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X-s-Permutable Subgroups

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Abstract Let X be a nonempty subset of a group G. A subgroup H of G is said to be X-s-permutable in G if, for every Sylow subgroup T of G, there exists an element $x \in X$ such that $HT^x = T^xH$. In this paper, we obtain some results about the X-s-permutable subgroups and use them to determine the structure of some finite groups.

Keywords finite groups; formations; X-s-permutable subgroups; Sylow subgroups; maximal subgroups.

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1. Introduction

All groups considered in this paper are finite.

It is well known that two subgroups H and T of a group G are said to be permutable if $HT = TH$. A subgroup H of a group G is said to be permutable (or quasinormal) in G if H is permutable with all subgroups of G. A subgroup H of a group G is said to be s-permutable or s-quasinormal in G if $HP = PH$ for all Sylow subgroups P of G.

The permutable subgroups have many interesting properties. For example, Ore[14] proved that every permutable subgroup H of a group G is subnormal in G. Ito and Szěp[10] proved that if H is a permutable subgroup of a group G, then H/H_G is nilpotent. In 1962, Kegel[12] proved that if H is an s-quasinormal subgroup of a soluble group G, then H is subnormal in G. In 1963, Deskins[2] further proved that every s-quasinormal subgroup H of any group G is subnormal. However, for two subgroups H and T of a group G, maybe they are not permutable but there exists an element $x \in G$ such that $HT^x = T^xH$. Recently, Guo, Shum and Skiba introduce the concept of X-permutable subgroup. Let H and T be subgroups of a group G and X a nonempty subset of group G. H is called X-permutable with T if there exists some $x \in X$ such that $HT^x = T^xH$. With this new concept, some new elegant results have been obtained on the structure of groups[3–7]. Later on, J. Huang and W. Guo call a subgroup H of a group

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G s-conditionally permutable in G if for every Sylow subgroup T of G, there exists an element $x \in G$ such that $HT^x = T^xH$[9].

As a continuation, in this paper, we introduce the following new concept:

Definition 1.1 *Let G be a group and X a nonempty subset of G. A subgroup H of G is said to be X-s-permutable in G if, for every Sylow subgroup T of G, there exists an element $x \in X$ such that $HT^x = T^xH$.*

In this paper, we determine the structures of some groups by using the X-s-permutability of some primary subgroups.

Recall that a normal factor H/K of a group G is said to be a Frattini factor if $H/K \subseteq \Phi(G/K)$. A factor H/K is said to be a pd-factor if $p \mid |H/K|$.

We use $\tilde{F}(G)$ to denote the subgroup of G such that $\tilde{F}(G)/\Phi(G) = \mathrm{Soc}(G/\Phi(G))$. $M < \cdot G$ denotes that M is a maximal subgroup of G.

All unexplained notations and terminologies are standard. The reader is referred to Refs. [8] and [15].

2. Preliminaries

Lemma 2.1 *Let G be a group and X a nonempty subset of G. Suppose that $K \trianglelefteq G$ and $H \leq G$. Then:*

(1) If H is X-s-permutable in G, then HK/K is XK/K-s-permutable in G/K.

(2) If HK/K is XK/K-s-permutable in G/K and $K \subseteq H$, then H is X-s-permutable in G.

(3) Assume that $K \subseteq X$, HK/K is X/K-s-permutable in G/K and $(|H|, |K|) = 1$. If G is soluble or K is nilpotent, then H is X-s-permutable in G.

(4) If H is X-s-permutable in G, then $H \cap K$ is X-s-permutable in G.

Proof (1), (2) are clear.

(3) Let $p \in \pi(G)$ and P be a Sylow p-subgroup of G. Then by the hypothesis and (2), HK is X-s-permutable in G. Thus, there exists $x \in X$ such that $HKP^x = P^xHK$. Assume that K is nilpotent and let $\pi = \pi(K) \setminus \{p\}$ and K_1 a Hall π-subgroup of K. Then K_1 is a normal Hall π-subgroup of P^xHK since $(|H|, |K|) = 1$. It follows from Shur-Zassenhass Theorem that there is a Hall π'-subgroup T of P^xHK such that $H \leq T$ and $P^{xy} \leq T$ for some $y \in K$. But, since $|HP^{xy}| = |T|$, $HP^{xy} = T = P^{xy}H$. Because $y \in K \subseteq X$, $xy \in X$. Hence H is X-s-permutable in G. Now assume that G is soluble. Then P^xHK is soluble. We can analogously prove that H is X-s-permutable in G.

(4) Let $p \in \pi(G)$ and P be a Sylow p-subgroup of G. Since H is X-s-permutable, there exists $x \in X$ such that $HP^x = P^xH$. Obviously, $(H \cap K)P^x \subseteq HP^x \cap KP^x = (H \cap KP^x)P^x$, $|H \cap KP^x| = |H||KP^x|/|HKP^x| = |H||K||P^x||HK \cap P^x|/(|K \cap P^x||HK||P^x|) = |HK \cap P^x||H \cap K|/|K \cap P^x|$. Hence $|H \cap KP^x|/|H \cap K| = |HK \cap P^x|/|K \cap P^x|$ is a p-number. It follows that $|HP^x \cap KP^x|/|(H \cap K)P^x|$ is a p-number. However, since P is a Sylow p-subgroup of G, $|HP^x \cap KP^x|/|(H \cap K)P^x|$ is a p'-number. This implies that $|HP^x \cap KP^x| = |(H \cap K)P^x|$, and

consequently $(H\cap K)P^x = HP^x \cap KP^x$ is a subgroup of G. Therefore $(H\cap K)P^x = P^x(H\cap K)$.

For the sake of convenience, we cite here some known results which will be useful in the sequel.

Lemma 2.2[15, IV, Theorem 3.4] *Let G be a group, $N \trianglelefteq G$ and $H \leq G$. If $N \leq \Phi(H)$, then $N \leq \Phi(G)$.*

Lemma 2.3[13, Theorem 3] *Let A and B be subgroups of G such that $G \neq AB$ and $AB^x = B^xA$ for all $x \in G$. Then G has a proper normal subgroup N such that either $A \leq N$ or $B \leq N$.*

Lemma 2.4[11, Lemma 2.8] *Let p be the minimal divisor of the order of a group G. Assume that G is A_4-free and L is a normal subgroup of G. If G/L is p-nilpotent and $p^3 \nmid |L|$, then G is nilpotent.*

Lemma 2.5[1, Theorem 1] *A group G is π-separable if and only if G has a Hall π-subgroup and a Hall π'-subgroup, and for any $p \in \pi$, $q \in \pi'$, G has a Hall $\{p, q\}$-subgroup.*

3. Main results

Theorem 3.1 *Let $\mathfrak{F}$ be saturated formation containing all supersoluble groups and let G be a group and X a soluble normal subgroup of G. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every Sylow subgroup of H is X-s-permutable in G.*

Proof The necessity part is clear and we only need to prove the sufficiency part. Suppose that it is false and let G be a counterexample of minimal order. Obviously, we can assume that $H \neq 1$. We carry out the proof via the following steps.

(1) If N is a minimal normal subgroup of G, then $G/N \in \mathfrak{F}$.

By Lemma 2.1 and the hypothesis, every maximal subgroup of every Sylow p-subgroup of HN/N is XN/N-s-permutable in G/N. Since $(G/N)/(HN/N) \cong G/HN \cong (G/H)/(HN/H) \in \mathfrak{F}$, we see that G/N satisfies the hypothesis. Hence $G/N \in \mathfrak{F}$ by the choice of G.

(2) G has a unique minimal normal subgroup $N = C_G(N) = O_p(G) = F(G)$ for some prime $p \in \pi(G)$, and $\Phi(G) = 1$.

Since $\mathfrak{F}$ is a saturated formation, by (1), we know that $\Phi(G) = 1$ and G has a unique minimal normal subgroup, N say. We first prove that N is soluble. If $N \cap X \neq 1$, then $N \subseteq X$ and so N is soluble. Hence we may assume that $X = 1$. Then, by the hypothesis we have that every maximal subgroup of every Sylow subgroup of H is s-quasinormal in G. Let H_1 be a maximal subgroup of some Sylow p-subgroup of H. Then by Deskins's result[2], H_1 is subnormal in G. If $H_1 \neq 1$, then $H_1 \subseteq O_p(G)$ and so $O_p(G) \neq 1$. Since N is the unique minimal normal subgroup of G, $N \subseteq O_p(G)$ and hence N is soluble. If every maximal subgroup of every Sylow subgroup of H is equal to 1, then $|H| = p_1p_2\cdots p_n$ and clearly H is soluble. It follows that $N \subseteq H$ is also soluble. Now, obviously, $N \subseteq O_p(G) \subseteq F(G) \subseteq C_G(N)$. Since $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $G = NM$. Let $C = C_G(N)$. Then $C = C \cap NM = N(C \cap M)$. It is easy to see that $C \cap M \lhd G$ and so $C \cap M = 1$. This induces that $N = C_G(N)$. Thus (2) holds.

(3) $|N| = p$.

By (2), $|N| = p^\alpha$ for some prime p and a positive integer α. Let P be a Sylow p-subgroup of G. Then $N \subseteq P$ and $N \nsubseteq \Phi(P)$ by Lemma 2.2. Hence there exists a maximal subgroup P_1 of P such that $N \nsubseteq P_1$. Since $N \subseteq H$, it is easy to see that $P_1 \cap H$ is a maximal subgroup of some Sylow p-subgroup of H. By the hypothesis, for any $q \in \pi(G)$ and every Sylow q-subgroup of G, there exists $x \in X$ such that $(P_1 \cap H)G_q^x = G_q^x(P_1 \cap H)$. If $q \neq p$, then $P_1 \cap H$ is a Sylow p-subgroup of $(P_1 \cap H)G_q^x$. By [8, Lemma 3.8.2], $N \cap P_1 = N \cap (P_1 \cap H) = N \cap (P_1 \cap H)G_q^x \trianglelefteq (P_1 \cap H)G_q^x$. It follows that $G_q^x \subseteq N_G(N \cap P_1)$. On the other hand, clearly $N \cap P_1 \trianglelefteq P$. This shows that $N \cap P_1 \trianglelefteq G$ and so $|N| = p$.

(4) The final contradiction:

Since $\mathfrak{F}$ is a saturated formation containing all supersoluble groups, $\mathfrak{F}$ has a formation function f such that $\mathfrak{A}(p-1) \subseteq f(p)$ for any $p \in \pi(\mathfrak{F})$. Hence $G/N = G/C_G(N) \in \mathfrak{A}(p-1) \subseteq f(p)$ by $|N| = p$. Then by (1), we obtain that $G \in \mathfrak{F}$. The proof is completed with the contradiction.

Corollary 3.1.1 *Let G be a group and X a soluble normal subgroup of G. Then G is supersoluble if and only if there exists a normal subgroup H of G such that G/H is supersoluble and every maximal subgroup of any Sylow subgroup of H is X-s-permutable in G.*

Theorem 3.2 *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group. Suppose that $H \trianglelefteq G$ and X is a soluble normal subgroup of G. If $G/H \in \mathfrak{F}$ and every maximal subgroup of every Sylow subgroup of $\tilde{F}(H)$ is X-s-permutable in G, then $G \in \mathfrak{F}$.*

Proof Suppose that the theorem is false and let G be a counterexample of minimal order. Then we proceed with the proof by proving the following claims.

(1) Every minimal normal subgroup of G is contained in $\tilde{F}(H)$.

Let N be a minimal normal subgroup of G. If $N \nsubseteq \tilde{F}(H)$, then $N \cap H = 1$. Obviously, $(G/N)/(HN/N) \cong G/HN \cong (G/H)/(HN/H) \in \mathfrak{F}$. Since $HN/N \cong H/H \cap N = H$, $\tilde{F}(HN/N) \cong \tilde{F}(H) \cong \tilde{F}(H)N/N$. Because $\tilde{F}(H)N/N \subseteq \tilde{F}(HN/N)$, $\tilde{F}(H)N/N = \tilde{F}(HN/N)$. By Lemma 2.1, every maximal subgroup of any Sylow p-subgroup of $\tilde{F}(HN/N)$ is XN/N-s-permutable in G/N. Hence by induction, $G/N \in \mathfrak{F}$. It follows that $G \cong G/(H \cap N) \in \mathfrak{F}$. This contradiction shows that (1) holds.

(2) If N is a minimal normal subgroup of G, then N is soluble.

Assume that N is not soluble. Then $N \nsubseteq X$ and $N \cap X = 1$. It follows that $X \subseteq C_G(N)$. Let P_1 be a maximal subgroup of some Sylow 2-subgroup of $\tilde{F}(H)$ and Q be a Sylow q-subgroup of N, where $q \neq 2$ is a prime divisor of $|N|$. If $P_1 \cap N = 1$, then $4 \nmid |N|$ and hence N is soluble, a contradiction. Suppose $P_1 \cap N \neq 1$. Then we claim that $P_1 \cap N$ permutes with Q^x, where Q^x is a conjugate subgroup of Q in N. In fact, let G_q be a Sylow q-subgroup of G containing Q^x. Then by the hypothesis, there exists $y \in X$ such that $P_1 G_q^y \leq G$. Now $P_1 G_q^y \cap N G_q^y = (P_1 \cap N G_q^y) G_q^y = (P_1 \cap N) G_q^y$ is a subgroup of G and so $(P_1 \cap N) G_q^y \cap N = (P_1 \cap N)(G_q^y \cap N) = (P_1 \cap N) Q^{xy} = (P_1 \cap N) Q^x$ since $X \subseteq C_G(N)$ is a subgroup of G. Thus,

$P_1 \cap N$ permutes with Q^x. If $(P_1 \cap N)Q^x = N$, then by Burnside $p^a q^b$-Theorem, N is soluble. If $(P_1 \cap N)Q^x \neq N$, then by Lemma 2.3, N has a proper normal subgroup M such that $P_1 \cap N \leq M$ or $Q^x \leq M$. If $P_1 \cap N \leq M$, then $4 \nmid |N/M|$ and hence N/M is soluble, which is impossible since N is a non-soluble minimal normal subgroup of G. If $Q^x \leq M$, then M contains a Sylow q-subgroup of N. This is also impossible since N is a direct product of some isomorphic simple groups. The contradiction shows that N is soluble.

(3) $\Phi(H) = 1$.

If $\Phi(H) \neq 1$, then there exists a minimal normal subgroup L of G, such that $L \subseteq \Phi(H)$. Obviously, $\tilde{F}(H)/L = \tilde{F}(H/L)$. It is easy to see that G/L satisfies the hypothesis. Thus $G/L \in \mathfrak{F}$ by the choice of G. Then, since $\mathfrak{F}$ is a saturated formation, $G \in \mathfrak{F}$, a contradiction.

(4) H is soluble and $F(H) = \mathrm{Soc}(H) = \tilde{F}(H) = \times_i N_i$, where N_i is any minimal normal subgroup of H and $|N_i| = p$.

Let R be a minimal normal subgroup of H. For any $x \in G$, R^x is still a minimal normal subgroup of H. Hence $R = R^x$ or $R \cap R^x = 1$. It follows that $R^G = R^{x_1} \times R^{x_2} \times \cdots \times R^{x_n}$. If R is non-soluble, then every minimal normal subgroup of G contained in R^G is also non-soluble by Ref. [15, p46, Example 7.9]. This is contrary to that all minimal normal subgroups of G is soluble. Thus, all minimal subgroups of H is soluble and hence $\tilde{F}(H) = \mathrm{Soc}(H) = F(H)$ since $\Phi(H) = 1$[8, Theorem 1.8.17]. Let $F(H) = N_1 \times N_2 \cdots \times N_n$, where N_i is a minimal normal subgroup of H, $i = 1, 2, \ldots, n$. We claim that $|N_i|$ is a prime. Assume that $|N_i| = p^\alpha$ for some prime p and a positive integer α. Let P be a Sylow p-subgroup of H. Then by Lemma 2.2, $N_i \nsubseteq \Phi(P)$, hence there exists a maximal subgroup P_1 of P such that $N_i \nsubseteq P_1$. Since $N_i \subseteq \tilde{F}(H)$, $P_1 \cap \tilde{F}(H)$ is a maximal subgroup of the Sylow p-subgroup of $\tilde{F}(H)$. By the hypothesis, $P_1 \cap \tilde{F}(H)$ is X-s-permutable in G, that is, for any $q \in \pi(G)$ and $G_q \in Syl_q(G)$, there exists $x \in X$ such that $(P_1 \cap \tilde{F}(H))G_q^x = G_q^x(\tilde{F}(H) \cap P_1)$. If $q \neq p$, then $(P_1 \cap \tilde{F}(H))$ is a Sylow p-subgroup of $(P_1 \cap \tilde{F}(H))(G_q^x \cap H)$. Hence $N_i \cap P_1 = N_i \cap (P_1 \cap \tilde{F}(H)) = N_i \cap (P_1 \cap \tilde{F}(H))H_q^x \trianglelefteq (P_1 \cap \tilde{F}(H))H_q^x$. It follows that $H_q^x \in N_G(N_i \cap P_1)$ for any $q \in \pi(H)$. Clearly $N_i \cap P_1 \trianglelefteq P$. This shows that $N_i \cap P_1 \trianglelefteq H$ and consequently $N_i \cap P_1 = 1$. Thus $|N_i| = p$ is a prime. It follows that $H/C_H(F(H)) = H/\bigcap_i C_H(N_i)$ is abelian. Since $\Phi(H) = 1$, by [8, Lemma 1.8.16], $F(H)$ has a complement M in H. Let $C = C_H(F(H))$. Since $F(H)$ is abelian, $F(H) \leq C$. Hence $C = C \cap [F(H)]M = F(H)(C \cap M)$. Since $C \cap M \trianglelefteq M$ and $[F(H), C] = 1$, $C \cap M \trianglelefteq H = F(H)M$. Since $F(H) = \mathrm{Soc}(H)$, $C \cap M = 1$ and so $C = F(H)$. This induces that $H/F(H) = H/C_H(F(H))$ is abelian. Therefore H is soluble and by Ref. [8, Theorem 1.8.17], $F(H) = \mathrm{Soc}(H) = \tilde{F}(H)$.

(5) $\Phi(G) = 1$.

Assume $\Phi(G) \neq 1$ and let $N \subseteq \Phi(G)$ be a minimal normal subgroup of G. Since H is soluble, by [8, Theorem 1.8.1 and Theorem 1.8.17], $\tilde{F}(H)/N = F(H)/N = F(H/N) = \tilde{F}(H/N)$. By Lemma 2.1, it is easy to see that the hypothesis still holds for the factor group G/N. Hence $G/N \in \mathfrak{F}$ by the choice of G. Then, since $\mathfrak{F}$ is a saturated formation, $G \in \mathfrak{F}$, a contradiction.

(6) Final contradiction:

Let N be a minimal normal subgroup of G. Then by (1), (2), (3) and (4), we see that

$N \subseteq \tilde{F}(H) = F(H)$ and $|N| = p^\alpha$ for some prime p and some positive integer α. Let P be a Sylow p-subgroup of G. Then by Lemma 2.2, $N \nsubseteq \Phi(P)$. Hence there exists a maximal subgroup P_1 of P such that $N \nsubseteq P_1$. Analogously to the above, we can see that $N \cap P_1 \trianglelefteq G$. This implies that $N \cap P_1 = 1$ and so $|N| = p$. Hence $\mathrm{Soc}(G) \subseteq \mathrm{Soc}(H) = F(H) \subseteq F(G) = \mathrm{Soc}(G)$. It follows that $F(G) = \mathrm{Soc}(G) = \mathrm{Soc}(H) = F(H) = \times_i N_i = C_G(F(H)) = \bigcap_i C_G(N_i)$, where N_i is any minimal normal subgroup of G. Since $\mathfrak{F}$ is a saturated formation containing all supersoluble groups, $\mathfrak{F}$ has a formation function f such that $\mathfrak{A}(p-1) \subseteq f(p) \subseteq \mathfrak{F}$ for every p. Because $|N_i| = p$, $G/C_G(N_i) \in \mathfrak{A}(p-1)$. Thus, $G/C_G(N_i) \in f(p) \subseteq \mathfrak{F}$. It follows that $G/F(G) = G/\bigcap_i C_G(N_i) \in \mathfrak{F}$. Now applying Theorem 3.1 leads to $G \in \mathfrak{F}$. With the final contradiction the proof is completed. □

Corollary 3.2.1 *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups, let G be a soluble group and X a normal subgroup of G. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup N of G such that $G/N \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of $F(N)$ is X-s-permutable in G.*

Theorem 3.3 *Let G be a group and p a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if there exists a p-soluble normal subgroup X of G such that for any non-Frattini pd-chief factor H/K of G, there exists a maximal subgroup P_1 of some Sylow p-subgroup of G not covering H/K such that P_1 is X-s-permutable in G.*

Proof **The necessity part:** If G is p-nilpotent and H/K is an arbitrary non-Frattini pd-chief factor of G, then $|H/K| = p$ and there exists a maximal subgroup M of G such that $H \nsubseteq M$, but $K \subseteq M$. Obviously $|G : M| = p$. Let P_1 be a Sylow p-subgroup of M. Then P_1 is a maximal subgroup of some Sylow p-subgroup of G and $H \nsubseteq P_1K$. Since G is p-nilpotent and certainly is p-soluble, we may choose $X = G$. In order to prove that P_1 is X-s-permutable in G, by Sylow theorem we need only to prove that there exists a Sylow q-subgroup Q of G such that P_1Q is a subgroup of G for any prime divisor q of $|G|$. If $q = p$, then $P_1 \subseteq Q$ for some Slow q-subgroup Q and so $P_1Q = Q$ is a subgroup of G. Now assume $q \neq p$. Then M has a Hall $\{p, q\}$-subgroup $P_1Q = QP_1$ by Lemma 2.5, where Q is a Sylow q-subgroup of M. Clearly, Q is also a Sylow q-subgroup of G. Thus we also have P_1Q is a subgroup of G.

The sufficiency part: Suppose that it is false and let G be a counterexample of minimal order. Let N be a minimal normal subgroup of G and $(H/N)/(K/N)$ be a non-Frattini pd-chief factor of G/N. Then H/K is a pd-chief factor of G/K. If $H/K \subseteq \Phi(G/K) = \bigcap_{K \subseteq M <\cdot G} M/K$, then $H \subseteq \bigcap_{K \subseteq M <\cdot G} M$. It follows that $H/N \subseteq \bigcap_{K \subseteq M <\cdot G} M/N$ and hence $(H/N)/(K/N) \subseteq \bigcap_{K \subseteq M <\cdot G} (M/N)/(K/N) = \Phi((G/N)/(K/N))$, a contradiction. This shows that H/K is also a non-Frattini chief pd-factor of G. Then, by the hypothesis, there exists a maximal subgroup P_1 of some Sylow p-subgroup of G such that P_1 is X-s-permutable in G and $H/K \nsubseteq P_1K/K$. By Lemma 2.1, P_1N/N is XN/N-s-permutable in G/N and clearly $(H/N)/(K/N) \nsubseteq (P_1K/N)/(K/N)$. This shows that the hypothesis holds on G/N. Hence, by the choice of G, G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$. We claim that N is p-soluble. Otherwise, we may suppose $X = 1$. Then, by the hypothesis, there exists a maximal subgroup P_1 of some Sylow p-subgroup of G such that P_1 is 1-s-permutable in G, that is, P_1 is s-quasinormal in G. Thus, P_1 is subnormal in G. If $P_1 = 1$, then $p^2 \nmid |G|$. Since $(|G|, p-1) = 1$, G is p-nilpotent, which contradicts the choice of G. If $P_1 \neq 1$, then $P_1 \subseteq O_p(G)$ and so $O_p(G) \neq 1$. Since N is the unique minimal normal subgroup of G, $N \subseteq O_p(G)$ and so N is soluble. Hence our claim holds. This implies that $O_p(N) \neq 1$ or $O_{p'}(N) \neq 1$. Hence N is a p-group or a p'-group. If N is a p'-group, then obviously G is p-nilpotent since G/N is p-nilpotent. Hence we may assume that N is a p-group. Since $\Phi(G) = 1$, N is a non-Frattini p-chief factor of G. By the hypothesis, there exists a maximal subgroup P_1 of some Sylow p-subgroup of G such that $N \nsubseteq P_1$ and P_1 is X-s-permutable in G. Let $q \in \pi(G)$ and Q be a Sylow q-subgroup of G. If $q = p$, then there exists $x \in G$ such that $P_1 < \cdot Q^x$. Hence $P_1 \trianglelefteq Q^x$ and so $N \cap P_1 \trianglelefteq Q^x$. On the other hand, if $q \neq p$, then by the hypothesis, there exists $x \in X$ such that $P_1Q^x = Q^xP_1$. This means that $N \cap P_1Q^x = N \cap P_1 \trianglelefteq P_1Q^x$. Thus $N \cap P_1 \trianglelefteq G$. Since $N \nsubseteq P_1$, $N \cap P_1 = 1$. But since NP_1 is a Sylow p-subgroup of G, we obtain that $|N| = p$. It is easy to see that $N = C_G(N)$. Hence $G/N = G/C_G(N)$ is isomorphic to some subgroup of $\mathrm{Aut}(N)$. Since $|\mathrm{Aut}(N)| \mid p-1$ and $(|G|, p-1)$=1, $G/N = 1$. It follows that $G = N$ is abelian. This final contradiction completes the proof. □

Theorem 3.4 *Let G be a group, p the smallest prime divisor of $|G|$ and X a p-soluble normal subgroup of G. If G/H is p-nilpotent, G is A_4-free and every 2-maximal subgroup of any Sylow p-subgroup of H is X-s-permutable in G, then G is p-nilpotent.*

Proof Assume that the assertion is false and let G be a counterexample of minimal order. Then we prove the theorem by following steps:

(1) $O_{p'}(G) = 1$.

Suppose $O_{p'}(G) \neq 1$. Then by Lemma 2.1, the hypothesis still holds on $G/O_{p'}(G)$. Thus, $G/O_{p'}(G)$ is p-nilpotent by the choice of G. It follows that G is p-nilpotent, a contradiction.

(2) G has a unique minimal normal subgroup L and G/L is p-nilpotent.

Let L be any minimal normal subgroup of G. Clearly, G/L satisfies the hypothesis of the theorem. Hence G/L is p-nilpotent by the choice of G. Since the class of all p-nilpotent groups is closed under subdirect product, clearly, G has a unique minimal normal subgroup, say, L.

(3) G is p-soluble.

Let H_p be a Sylow p-subgroup of H and P_1 a 2-maximal subgroup of H_p. If L is p-soluble, then by (2) G is p-soluble. We may, therefore, assume that L is not p-soluble. Then clearly $X = 1$. By the hypothesis, P_1 is permutable with every Sylow subgroup of G, and consequently P_1 is subnormal subgroup of G. This implies that $P_1 \subseteq O_p(G)$. If $P_1 = 1$, then $|H_p| = p^2$. It follows from Lemma 2.4 that G is p-nilpotent. If $P_1 \neq 1$, then $O_p(G) \neq 1$ and so $L \leq O_p(G)$, a contradiction.

(4) $L = O_p(G) = F(G) = C_G(L)$ and $\Phi(G) = 1$.

Since the class of all p-nilpotent groups is a saturated formation and G/L is p-nilpotent,

$\Phi(G) = 1$. Since G is p-soluble, L is p-soluble. Then by (1), we know that $O_p(G) \neq 1$. Thus $L \subseteq O_p(G)$ and consequently we have $L = O_p(G) = F(G) = C_G(L)$.

(5) $G = [L]M$, where $p^3 \mid |L|$ and M is p-nilpotent.

By (4), L has a complement M in G. Then $G = [L]M$ and $M \cong G/L$ is p-nilpotent. If $p^3 \nmid |L|$, then G is p-nilpotent by Lemma 2.4 which contradicts the choice of G.

(6) Final contradiction.

Let M_p be a Sylow p-subgroup of M and G_p a Sylow p-subgroup of G containing M_p. Clearly $|G_p : M_p| = |L| \geq p^3$. So there exists a 2-maximal subgroup P_1 of G_p such that $M_p \leq P_1$. Put $P = P_1 \cap H$. Since $H_p = G_p \cap H$ is a Sylow p-subgroup of H, $H \cap P_1 = H_p \cap P_1$. Obviously $G_p = LM_p = LP_1 = H_pP_1$. Hence $|H_p : P| = |H_p : H \cap P_1| = |H_p : H_p \cap P_1| = |H_pP_1 : P_1| = |G_p : P_1| = p^2$. This means that $P = P_1 \cap H$ is a 2-maximal subgroup of H_p. By the hypothesis, P is X-s-permutable in G. Thus, for arbitrary $q \in \pi(G)$ and $q \neq p$, there exists a Sylow q-subgroup G_q of G such that $PG_q^x = G_q^xP$ for some $x \in X$. Since $L \cap P = L \cap (P_1 \cap H) = L \cap (P_1 \cap H)G_q^x \trianglelefteq (P_1 \cap H)G_q^x$, $G_q \subseteq N_G(L \cap P)$. On the other hand, since $L \cap P = L \cap (P_1 \cap H) \trianglelefteq P_1$ and $L \cap P \trianglelefteq L$, $L \cap P \trianglelefteq LP_1 = G_p$. This shows that $L \cap P \trianglelefteq G$. If $L \cap P = 1$, then $|LP| \geqslant p^3|P|$ which is impossible since $|H_p| \geqslant |LP|$ and $|H_p : P| = p^2$. If $L \cap P = L$, then $L \subseteq P$ and so $|G_p| = |LP_1| = |P_1|$ which is also impossible. Thus $1 \neq L \cap P \neq L$. The contradiction completes the proof. □

Theorem 3.5 *Suppose that $\mathfrak{F}$ is a saturated formation containing all supersoluble groups. Let G be a group and X a soluble normal subgroup of G. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every primary cyclic subgroup of H is X-s-permutable in G.*

Proof We need only to prove the sufficiency part since the necessity part is clear.

Assume that the assertion is false and let G be a counterexample of minimal order. Then obviously $H \neq 1$. We proceed with the proof by the following steps.

(1) For any non-trivial normal subgroup N of G, we have that $G/N \in \mathfrak{F}$.

By isomorphic theorems, $(G/N)/(HN/N) \cong G/HN \cong (G/H)/(HN/H) \in \mathfrak{F}$. Let T/N be any primary cyclic subgroup of HN/N. Then there exists a cyclic subgroup $\langle x\rangle$ of T such that $T/N = \langle x\rangle N/N$. Suppose that T/N is a p-subgroup of HN/N, then there exists a Sylow p-subgroup H_p of H such that $\langle x\rangle N/N \leq H_pN/N$. Put $x = hn$, where $h \in H_p$, $n \in N$. Then $\langle x\rangle N = \langle hn\rangle N = \langle h\rangle N$. Hence by the hypothesis and Lemma 2.1, T/N is X-s-permutable in G/N. This shows that G/N satisfies the condition of the theorem and so $G/N \in \mathfrak{F}$ by the choice of G.

(2) $\Phi(G) = 1$ and G has a unique minimal normal subgroup L such that $L = O_p(G) = C_G(L)$.

Since $\mathfrak{F}$ is a saturated formation, $\Phi(G) = 1$ and G has a unique minimal normal subgroup. We need only to prove that L is soluble. If $L \subseteq X$, then L is soluble. If $L \nsubseteq X$, then $L \cap X = 1$ and hence $X \subseteq C_G(L)$. Let M be a minimal subnormal subgroup of G contained in L. If M is abelian. Then L is soluble. Assume M is a non-abelian simple group. Then $|\pi(M)| > 2$. Let $p,\ q$ be two different prime divisors of $|M|$ and $\langle m\rangle \neq 1$ be a cyclic p-subgroup of M. We claim that $\langle m\rangle$ permutes with any Sylow q-subgroup Q of M. Assume that G_q is a Sylow q-subgroup of G containing Q. Since $H\neq 1$, $\langle m\rangle\subseteq L\subseteq H$. By the hypothesis, there exists $x\in X$ such that $\langle m\rangle G_q^x=G_q^x\langle m\rangle$. Hence $\langle m\rangle G_q^x\cap M=\langle m\rangle(G_q^x\cap M)=\langle m\rangle Q^x$ is a subgroup of G. But since $X\subseteq C_G(L)$, $Q^x=Q$. It follows that $\langle m\rangle Q=Q\langle m\rangle$. Hence our claim holds. If $\langle m\rangle Q=M$, then by Burnside p^aq^b-Theorem, M is soluble, a contradiction. If $\langle m\rangle Q\neq M$, then by Lemma 2.3, M is not simple. The contraction shows that L is soluble.

(3) $|L|=p$.

Let P be a Sylow p-subgroup of G. Then $L\cap Z(P)\neq 1$. Let L_1 be a subgroup of $L\cap Z(P)$ with order p. Since $L\leq H$, by the hypothesis, L_1 is X-s-permutable in G. Let $q\in\pi(G)$ and G_q be a Sylow q-subgroup of G. Then there exists $x\in X$ such that $L_1G_q^x=G_q^xL_1$. Assume that $p\neq q$. Since $L_1\lhd L\lhd G$, L_1 is a subnormal Hall subgroup of $L_1G_q^x$. Hence $L_1\trianglelefteq L_1G_q^x$ for any $q\in\pi(G)$ and $q\neq p$. This means that $G_q^x\leq N_G(L_1)$. On the other hand, $P\leq N_G(L_1)$ since $L_1\subseteq L\cap Z(P)$. This induces that $L_1\trianglelefteq G$ and consequently $L=L_1$ with order p.

(4) Final contradiction:

By (2) and (3), $G/L=G/C_G(L)\lesssim \mathrm{Aut}(L)$ is a cyclic subgroup of order $p-1$. Then, since $G/L\in\mathfrak{F}$, we obtain that $G\in\mathfrak{F}$. The proof is completed due to the final contradiction. □

Corollary 3.5.1 *Let G be a soluble group, $H\trianglelefteq G$ and X be a normal subgroup of G. If G/H is supersoluble and every primary cyclic subgroup of H is X-s-permutable in G, then G is supersoluble.*

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