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Global Existence and Blow-up for a Dissipative Nonlinear Schrödinger Equation with Harmonic Potential

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Abstract: This paper is concerned with a dissipative nonlinear Schrödinger equation with a harmonic potential. By using some intricate inequalities and the argument of priori estimates, some behaviors of the solution are investigated.

Key words: dissipative; nonlinear Schrödinger equation; harmonic potential; global existence; blow up.
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1. Introduction

In this paper we investigate the dissipative nonlinear Schrödinger equation with a harmonic potential

$$iu_t - u_{xx} + \alpha|x|^2u - |u|^2u - iu + iu\int_{-\infty}^{x}|u(t,\xi)|^2\mathrm{d}\xi = 0, \quad x \in \mathbf{R}, \quad t \in \mathbf{R}. \tag{1.1}$$

Here $\alpha > 0$ is a constant and u is a complex value function.

From Vázquez[12] and Ciplatti[3], we know that the nonlinear Schrödinger equation

$$iu_t - u_{xx} - |u|^2u - iu + iu\int_{-\infty}^{x}|u(t,\xi)|^2\mathrm{d}\xi = 0 \tag{1.2}$$

describes the interaction of an intense electromagnetic radiation with nonlinear dispersive medium in the ultra-short laser pulses. Note that the classical cubic Schrödinger equation with a harmonic potential

$$iu_t - u_{xx} + \alpha|x|^2u - |u|^2u = 0 \tag{1.3}$$

is quite different from the classical cubic Schrödinger equation without any potential. For example, for the critical case of Equation (1.3), the soliton always possesses positive energy[1,14,15] (for the existence, see [4], [8]); but for the critical case of the classical cubic Schrödinger equation without any potential, the soliton possesses zero energy.

On the other hand, Equation (1.1) without the nonlinear damped form describes the attractive Bose-Einstein condensate[5,13]. Shu and Zhang[9] investigated it and obtained the property of collapse.

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In this paper, from the view point of mathematic significance, we investigate Equation (1.1) and analyze the behavior of the solution. Originated by Ciplatti[3], we prove that for any initial value, solutions of Equation (1.1) exist on $t \in (T_1, +\infty)$. Furthermore, if the initial value $\|u_0\|_{L^2} \leq \sqrt{2}$ then $T_1 = -\infty$; but if the initial value $\|u_0\|_{L^2} > \sqrt{2}$, then the solutions blow up with L^2 norm as $t \to -\frac{1}{2} \ln \frac{\|u_0\|_2^2}{\|u_0\|_2^2 - 2}$ from the right side. On the other hand, we also obtain that, if $u(x,0) \neq 0$, no matter how small is the norm of the initial value in any space, $\|u(t)\|_{L^2}$ converges to $\sqrt{2}$ as $t \to +\infty$.

This paper is organized as follows. In the second section, we give some necessary preliminaries. In the third section, we give the main results and their proofs.

2. Preliminaries

For Equation (1.1), we impose the initial data as follows

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}. \tag{2.1}$$

Set

$$H := \{u \in H^1(\mathbf{R}) : \int |x|^2 |u|^2 \mathrm{d}x < \infty\}. \tag{2.2}$$

Here and hereafter, for simplicity, we denote $\int_{\mathbf{R}} \mathrm{d}x$ by $\int \mathrm{d}x$. H becomes a Hilbert space, continuously embedded in $H^1(\mathbf{R})$, when endowed with the inner product

$$\langle \varphi, \psi \rangle_H = \int [\nabla \varphi \nabla \overline{\psi} + \varphi \overline{\psi} + |x|^2 \varphi \overline{\psi}] \mathrm{d}x, \tag{2.3}$$

whose associated norm we denote by $\| \cdot \|_H$.

We define two functionals on H as follows

$$M(t) = \int |u(t,x)|^2 \mathrm{d}x, \tag{2.4}$$

and

$$E(t) = \int [\frac{1}{2} |u_x(t,x)|^2 + \frac{1}{2} \alpha |x|^2 |u(t,x)|^2 - \frac{1}{4} |u(t,x)|^4] \mathrm{d}x. \tag{2.5}$$

Using the method and the framework of Oh[7] and Tsutsumi[10,11](also see Cazenave[2]) and combining the argument of Laurencot[6], we can obtain the local well-posedness of the Cauchy problem (1.1)–(2.1).

Proposition 2.1 *Assume initial value $u_0 \in H$. Then there exists a unique solution $u(x,t)$ of the Cauchy problem (1.1)–(2.1) in $C((T_1(u_0), T_2(u_0)); H)$. Here $T_1(u_0) < 0$ and $T_2(u_0) > 0$ are maximal existence times in the sense that if $T_1(u_0) > -\infty$ (respectively, $T_2(u_0) < +\infty$) then $\|u(t)\|_H \longrightarrow +\infty$ as $t \to T_1(u_0)$ and $t > T_1(u_0)$ (respectively, $\|u(t)\|_H \longrightarrow +\infty$ as $t \to T_2(u_0)$ and $t < T_2(u_0)$).*

Proposition 2.2 *For all $u_0 \in H$ and $t \in (T_1, T_2)$, if u is the solution of Equation (1.1) corresponding the initial value u_0, we have*

1^0
$$M(t)=\frac{2\|u_0\|_2^2}{\|u_0\|_2^2+(2-\|u_0\|_2^2)e^{-2t}};\tag{2.6}$$

2^0
$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=\int(|u_x|^2+\alpha|x|^2|u|^2-|u|^4)(1-F(u))\mathrm{d}x.\tag{2.7}$$

Here $F(u)=\int_{-\infty}^{x}|u(t,\xi)|^2\mathrm{d}\xi$.

Proof 1^0. Multiplying Equation (1.1) with $-i\overline{u}$ and integrating on $\mathbf{R}$, then taking the real part, we get
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|u|^2\mathrm{d}x=\int|u|^2\mathrm{d}x-\int|u|^2\int_{-\infty}^{x}|u(t,\xi)|^2\mathrm{d}\xi\mathrm{d}x.\tag{2.8}$$
At the same time, noting that
$$|u|^2\int_{-\infty}^{x}|u(t,\xi)|^2\mathrm{d}\xi=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\int_{-\infty}^{x}|u(t,\xi)|^2\mathrm{d}\xi)^2,$$
we easily have
$$\frac{\mathrm{d}}{\mathrm{d}t}\int|u|^2\mathrm{d}x=2\int|u|^2\mathrm{d}x-(\int|u|^2\mathrm{d}x)^2.$$
Therefore,
$$M(t)=\|u(t)\|_2^2=\frac{2\|u_0\|_2^2}{\|u_0\|_2^2+(2-\|u_0\|_2^2)e^{-2t}}$$
is true.

2^0. Multiplying Equation (1.1) with $\overline{u_t}$ and integrating on $\mathbf{R}$, then taking the real part, we have
$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{2}|u_x|^2+\frac{1}{2}\alpha|x|^2|u|^2-\frac{1}{4}|u|^4\mathrm{d}x=-\Im\int u\overline{u_t}\mathrm{d}x+\Im\int u\overline{u_t}\int_{-\infty}^{x}|u(t,\xi)|^2\mathrm{d}\xi\mathrm{d}x.\tag{2.9}$$
On the other hand, multiplying Equation (1.1) with $-i\overline{u}$ and integrating on $\mathbf{R}$, then taking the imaginary part, we get
$$\Im\int\overline{u}u_t\mathrm{d}x=\int|u_x|^2+\alpha|x|^2|u|^2-|u|^4\mathrm{d}x.\tag{2.10}$$
Therefore, it follows from (2.9) and (2.10) that
$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{2}|u_x|^2+\frac{1}{2}\alpha|x|^2|u|^2-\frac{1}{4}|u|^4\mathrm{d}x\\&=\int|u_x|^2+\alpha|x|^2|u|^2-|u|^4\mathrm{d}x+\Im\int u\overline{u_t}\int_{-\infty}^{x}|u(t,\xi)|^2\mathrm{d}\xi\mathrm{d}x.\end{aligned}\tag{2.11}$$
Taking the complex conjugate of Equation (1.1), then multiplying with $iuF(u)$ and taking the imaginary part, we get
$$\Im\int u\overline{u_t}F(u)\mathrm{d}x=\Re\int\overline{u_{xx}}uF(u)\mathrm{d}x-\int\alpha|x|^2|u|^2F(u)\mathrm{d}x+\int|u|^4F(u)\mathrm{d}x.\tag{2.12}$$

After integrating $\int \overline{u_{xx}} uF(u)\mathrm{d}x$ by part, then it follows from (2.11) and (2.12) that the conclusion is true. $\square$

3. Main results and proofs

In this section, we give the main results.

Theorem 3.1 *If the initial value $u_0 \in H$, then $T_2(u_0) = +\infty$.*

Proof It follows from (2.5) that

$$2E(t) \geq \int |u_x|^2 + \alpha|x|^2|u|^2 - |u|^4 \mathrm{d}x. \tag{3.1}$$

Then by (2.7) and (3.1), we have

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq 2E(t) + \int |u|^4 F(u)\mathrm{d}x. \tag{3.2}$$

Noting that $F(u) = \int_{-\infty}^{x} |u(t,\xi)|^2 \mathrm{d}\xi \leq \int_{-\infty}^{+\infty} |u(t,\xi)|^2 \mathrm{d}\xi = M(t) \leq \max\{2, \|u_0\|_2^2\} := K$ from (2.6), then we have

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq 2E(t) + K\|u\|_4^4. \tag{3.3}$$

At the same time, it follows from Gagliardo-Nirenberg inequality and Young inequality that

$$\|u\|_4^4 \leq C\|u_x\|_2\|u\|_2^3 \leq C\|u_x\|_2 \leq \frac{C^2}{2} + \frac{1}{2}\|u_x\|_2^2. \tag{3.4}$$

Here and hereafter, for simplicity, we use C to denote various positive constants. Therefore, from (3.3) and (3.4), we have

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq 2E(t) + \frac{C^2}{2} + \frac{1}{2}\|u_x\|_2^2. \tag{3.5}$$

Integrating (3.5) on $(0,t)$, we get

$$E(t) \leq E(0) + 2\int_0^t E(s)\mathrm{d}s + Ct + \frac{1}{2}\int_0^t \|u_x(s)\|_2^2 \mathrm{d}s. \tag{3.6}$$

Therefore, it follows from (2.5) and (3.6) that

$$\frac{1}{2}\|u_x\|_2^2 + \frac{1}{2}\|xu\|_2^2 \leq C + Ct + 2\int_0^t \frac{1}{2}\|u_x(s)\|_2^2 + \frac{1}{2}\|xu(s)\|_2^2 \mathrm{d}s + \frac{1}{2}\int_0^t \|u_x(s)\|_2^2 \mathrm{d}s + \frac{1}{4}\|u\|_4^4. \tag{3.7}$$

Then from (3.4) and (3.7), we obtain

$$\|u_x\|_2^2 + \|xu\|_2^2 \leq C(1+t) + C\int_0^t (\|u_x(s)\|_2^2 + \|xu(s)\|_2^2)\mathrm{d}s. \tag{3.8}$$

Using the Gronwall inequality, we get $\|u_x\|_2^2 + \|xu\|_2^2 \leq C(1+\eta)e^{C\eta}$ for all $t \in (0,\eta)$. Therefore we conclude that $T_2(u_0) = +\infty$.

Theorem 3.2 *If the initial value $u_0 \in H$ satisfies $\|u_0\|_2 \leq \sqrt{2}$, then $T_1(u_0) = -\infty$.*

Proof If $\|u_0\|_2 \le \sqrt{2}$, for $t > T_1(u_0)$, it follows from (2.6) that $\|u(t)\|_2 \le \sqrt{2}$. From (2.5), (2.7) and

$$F(u) = \int_{-\infty}^{x} |u(t,\xi)|^2 \mathrm{d}\xi \le \int_{-\infty}^{+\infty} |u(t,\xi)|^2 \mathrm{d}\xi = \|u\|_2^2,$$

it yields that

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &\ge -\|u\|_4^4 - \int F(u)|u_x|^2 \mathrm{d}x - \alpha \int F(u)|x|^2|u|^2 \mathrm{d}x \\ &\ge -\|u\|_4^4 - \int \|u\|_2^2 |u_x|^2 \mathrm{d}x - \alpha \int \|u\|_2^2 |x|^2|u|^2 \mathrm{d}x \\ &\ge -\|u\|_4^4 - \|u\|_2^2\|u_x\|_2^2 - \alpha\|u\|_2^2\|xu\|_2^2. \end{aligned} \tag{3.9}$$

Then it follows from (3.4) and $\|u\|_2^2 \le \max\{2, \|u_0\|_2^2\} := K$ that for all $t \in (T_1, T_2)$

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &\ge -C\|u_x\|_2 - C\|u_x\|_2^2 - C\|xu\|_2^2 \\ &\ge -(\frac{C^2}{2} + \frac{1}{2}\|u_x\|_2^2) - C\|u_x\|_2^2 - C\|xu\|_2^2 \\ &= -C(\|u_x\|_2^2 + \|xu\|_2^2) - C. \end{aligned} \tag{3.10}$$

For $T_1(u_0) < -\eta < 0$, integrating (3.10) on $(-\eta,0)$, we get

$$E(-\eta) \le E(0) + C\int_{-\eta}^{0} \|u_x(s)\|_2^2 + \|xu(s)\|_2^2 \mathrm{d}s + C\eta. \tag{3.11}$$

On the other hand, it follows from (2.5) that

$$\|u_x\|_2^2 + \|xu\|_2^2 \le 2E(t) + \|u\|_4^4. \tag{3.12}$$

Hence by (3.11), (3.12) and (3.4), we have

$$\begin{aligned} &\|u_x(-\eta)\|_2^2 + \|xu(-\eta)\|_2^2 \\ &\quad \le 2E(-\eta) + \|u(-\eta)\|_4^4 \\ &\quad \le 2E(0) + 2C\int_{-\eta}^{0} [\|u_x\|_2^2 + \|xu\|_2^2] \mathrm{d}s + 2C\eta + \|u(-\eta)\|_4^4 \\ &\quad \le C(1+\eta) + C\int_0^{\eta} [\|u_x(-s)\|_2^2 + \|xu(-s)\|_2^2] \mathrm{d}s + \|u(-\eta)\|_4^4 \\ &\quad \le C(1+\eta) + C\int_0^{\eta} [\|u_x(-s)\|_2^2 + \|xu(-s)\|_2^2] \mathrm{d}s + \frac{C^2}{2} + \frac{\|u_x(-\eta)\|_2^2}{2} \end{aligned} \tag{3.13}$$

which yields that

$$\|u_x(-\eta)\|_2^2 + \|xu(-\eta)\|_2^2 \le C(1+\eta) + C\int_0^{\eta} \|u_x(-s)\|_2^2 + \|xu(-s)\|_2^2 \mathrm{d}s. \tag{3.14}$$

Again from Gronwall inequality, it follows that $u(t)$ remains bounded in H on any interval $[-\eta, 0]$. Hence $T_1(u_0) = -\infty$. □

Theorem 3.3 *If the initial value $u_0 \in H$ satisfies $\|u_0\|_2 > \sqrt{2}$, then* $T_1(u_0) = -\frac{1}{2}\ln\frac{\|u_0\|_2^2}{\|u_0\|_2^2 - 2}$

and the solution $u(t)$ blows up with L^2 norm, which means that $\|u(t)\|_2 \longrightarrow +\infty$ as $t \to T_1(u_0)$ from the right side.

Proof From (2.6) and $u_0 \in H$ satisfying $\|u_0\|_2 > \sqrt{2}$, we easily obtain $t > -\frac{1}{2}\ln\frac{\|u_0\|_2^2}{\|u_0\|_2^2-2}$. Therefore, $T_1 \geq -\frac{1}{2}\ln\frac{\|u_0\|_2^2}{\|u_0\|_2^2-2}$. In order to prove our claim, assume that we have a strict inequality in the estimate $T_1 > -\frac{1}{2}\ln\frac{\|u_0\|_2^2}{\|u_0\|_2^2-2}$. For all $t > T_1(u_0)$, we easily have

$$\|u(t)\|_2^2 = \frac{2\|u_0\|_2^2}{\|u_0\|_2^2 - (\|u_0\|_2^2 - 2)e^{-2t}} \leq \frac{2\|u_0\|_2^2}{\|u_0\|_2^2 - (\|u_0\|_2^2 - 2)e^{2T_1(u_0)}} := K_*. \tag{3.15}$$

As we already noticed in the proof of Theorem 3.2, we have

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \geq -C(K_*)[\|u_x\|_2^2 + \|xu\|_2^2] - C(K_*). \tag{3.16}$$

Now using the same argument as we used in the proof of Theorem 3.2, we get that for all $T_1(u_0) < -\eta < 0$, $\|u_x(-\eta)\|_2^2 + \|xu(-\eta)\|_2^2 \leq C(K_*)$ for some constant depending on K_*, which in fact depends on $T_1(u_0)$ and $\|u_0\|_2$. Indeed this is in contradiction with the fact that $T_1(u_0)$ is finite. □

Theorem 3.4 *The sphere $B = \{u \in H : \|u\|_2^2 = 2\}$ is invariant under the flow of the Cauchy problem (1.1)–(2.1). Furthermore, the sphere B attracts every positive orbit of Equation (1.1) starting from u_0 only if $u_0 \neq 0$.*

Proof Indeed, by Theorem 3.1, for any $u_0 \in H$, the solution $u(t)$ of the Cauchy problem (1.1)–(2.1) exists in $(T_1(u_0), +\infty)$. Furthermore, the equation for the $M(t)$ has the explicit solution from (2.6) given by

$$M(t) = \|u(t)\|_2^2 = \frac{2\|u_0\|_2^2}{\|u_0\|_2^2 + (2 - \|u_0\|_2^2)e^{-2t}}. \tag{3.17}$$

It is easily got that $\|u(t)\|_2^2 \longrightarrow 2$ as $t \to +\infty$. □

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一类带调和势的耗散非线性 Schrödinger 方程的全局和爆破性质

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摘要: 针对一类带调和势的耗散非线性 Schrödinger 方程，本文运用一些不等式和先验估计方法研究了其解的行为特征.

关键词: 耗散；非线性 Schrödinger 方程；调和势；整体存在；爆破.