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A Random Walk Problem Involving Generalized Binomial Series

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Abstract In this paper, based on the method of generalized binomial series, the probability that a random lattice point touches a given boundary line is obtained.

Keywords generalized binomial series; Dwass theorem; random walk.

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The generalized binomial series $B_t(z)$ was discovered by J. H. Lambert and defined in [1]

$$B_t(z) = \sum_{k=0}^{\infty} \frac{(tk)!}{(tk-k+1)!}\frac{z^k}{k!} = \sum_{k=0}^{\infty} \binom{tk+1}{k}\frac{1}{tk+1} z^k. \tag{1}$$

$B_t(z)$ satisfies the following identities for all integers t, r

$$B_t(z)^{1-t} - B_t(z)^{-t} = z, \quad B_0(z) = 1 + z, \tag{2.1}$$

$$B_t(z)^r = \sum_{k=0}^{\infty} \binom{tk+r}{k} \frac{r}{tk+r} z^k, \tag{2.2}$$

$$\frac{B_t(z)^r}{1-t+tB_t(z)^{-1}} = \sum_{k=0}^{\infty} \binom{tk+r}{k} z^k. \tag{2.3}$$

It is known that large number of binomial coefficient identities can be derived by the generalized binomial series, and we show that the generalized binomial series can be applied in random walk.

Consider a random point starting at the origin $(0,0)$. The random point moves with probability $1-p$ $(0<p<1)$ one unit to the right and with probability p one unit up, and the random point stops if it touches one of the points of the boundary line l

$$l:\ y = (t-1)x + s,\ t > 1,\ t \in \mathbf{N}, s \in \mathbf{N}.$$

The research for the random lattice point is a classical problem in combinatorics and probability theory, such as the enumeration of lattice paths, limit probability of reaching head-to-tail ratio for coins [2]–[4], etc.. In this paper we obtain the probability P_T that the random point

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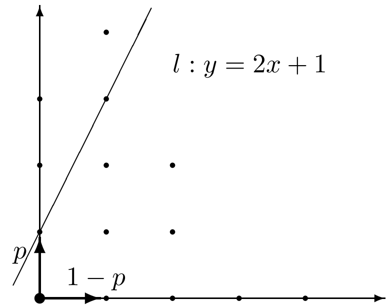

Figure 1

touches the line l with integer slope, and the general result for arbitrary positive slope and intercept of the boundary line l will be given elsewhere.

Theorem *The probability P_T is α^s, where α is the solution of the equation*

$$(1-p)\alpha^t - \alpha + p = 0, \quad 0 < \alpha \leq 1 \tag{3}$$

and

$$P_T \equiv 1, \quad \textit{when} \ \ p \geq \frac{t-1}{t}.$$

Proof Denote by P_{Tk} the probability that the random lattice point first touches the boundary line l with k steps to the right and $(t-1)k+s$ steps up. Then

$$P_T = \sum_{k=0}^{\infty} P_{Tk}.$$

By the Dwass theorem [5], a random sequence $\{X_n, n \geq 1\}$ is independent and identically distributed and the values of X are the integers not more than 1. Let $S_n = m_0 + X_1 + X_2 + \cdots + X_n$. Then

$$P(S_n = m, S_j < m, j < n) = \frac{m - m_0}{n} P(S_n = m), \quad m > m_0,$$

that is, when a random point on the x-axis moves one unit to the right or arbitrary unit to the left or stays at each step, Dwass theorem gives the probability that the random point first reaches the point $(m, 0)$ starting from the point $(m_0, 0)$ after n steps.

Let

$$P(X = 1) = p, \ \ P(X = 1 - t) = 1 - p, \ \ m - m_0 = s,$$

we have

$$P_{Tk} = \frac{s}{tk+s}\binom{tk+s}{k} p^{(t-1)k+s}(1-p)^k, \tag{4}$$

here P_{Tk} implies the known result in [6], [7], that the number of lattice paths starting at the origin first touching the point $(k, (t-1)k+s)$ is $\frac{s}{tk+s}\binom{tk+s}{k}$, and in [7] Gessel considered the sufficiently small probability $1-p$ to the right.

So

$$P_T = p^s \sum_{k=0}^{\infty} \binom{tk+s}{k} \frac{s}{tk+s} [p^{(t-1)}(1-p)]^k = p^s B_t(p^{(t-1)}(1-p))^s \triangleq \alpha^s,$$

where $\alpha = pB_t(p^{(t-1)}(1-p))$, which implies that

$$B_t(p^{(t-1)}(1-p))^{1-t} - B_t(p^{(t-1)}(1-p))^{-t} = p^{(t-1)}(1-p).$$

Then

$$\alpha - p = (1-p)\alpha^t.$$

Obviously, there is a root $\alpha = 1$ of the Equation (3), and the other solutions satisfy

$$\alpha^{t-1} + \alpha^{t-2} + \cdots + \alpha^2 + \alpha = \frac{p}{1-p}, \quad 0 < \alpha < 1. \tag{5}$$

Since $\frac{p}{1-p} > t-1$ and $\alpha^{t-1} + \alpha^{t-2} + \cdots + \alpha^2 + \alpha < t-1$ for $\alpha < 1$ when $p \geq \frac{t-1}{t}$, we have $P_T \equiv 1$ when $p \geq \frac{t-1}{t}$.

There is only a root of the Equation (5) in the interval (0,1) when $p < \frac{t-1}{t}$.

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