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Nested Chain Order

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Abstract Let $X_1, X_2, \ldots, X_k$ be k disjoint subsets of S with the same cardinality m. Define $H(m,k) = \{X \subseteq S : X \not\subseteq X_i \text{ for } 1 \le i \le k\}$ and $P(m,k) = \{X \subseteq S : X \cap X_i \ne \emptyset \text{ for at least two } X_i\text{'s}\}$. Suppose $S = \bigcup_{i=1}^{k} X_i$, and let $Q(m,k,2)$ be the collection of all subsets K of S satisfying $|K \cap X_i| \ge 2$ for some $1 \le i \le k$. For any two disjoint subsets Y_1 and Y_2 of S, we define $\mathcal{F}_{1,j} = \{X \subseteq S : \text{either } |X \cap Y_1| \ge 1 \text{ or } |X \cap Y_2| \ge j\}$. It is obvious that the four posets are graded posets ordered by inclusion. In this paper we will prove that the four posets are nested chain orders.

Keywords poset; normalized matching property; sperner property; nested chain decomposition.

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1. Introduction

Let P be a finite poset. A rank function on P is a function $r : P \to \mathbb{N}$ such that $r(p) = 0$ for the minimal element of P and $r(q) = r(p) + 1$ whenever q covers p. If P admits a rank function, then P is said to be ranked. The number $r(P) = \max\{r(p) : p \in P\}$ is called the rank of P. For $0 \le i \le n$, where $n = r(P)$, the ith level set of P is defined by $P_i = \{p \in P : r(p) = i\}$ and the number $W_i = |P_i|$ is called the ith Whitney number. The sequence $R(P) = \{W_0, W_1, \ldots, W_n\}$ is called the rank sequence of P. We say that P is rank symmetric if $W_i = W_{n-i}$, $0 \le i \le n$, and rank unimodal if $W_j \ge \min\{W_i, W_k\}$, $0 \le i \le j \le k \le n$. Similarly, we say P is log concave if $W_j^2 \ge W_{j-1}W_{j+1}$, $1 \le j \le n-1$.

A ranked poset P is Sperner if no antichain of P has cardinality greater than the largest Whitney number. More generally, P is k-Sperner if no union of k antichains has cardinality greater than the sum of the k largest Whitey numbers, and is strongly Sperner if it is k-Sperner for $1 \le k \le r(P) + 1$. Let $\mathcal{B} \subseteq P_i$, $0 \le i \le n-1$. Then the collection $\nabla\mathcal{B} = \{D \in P_{i+1} : D \text{ covers } B \text{ for some } B \in \mathcal{B}\}$ is called the shade of $\mathcal{B}$. If

$$\frac{|\nabla\mathcal{B}|}{W_{i+1}} \ge \frac{|\mathcal{B}|}{W_i},$$

then we say P has the normalized matching property.

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A chain $p_1 < p_2 < \cdots < p_t$ in a ranked poset P is saturated if p_i covers p_{i-1} for $2 \le i \le t$. Two chains in a ranked poset P are nested if they are saturated and the chain containing the element of least rank in their union also contains the element of greatest rank. Also, P is called a nested chain order if P can be partitioned into pairwise nested chains. It is obvious that nested chain order implies the unimodality and strong spernerity.

Let B_n be the Boolean lattice over an n-set S. It is well known that B_n has the strong spernerity, normalized matching property and is log concave. Let $C(n,k)$ be the collection of all subsets of an n-set S which intersect a fixed k-subset of S. Then $C(n,k)$ is a natural generalization for the subset lattice. Lih[7] first observed this and showed that $C(n,k)$ has the Sperner property. Griggs[5] further showed that $C(n,k)$ has several strong properties.

Let $X_1, X_2, \ldots, X_k$ be k pairwise disjoint subsets of S and with the same cardinality m. Two posets are defined by : $P(m,k) = \{X \subseteq S : X \cap X_i \ne \emptyset \text{ for at least two } X_i\text{'s}\}$ and $H(m,k) = \{X \subseteq S : X \not\subseteq X_i \text{ for } 1 \le i \le k\}$. If $S = \bigcup_{i=1}^{k} X_i$, let $Q(m,k,2)$ be the collection of all subsets K of S such that $|K \cap X_i| \ge 2$ for some $1 \le i \le k$. Note that $P(m,k)$ and $H(m,k)$ are identical when $S = \bigcup_{i=1}^{k} X_i$. For any two disjoint subsets Y_1 and Y_2 of S, let $\mathcal{F}_{1,j}$ be the collection of all subsets X of S such that either $|X \cap Y_1| \le 1$ or $|X \cap Y_2| \le j$. It is obvious that the four posets are graded ordered by inclusion. Moreover, $H(m,k)$ and $P(m,k)$ can be regarded as the generalization of $C(n,k)$. Horrocks[3] proved that $Q(m,k,2)$ has the normalized matching property. In this paper we will prove that the four posets are nested chain orders.

2. Nested chain decomposition

In 1951, de Bruijin et al.[1] discovered an inductive way to decompose B_n into symmetric chains. Subsequently, Greene and Kleitman[4] produced an explicit symmetric chain decomposition of B_n. We now describe this method.

For each subset $X \in B_n$ we associate it with a (0,1)-sequence $\chi(X) = \varepsilon_1\varepsilon_2\cdots\varepsilon_n$, where $\varepsilon_i = 1$ if $i \in X$ and $\varepsilon_i = 0$ otherwise. Whenever a 1 immediately follows a zero in $\chi(X)$, bracket or join them by placing parentheses around them. Continue this pairing procedure as long as possible by pairing an unpaired zero with an unpaired 1 which follows it immediately, or which is separated from the zero only by previously paired digits. For instance, for $n = 12$ and $X = \{1,4,9,10\}$, $\chi(X)$ is bracketed into

$$10(01)00(0(01)1)00.$$

This bracketing is unique for all $X \in B_n$, and, the unpaired zeros must appear on the right hands of unpaired 1's (if they exist). Sets in B_n with the same bracketing (joined pairs) form a saturated chain which is symmetric about middle rank. Such bracketing induces a symmetric chain decomposition of B_n. We denote this chain partition by $\mathcal{E}_n$.

Now suppose that $X_1 \cup X_2 \cup \cdots \cup X_k \subseteq [n]$, where $X_i = \{i + jk : j = 0,1,2,\ldots,m-1\}$ for $i = 1,2,\ldots,k$.

Lemma 2.1 *Let $C = \{A_1 \subset A_2 \subset \cdots \subset A_r\}$ be a chain in $\mathcal{E}_n$ satisfying $C \cap H(m,k) \ne \emptyset$. Then*

$C \subset H(m,k)$ or $C \setminus \{A_1\} \subset H(m,k)$ or $C \setminus \{A_1, A_2\} \subset H(m,k)$.

Proof We first observe that if $A_i \in H(m,k)$, then $A_j \in H(m,k)$ whenever $j \geq i$. Therefore the lemma holds if $A_1 \in H(m,k)$. Thus it remains to consider the case $A_1 \notin H(m,k)$.

Suppose $A_1 \notin H(m,k)$. If the zeros in the first km digits of $\chi(A_1)$ are all paired, then $A_2 \in H(m,k)$. Thus C appears in $H(m,k)$ missing only its first element. Now we distinguish three cases to consider that $\chi(A_1)$ contains at least one unpaired zero in the first km digits.

Case 1. $A_1 = \emptyset$. Then $A_i = \{1, 2, \ldots, i-1\}$. Thus C appears in $H(m,k)$ only missing its first two elements.

Case 2. $A_1 \subseteq X_1$. If $k \geq 3$, the first two digits of $\chi(A_1)$ are unpaired zeros, then $A_2 = A_1 \cup \{1\} \notin H(m,k)$ and $A_3 = A_2 \cup \{2\} \in H(m,k)$. If $k = 2$, the leftmost two unpaired zeros in $\chi(A_1)$ are in position 1 and position j where $j \in X_2$, then $A_2 = A_1 \cup \{1\} \notin H(m,k)$ and $A_3 = A_2 \cup \{j\} \in H(m,k)$. Thus in this case, C appears in $H(m,k)$ only missing its first two elements.

Case 3. $A_1 \cap X_1 = \emptyset$. Let r be the position of the leftmost unpaired zero in $\chi(A_1)$. Suppose $A_1 \subseteq X_2$. If $r \notin X_2$, then $A_2 = A_1 \cup \{r\} \in H(m,k)$. If $r \in X_2$, then the digit in position $(r-1)$ is also an unpaired zero. That's a contrary. If $A_1 \subseteq X_k$ $(k \geq 2)$, then $r = 1$ and $A_2 = A_1 \cup \{1\} \in H(m,k)$. Therefore, C appears in $H(m,k)$ only missing its first element. $\square$

Let $\mathcal{E}_H$ be the chain decomposition of $H(m,k)$ obtained by using the chains of $\mathcal{E}_n$. From the previous lemma, every chain in $\mathcal{E}_H$ is saturated, beginning with a set of size $j, j+1$ or $j+2$ and ending with a set of size $(n-j)$. It is easy to observe that chains in $\mathcal{E}_H$ need not be nested.

As in Ref.[2], we transform $\mathcal{E}_H$ into a nested chain partition as follows. For every chain C from a $(j+2)$-set to an $(n-j)$-set with $j \geq 1$, we will find a uniquely determined chain $\widehat{C}$ from a $(j+1)$-set to an $(n-j-1)$-set with the property that the set at the top of $\widehat{C}$ is contained in the set at the top of C. We form chain $C^{'}$ from a $(j+2)$-set to an $(n-j-1)$-set and $\widehat{C}^{'}$ from a $(j+1)$-set to an $(n-j)$-set. Now, we introduce a result[2].

Lemma 2.2 *Let A and B be elements of B_n and let C_A and C_B be the chains which contain A and B, respectively. If, for all $i \leq n$, $\chi(B)$ contains a paired zero in position i whenever $\chi(A)$ contains a paired zero in position i, then*

$$T(C_B) \subseteq T(C_A)$$

where, for any chain $C \in \mathcal{E}_n$, $T(C)$ denotes the top element of C.

Theorem 2.3 *$H(m,k)$ is a nested chain order.*

Proof Let C be a chain in $\mathcal{E}_H$ which, when considered as an element of $\mathcal{E}_n$, is missing its first two elements. Let X and Y be the first and second elements of C respectively, and Z be the above element of Y when viewed as a chain of $\mathcal{E}_n$. From the above process, we obtain that either $X = \emptyset$ or $X \subseteq X_1$, $Y = X \cup \{1\}$ and $Z = Y \cup \{s\}$ where $s \notin X_1$. Now we distinguish two cases to construct the nested chain decomposition.

Case i. $S = \bigcup X_i$. If $X \neq \emptyset$, let $T = X \cup \{s\}$. Then $T \in H(m,k)$ is uniquely determined by C. Let $\widehat{C}$ be the chain in $\mathcal{E}_n$ which contains T. Then the bottom element of $\widehat{C}$ is T, so $\widehat{C}$ appears entirely in $H(m,k)$. By Lemma 2.2, we obtain that $T(\widehat{C}) \subseteq T(C)$. Construct C' and $\widehat{C}'$ as follows. Let $C' = C \backslash T(C)$ and $\widehat{C}' = \widehat{C} \cup T(C)$. In $\mathcal{E}_H$, replace each such pair $(C, \widehat{C})$ with $(C', \widehat{C}')$ and let $\mathcal{E}'_H$ be the collection of chains so obtained. Now observe that each chain in $\mathcal{E}'_H$ runs from a 2-set to an n-set, or from a j-set to an $(n-j)$-set for some $j \geq 2$ or from a $(j+1)$-set to an $(n-j)$-set for some $j \geq 1$, so $\mathcal{E}'_H$ is a nested chain partition of $H(m,k)$.

Case ii. $\bigcup X_i \subset S$. If $X = \emptyset$, let $T = X \cup \{n\}$. Otherwise, let $T = X \cup \{s\}$. Then $T \in H(m,k)$ is uniquely determined by C. Let $\widehat{C}$ be the chain in $\mathcal{E}_n$ which contains T. Then the bottom element of $\widehat{C}$ is T, so $\widehat{C}$ appears entirely in $H(m,k)$. By Lemma 2.2, we obtain that $T(\widehat{C}) \subseteq T(C)$. Construct C' and $\widehat{C}'$ as follows. Let $C' = C \backslash T(C)$ and $\widehat{C}' = \widehat{C} \cup T(C)$. In $\mathcal{E}_H$, replace each such pair $(C, \widehat{C})$ with $(C', \widehat{C}')$ and let $\mathcal{E}'_H$ be the collection of chains so obtained. Now observe that each chain in $\mathcal{E}'_H$ runs from a j-set to an $(n-j)$-set for some $j \geq 1$ or from a $(j+1)$-set to an $(n-j)$-set for some $j \geq 1$, so $\mathcal{E}'_H$ is a nested chain partition of $H(m,k)$.

Thus we complete the proof. □

Theorem 2.4 *$P(m,k)$ is a nested chain order.*

Proof Let $C = \{A_1 \subset A_2 \subset \cdots \subset A_r\}$ be a chain in $\mathcal{E}_n$ satisfying $C \cap P(m,k) \neq \emptyset$. We first prove that $C \subset P(m,k)$ or $C \setminus \{A_1\} \subset P(m,k)$ or $C \setminus \{A_1, A_2\} \subset P(m,k)$. Similarly to the proof of Lemma 2.1, we only need to consider the case $A_1 \notin P(m,k)$. Suppose $A_1 \notin P(m,k)$. Then $A_1 \cap X_i \neq \emptyset$ for at most one i.

Case i. $A_1 \cap X_1 \neq \emptyset$. Then in $\chi(A_1)$, the leftmost two unpaired zeros are in position 1 and position j where $j \in X_2$, which implies that $A_2 = A_1 \cup \{1\} \notin P(m,k)$ and $A_3 = A_2 \cup \{j\} \in P(m,k)$. Thus C appears in $P(m,k)$ only missing its first two elements.

Case ii. $A_1 \cap X_j \neq \emptyset$ for some $j \geq 2$. If $j = 2$ and $k = 2$, then the position of the leftmost unpaired zero in $\chi(A_1)$ is in X_1; and if $k \geq 3$, the position of the leftmost unpaired zero is either in X_1 or in X_3. If $j \geq 3$, the first digit of $\chi(A_1)$ is an unpaired zero. Thus $A_2 \in P(m,k)$ and C appears in $P(m,k)$ only missing its first element.

Case iii. $A_1 \cap X_j = \emptyset$ for $j = 1, 2, \ldots, k$. It is obvious that the leftmost two unpaired zeros lie in position 1 and position 2. Then $A_2 = A_1 \cup \{1\} \notin P(m,k)$ and $A_3 = A_2 \cup \{2\} \in P(m,k)$. Thus C appears in $P(m,k)$ only missing its first two elements.

Let $\mathcal{E}_P$ be the chain decomposition of $P(m,k)$ obtained by using the chains of $\mathcal{E}_n$. From the previous process, every chain in $\mathcal{E}_P$ is saturated, beginning with a set of size $j, j+1$ or $j+2$ and ending with a set of size $(n-j)$. It is easy to observe that chains in $\mathcal{E}_P$ need not be nested. Subsequently, we transform it into a nested chain.

Suppose C is a chain in $\mathcal{E}_P$ which, when considered as an element of $\mathcal{E}_n$, is missing its first two elements. Let X and Y be the first and second elements of C respectively when viewed as a chain of $\mathcal{E}_n$ and Z be the above element of Y in C. From the above process, we obtain that $Y = X \cup \{1\}$ and $Z = Y \cup \{s\}$, where $s \notin X_1$. If $X \neq \emptyset$, let $T = X \cup \{s\}$. Then $T \in P(m,k)$ is uniquely determined by C. Let $\widehat{C}$ be the chain in $\mathcal{E}_n$ which contains T. Then the bottom element of $\widehat{C}$ is T, so $\widehat{C}$ appears entirely in $P(m,k)$. By Lemma 2.2, we obtain that $T(\widehat{C}) \subseteq T(C)$. Construct C' and $\widehat{C}'$ as follows. Let $C' = C \backslash T(C)$ and $\widehat{C}' = \widehat{C} \cup T(C)$. In $\mathcal{E}_P$, replace each such pair $(C, \widehat{C})$ with $(C', \widehat{C}')$ and let $\mathcal{E}'_P$ be the collection of chains so obtained. Now observe that every chain in $\mathcal{E}'_P$ runs from a 2-set to an $(n-2)$-set, or from a j-set to an $(n-j)$-set for some $j \geq 2$ or from a $(j+1)$-set to an $(n-j)$-set for some $j \geq 1$, so $\mathcal{E}'_P$ is a nested chain partition of $P(m,k)$. Thus we complete the proof. $\square$

Theorem 2.5 *$Q(m,k,2)$ is a nested chain order.*

Proof Suppose that $[n] = X_1 \cup X_2 \cup \cdots \cup X_k$, where $X_i = \{(i-1)m+1, (i-1)m+2, \ldots, im\}$, $i = 1, 2, \ldots, k$. Let $C = \{A_1 \subset A_2 \subset \cdots \subset A_r\}$ be a chain in $\mathcal{E}_n$ satisfying $C \cap Q(m,k,2) \neq \emptyset$. We first prove that $C \subset Q(m,k)$ or $C \setminus \{A_1\} \subset Q(m,k)$ or $C \setminus \{A_1, A_2\} \subset Q(m,k,2)$. If $A_1 \in Q(m,k,2)$, then $C \subseteq Q(m,k,2)$. Thus we may suppose that $A_1 \notin Q(m,k,2)$. We consider two cases.

Case 1. $m > 3$. If $|A_1 \cap X_1| = 1$, suppose that the leftmost unpaired zero in $\chi(A_1)$ is in position i. Then $i \in X_1$ and $A_2 = A_1 \cup \{i\} \in Q(m,k,2)$. Thus C appears in $Q(m,k,2)$ missing only its first element. If $A_1 \cap X_1 = \emptyset$, then the first two digits of $\chi(A_1)$ are unpaired zeros, which deduces that $A_2 = A_1 \cup \{1\} \notin Q(m,k,2)$ and $A_3 = A_2 \cup \{2\} \in Q(m,k,2)$. Thus C appears in $Q(m,k,2)$ missing only its first two elements.

Case 2. $m = 3$. If $|A_1 \cap X_1| = 1$, suppose that the leftmost unpaired zero in $\chi(A_1)$ is in position i. Then either $i \in X_1$ or $i \in X_2$. If $i \in X_1$, then $A_2 = A_1 \cup \{i\} \in Q(m,k,2)$. If $i \in X_2$, then $A_1 \cap X_2 \neq \emptyset$ and $A_2 = A_1 \cup \{i\} \in Q(m,k,2)$. Thus C appears in $Q(m,k,2)$ missing only its first element. If $A_1 \cap X_1 = \emptyset$, then the first two digits are unpaired zeros in $\chi(A_1)$, which deduces that $A_2 = A_1 \cup \{1\} \notin Q(m,k,2)$ and $A_3 = A_2 \cup \{2\} \in Q(m,k,2)$. Thus C appears in $Q(m,k,2)$ missing only its first two elements.

Let $\mathcal{E}_Q$ be the chain decomposition of $Q(m,k,2)$ obtained by using the chains of $\mathcal{E}_n$. Let C be a chain in $\mathcal{E}_Q$ which, when considered as an element of $\mathcal{E}_n$, is missing its first two elements. Let X and Y be the first and second elements of C respectively when viewed as a chain of $\mathcal{E}_n$ and Z be the above element of Y in C. Then C is a chain from a $(j+2)$-set to an $(n-j)$-set. From the above process, we obtain that $Y = X \cup \{1\}$ and $Z = Y \cup \{2\}$. If $j \geq 1$, then $\chi(X)$ contains at least one 1. For any i we define the right position i to be position r, where

$$r = \begin{cases} i+1 & \text{if } m \nmid i \\ i-m+1 & \text{otherwise.} \end{cases}$$

It is obvious that if the digit of $\chi(X)$ in the position i is 1, then the digit in the right position i is zero. Let i be the least integer such that the digit in the position i is 1 and the zero in the right position i is unpaired. Let $T = X \cup \{r\}$. Then $T \in Q(m,k,2)$ is uniquely determined by C. Let $\widehat{C}$ be the chain in $\mathcal{E}_n$ which contains T. Then the bottom element of $\widehat{C}$ is T, so $\widehat{C}$ appears entirely in $Q(m,k,2)$. By Lemma 2.2, we obtain that $T(\widehat{C}) \subseteq T(C)$. Constructing C' and $\widehat{C}'$ as follows. Let $C' = C \backslash T(C)$ and $\widehat{C}' = \widehat{C} \cup T(C)$. In $\mathcal{E}_Q$, replace each such pair $(C, \widehat{C})$ with $(C', \widehat{C}')$ and let $\mathcal{E}'_Q$ be the collection of chains so obtained. Now observe that a chain in $\mathcal{E}'_Q$ runs from a 2-set to an n-set, or from a j-set to an $(n-j)$-set for some $j \geq 2$ or from a $(j+1)$-set to an $(n-j)$-set for some $j \geq 1$, so $\mathcal{E}'_Q$ is a nested chain partition of $Q(m,k,2)$. □

Theorem 2.6 *Let Y_1, Y_2 be two disjoint subsets of S and $|Y_1| = k_1, |Y_2| = k_2$, and $\mathcal{F}_{1,2}$ be the collection of all subsets X of S such that either $|X \cap Y_1| \geq 1$ or $|X \cap Y_2| \geq k$. Then $\mathcal{F}_{1,2}$ is a nested chain order if $k_1 \geq k-1$.*

Proof In $\mathcal{F}_{1,2}$ we bracket and form chains in the same way as B_n, so that $\mathcal{F}_{1,2}$ is partitioned into chains with each lying in one of the chains induced by bracketing B_n. We now consider how a chain in the bracketing of B_n produces a chain in $\mathcal{F}_{1,2}$.

Suppose $Y_1 = \{1, 2, \ldots, k_1\}$ and $Y_2 = \{k_1+1, \ldots, k_1+k_2\}$. Then $X \in B_n$ belongs to $\mathcal{F}_{1,2}$ only if that either $|X \cap Y_1| \geq 1$ or $|X \cap Y_2| \geq k$. That is, only if either one of the first k_1 digits in $\chi(X)$ is a 1 or there exist k 1's in the digits from (k_1+1)th to (k_1+k_2)th. Let C be a chain in the bracketing of B_n and X be its bottom set. Now we distinguish two cases to consider $\chi(X)$.

Case 1. Suppose $\chi(X)$ contains at least k $1's$ in its first (k_1+k_2) digits. Then there exists at least either one 1 in the first k_1 digits or k $1's$ in the digits from (k_1+1)th to (k_1+k_2)th. Otherwise it is contrary to the fact that there are at least k $1's$ in the first (k_1+k_2) digits. Then we obtain that $X \in \mathcal{F}_{1,2}$, which implies that all sets in C belong to $\mathcal{F}_{1,2}$, so C is a chain induced by bracketing $\mathcal{F}_{1,2}$.

Case 2. Suppose $\chi(X)$ begins with at most $(k-1)$ $1's$ in the first (k_1+k_2) digits. If there exists one 1 in the first k_1 digits, then $X \in \mathcal{F}_{1,2}$. This implies that all sets in C belong to $\mathcal{F}_{1,2}$, so C is a chain induced by bracketing $\mathcal{F}_{1,2}$.

Otherwise there is no 1 in the first k_1 digits of $\chi(X)$. If all zeros in the first (k_1+k_2) digits are paired, then this will be true all the way up the chain C, and no set in C belongs to $\mathcal{F}_{1,2}$. Otherwise there is at least one unpaired zero in the first (k_1+k_2) digits. Let T be the set in C which covers X. If there is one unpaired zero in the first k_1 digits, then $T \in \mathcal{F}_{1,2}$. Otherwise there is no unpaired zero in the first k_1 digits, then $k_1 = k-1$ and there are $(k-1)$ $1's$ in the digits from the (k_1+1)th to (k_1+k_2)th. Moreover $\chi(T)$ is formed by changing the leftmost unpaired zero in $\chi(X)$ to a 1. Thus $\chi(T)$ contains k $1's$ in the first (k_1+k_2) digits, which implies $T \in \mathcal{F}_{1,2}$. So all sets above X in C belong to $\mathcal{F}_{1,2}$. Deleting X from C produces a chain in the bracketing of $\mathcal{F}_{1,2}$.

Let $\mathcal{E}'$ be the chain decomposition of $\mathcal{F}_{1,2}$ obtained by using the chains of $\mathcal{E}_n$. From the above two cases, every chain in $\mathcal{E}'$ is saturated, beginning with a set of size j or $j+1$ and ending with a set of size $(n-j)$. Then $\mathcal{E}'$ is a nested chain decomposition of $\mathcal{F}_{1,2}$. □

3. Remarks

From Section 2, we know that the posets $H(m,k)$, $P(m,k)$, $Q(m,k,2)$ and $\mathcal{F}_{1,j}$ have the strong sperner property. Usually, we will ask whether the poset has the normalized matching property or not when it has the sperner property. It is not difficult to verify that $\mathcal{F}_{1,j}$ doesn't employ the normalized matching property if $j \geq 2$. Using the method of Ref. [6], we have verified that $P(m,k)$ is log concave and has the normalized matching property for $k \geq 1$ and $m = 2, 3, 4$. Though we can't verify that $P(m,k)$, $H(m,k)$ and $Q(m,k,2)$ have the normalized matching property for every $k \geq 1$ and $m \geq 1$, we also confirm that the result holds true.

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