussion, we have

$$\nu(\varphi_{k+1}/\prod_{j=1}^{k}(w-c_j),\infty)\leq 2\nu(w,c_j)-1.$$

Thus, its contribution is at most $\sum_{j=1}^{k}[2N(r,\nu(w,c_j))-\overline{N}(r,\nu(w,c_j))]$.

Case (iv). If z_0 is a pole of w with multiplicity τ, then it is the poles of the denominator of right-side of the equality (14) with multiplicity $(2\Delta-1)\tau$. But z_0 is at most the poles of the numerator of right-side of the equality (14) with multiplicity $(2\Delta-1)\tau$. Hence, it follows that the poles of $w(z)$ does not arise from the poles of $\varphi_{k+1}/\prod_{j=1}^{k}(w-c_j)$.

From Cases (i)–(iv) if follows that

$$\begin{aligned}&N(r,\varphi_{k+1}/\prod_{j=1}^{k}(w-c_j))\\&\quad\leq\sum_{j=1}^{k}[2N(r,\nu(w,c_j))-\overline{N}(r,\nu(w,c_j))]+\sum_{j=1}^{k}N(r,\nu(H_j,\infty))+\\&\quad\quad\sum_{j=1}^{k}N(r,\nu(H_j,0))+\sum_{(i)}N(r,\nu(a_{(i)},\infty))+\sum_{(j)}N(r,\nu(b_{(j)},\infty)).\end{aligned} \tag{15}$$

In a similar fashion, we have

$$\begin{aligned}&N(r,\varphi_{k+1}/\prod_{j=1}^{k+1}(w-c_j))\\&\quad\leq\sum_{j=1}^{k+1}[2N(r,\nu(w,c_j))-\overline{N}(r,\nu(w,c_j))]+\sum_{j=1}^{k+1}N(r,\nu(H_j,\infty))+\end{aligned}$$

$$\sum_{j=1}^{k+1}N(r,\nu(H_j,0))+\sum_{(i)}N(r,\nu(a_{(i)},\infty))+\sum_{(j)}N(r,\nu(b_{(j)},\infty)). \tag{16}$$

Combining (6), (12), (13), (15) and (16), we obtain

$$\begin{aligned}T(r,w)\leq &8\sum_{j=1}^{k+1}m(r,\frac{1}{w-c_j})+\sum_{j=1}^{k+1}[4N(r,\nu(w,c_j))-2\overline{N}(r,\nu(w,c_j))]+2\sum_{j=1}^{k+1}T(r,H_j)+\\ &2\sum_{j=1}^{k+1}T(r,\frac{1}{H_j})+2\sum_{(i)}T(r,a_{(i)})+2\sum_{(j)}T(r,b_{(j)})+S(r,w). \end{aligned}\tag{17}$$

We choose 17 systems which are distinct from each other $\{c_j\}$ $(j=1,2,\ldots,17(k+1))$ and apply Inequality (17) to every system. Combining the above seventeen inequalities, we deduce

$$\begin{aligned}17T(r,w)\leq &8\sum_{j=1}^{17(k+1)}m(r,\frac{1}{w-c_j})+\sum_{j=1}^{17(k+1)}[4N(r,\nu(w,c_j))-2\overline{N}(r,\nu(w,c_j))]+\\ &2\sum_{j=1}^{17(k+1)}T(r,H_j)+2\sum_{j=1}^{17(k+1)}T(r,\frac{1}{H_j})+34\sum_{(i)}T(r,a_{(i)})+\\ &34\sum_{(j)}T(r,b_{(j)})+S(r,w).\end{aligned}$$

By Lemma 2, we have

$$\begin{aligned}17T(r,w)\leq &16T(r,w)+2\sum_{j=1}^{17(k+1)}T(r,H_j)+\\ &2\sum_{j=1}^{17(k+1)}T(r,\frac{1}{H_j})+34\sum_{(i)}T(r,a_{(i)})+34\sum_{(j)}T(r,b_{(j)})+S(r,w),\end{aligned}\tag{18}$$

By $\sum_{(i)}T(r,a_{(i)})+\sum_{(j)}T(r,b_{(j)})+T(r,H_j)=S(r,w)(j=1,2,\ldots)$ and Inequality (18), we deduce $1\leq 0$. This is a contradiction. It follows that $\varphi_{k+1}\equiv 0$.

It follows that w satisfies the following equation

$$\Omega_1Q_k(z,w)=\Omega_2Q_{k-1}(z,w).$$

Define

$$R(z,w)=H(z,w)-\frac{Q_k(z,w)}{Q_{k-1}(z,w)}.$$

We claim that $R(z,c_j)\equiv R_j(z)\equiv 0$ for $j=1,2,\ldots$. Assume to the contrary that $R_j\not\equiv 0$. Then

$$\begin{aligned}\overline{N}(r,w=c_j)\leq &N(r,R_j=0)\leq T(r,R_j)+O(1)\\ \leq &T(r,H_j)+\sum_{l=1}^{k+1}T(r,H_l)+O(1)=S(r,w).\end{aligned}$$

By Lemma 2, there are at most two values c_j such that the inequality above holds. Hence $R(z, c_j) \equiv 0$, or

$$H(z, c_j) = \frac{Q_k(z, c_j)}{Q_{k-1}(z, c_j)}, \quad \text{for all} \quad z \in \mathbb{C}^n.$$

Hence, the identity theorem implies $H(z, w) = \frac{Q_k(z,w)}{Q_{k-1}(z,w)}$. This completes the proof.

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一类高阶偏微分方程的亚纯解

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摘要: 利用多复变值分布理论，我们将 Steinmetz 的代数微分方程的 Malmquist 型定理推广到复偏微分方程中.

关键词: 亚纯解; 偏微分方程; Malmquist 型定理.