*Journal of Mathematical Research & Exposition*
May, 2008, Vol. 28, No. 2, pp. 273–278
DOI:10.3770/j.issn:1000-341X.2008.02.005
Http://jmre.dlut.edu.cn

# Finite Groups in Which Every Subgroup Is Abelian or Normal

*TANG Feng*[1,2], *QIAN Guo Hua*[1]
(1. Department of Mathematics, Changshu Institute of Technology, Jiangsu 215500, China;
2. School of Mathematics and Computer Science, Nanjing Normal University, Jiangsu 210097, China)
(E-mail: tangff-04@163.com)

**Abstract** The main object of this paper is to investigate the finite groups in which every subgroup is either abelian or normal. We obtain a characterization of the groups for the nonnilpotent case, and we also give some properties for the nilpotent case.

**Keywords** abelian subgroup; normal subgroup; Frobenius group; nilpotent group

**Document code** A
**MR(2000) Subject Classification** 20D10
**Chinese Library Classification** O152.1

## 1. Introduction and main results

All groups considered in this paper are finite. Finite nonabelian groups in which every subgroup is abelian were classified by Miller and Moreno in 1903[1]. Also, the structure of finite groups in which every subgroup is normal were given by Dedekind and Bear[2, Theorem 5.3.7]. The aim of this paper is to classify the finite groups in which every subgroup is either abelian or normal.

We write $G=H[F]$ to denote a semidirect product of a subgroup $H$ and a normal subgroup $F$ of $G$. Let $G=H[F]$ be a Frobenius group with the kernel $F$ and a cyclic complement $H$. If $F$ is always an irreducible $H_1$-subgroup for any $1<H_1\leq H$, then $G$ is called a $(*)$-Frobenius group.

Recall that a finite $p$-group $G$ is called extraspecial if $G'=Z(G)=\Phi(G)$ is of order $p$.

Suppose that $1=F_0<F_1<\cdots<F_n=G$ is a normal series of a solvable group $G$ such that $G_{i+1}/G_i$ is the largest normal nilpotent subgroup of $G/G_i$, that is, $G_{i+1}/G_i$ is the Fitting subgroup of $G/G_i$. Then the Fitting length of the solvable group $G$ is defined by $n$, and we write $nl(G)=n$.

In this paper, $p$ always denotes a prime integer, and the other notations are standard and are taken from the [2]. For example, $F(G)$, $\Phi(G)$ and $G'$ denote the Fitting subgroup, the Frattini subgroup and the derived subgroup, respectively, of $G$. The main results of this paper are as follows.

**Received date**: 2006-01-04; **Accepted date**: 2006-07-02
**Foundation item**: the National Natural Science Foundation of China (No. 10571128); the Natural Science Foundation of Jiangsu Education Committee (No. 05KJB110002).

**Theorem 1.1** *Let $G$ be a nonnilpotent group. Then every subgroup of $G$ is abelian or normal if and only if $G/Z(G)$ is a $(*)$-Frobenius group and $G$ satisfies one of the following properties:*

*(1) $F(G)$ is abelian;*

*(2) $F(G) = P \times Q$, where $P$ is an extraspecial $p$-group of order $p^3$, and $Q = O_{p'}(Z(G))$. In particular, $G/Q \cong SL(2,3)$ for the case when $p = 2$.*

**Theorem 1.2** *Let $G$ be a nilpotent but nonabelian group. If every subgroup of $G$ is abelian or normal, then $G = P \times A$, where $P \in \mathrm{Syl}_p(G)$ is nonabelian, and $A$ is an abelian $p'$-Hall subgroup of $G$. Moreover, there exists an abelian normal subgroup $N$ of $P$ such that every subgroup of $P/N$ is normal in $P/N$.*

## 2. Proofs

**Lemma 2.1** *Let $G$ be a finite group in which every subgroup is abelian or normal. Then*

*(1) The property is closed under taking subgroups and quotient groups;*

*(2) $G$ is solvable;*

*(3) $nl(G) \le 2$.*

**Proof** (1) This is clear.

(2) Suppose that $G$ is simple. Then any proper subgroup of $G$ is abelian, and this implies by [3, Ch1, Example 7.1] that $G$ is a cyclic group of prime order. Suppose that $G$ is not simple and let $N$ be a minimal normal subgroup of $G$. We conclude by (1) and by induction on group order that $N$ and $G/N$ are both solvable, and so is $G$.

(3) Note that the hypothesis is inherited by any quotient group of $G$. Suppose that $G$ possesses different minimal normal subgroups $M, N$. We have by induction that $nl(G/M), nl(G/N) \le 2$, and so

$$nl(G) = \max\{nl(G/M), nl(G/N)\} \le 2,$$

and we are done. Consequently, we may assume that there is only one minimal normal subgroup, say, $N$ in $G$.

Assume that $\Phi(G) > 1$. Then $nl(G/\Phi(G)) \le 2$ by induction, and so $nl(G) = nl(G/\Phi(G)) \le 2$, and we are done. Assume that $\Phi(G) = 1$. Then there exists a subgroup $A$ of $G$ such that $G = A[N]$. Clearly $A$ is not normal in $G$, so $A$ is abelian, and then $nl(G) \le 2$ as desired.

**Lemma 2.2**[3, Ch 7, Theorem 2.7] *If a $\pi'$-group $H$ acts on an abelian $\pi$-group $G$, then $G = C_G(H) \times [G, H]$.*

**Lemma 2.3**[3, Ch 5, Theorem 4.3(3)] If $G$ is a solvable group, then $C_G(F(G)) \le F(G)$.

**Proof of Theorem 1.1** $\Longrightarrow$. Suppose that $G$ satisfies the hypothesis of the theorem. By Lemma 2.1, $G/F(G)$ is nilpotent. Write $F = F(G)$.

**Step 1.** There exists a chief factor $F/L$ of $G$ such that $G/L$ is nonnilpotent.

Clearly, $F/\Phi(G)$ is the Fitting subgroup of $G/\Phi(G)$, and $F/\Phi(G) = V_1 \times \cdots \times V_s$, where $V_i$ is minimal normal subgroup of $G/\Phi(G)$ $(i = 1, 2, \ldots s)$. Also, we know that $F/\Phi(G)$ has a complement, say, $Y/\Phi(G)$, in $G/\Phi(G)$. Then there exists some $V_i$ such that $Y/\Phi(G)$ acts nontrivially on $V_i$. In fact, if $Y/\Phi(G)$ always acts on $V_i$ trivially for any $1 \le i \le s$, then $Y/\Phi(G)$ acts trivially on $F/\Phi(G)$, and so $G/\Phi(G)$ and $G$ are nilpotent, a contradiction. Assume that $Y/\Phi(G)$ acts nontrivially on $V_i$ and set $L = V_1 \times \cdots \times V_{i-1} \times V_{i+1} \times \cdots \times V_s$. We see that $G/L$ is not nilpotent.

**Step 2.** $L = Z(G)$, $G = AF$, where $A$ is a maximal and abelian subgroup of $G$ with $A \cap F = L$.

It is easy to see that $F/L$ has a complement, say, $A/L$, in $G/L$, and that $A/L \cong G/F$ is nilpotent. Observe that if $A/L$ is normal in $G/L$, then $G/L = A/L \times F/L$ is nilpotent which contradicts the Step 1. Thus $A/L$ is not normal in $G/L$, so $A$ is not normal in $G$, and thus $A$ is abelian by the hypothesis. Note that $G/L = A/L[F/L]$, where $F/L \cong V_i$ is a chief factor of $G$. This implies that $A/L$ is a maximal subgroup of $G/L$, and so that $A$ is a maximal and abelian subgroup of $G$.

Clearly, $C_G(L) \ge A$, and $C_G(L) \lhd G$ as $L$ is normal in $G$. Since $A$ is maximal but not normal in $G$, we conclude that $C_G(L) = G$, that is, $L \le Z(G)$. Observe that $F > Z(G) \ge L$ and $F/L$ is a chief factor of $G$, it follows that $L = Z(G)$.

**Step 3.** $G/L$ is a $(*)$-Frobenius group with kernel $F/L$ .

As $F(G/Z(G)) = F(G)/Z(G)$, we have by Step 2 that $G/L = A/L[F/L]$, where $A/L$ is abelian and $F/L$ is the Fitting subgroup of $G/L$. Since $F/L$, as a chief factor of $G$, is an elementary abelian $p$-group for some prime $p$, we see that $F/L$ is a normal Sylow $p$-subgroup of $G/L$ (Indeed, if $U/L$ is a Sylow $p$-subgroup of $G/L$, then $U/L$ is a normal $p$-subgroup of $G/L$ because $G/F$ is nilpotent, and so $U/L \le F(G/L)$, and $U \le F$). Now $A/L$ acts coprimely on $F/L$.

For each $h \in A/L - \{1\}$, by Lemma 2.2 we have $F/L = C_{F/L}(h) \times [F/L, \langle h \rangle]$. If there exists some $h \in A/L - \{1\}$ such that $[F/L, \langle h \rangle] = 1$, then $F/L = C_{F/L}(h)$ and $h \in C_{G/L}(F/L) \le F/L = F(G/L)$, which contradicts Lemma 2.3. Thus $[F/L, \langle h \rangle] > 1$ for any $h \in A/L - \{1\}$. Clearly, $\langle h \rangle [F/L, \langle h \rangle]$ is not abelian, and it follows by our hypothesis that $\langle h \rangle [F/L, \langle h \rangle] \lhd G/L$. Then

$$[F/L, \langle h \rangle] = F/L \cap \langle h \rangle [F/L, \langle h \rangle] \lhd G/L.$$

Since $F/L$ is a chief factor of $G$, we have $F/L = [F/L, \langle h \rangle]$, thus $C_{F/L}(h) = 1$, and consequently $G/L$ is a Frobenius group with the kernel $F/L$ and a complement $A/L$. Moreover, by [3, Ch8, Theorem 7.9], $A/L$ is cyclic.

Again we claim that $F/L$ is an irreducible $A_1/L$-subgroup for any $1 < A_1/L < A/L$. Suppose that $F_1/L$ is an irreducible $A_1/L$-subgroup of $F/L$, where $1 < F_1/L \le F/L$ and $1 < A_1/L < A/L$. Since $G_1/L = A_1/L[F_1/L]$ is not abelian, it follows that $G_1/L \lhd G/L$. This implies that $F_1/L = G_1/L \cap F/L$ is normal in $G/L$, and so $F_1/L = F/L$ as wanted.

**Step 4.** Final proof of the "only if" part.

We need only to consider the case when $F$ is not abelian.

We claim first that $L = Z(F) = Z(G)$, and $|F/L| = p^2$ for some prime $p$. Recall that $L = Z(G)$ and $F/L$ is a chief factor of $G$. Since $Z(F)$ is normal in $G$ with $L \le Z(F) < F$, we have

$$Z(F) = L.$$

Assume that $|F/L| = p$. Then $F$ is abelian by [3, Ch4, Theorem 5.8], a contradiction. Thus $|F/L| \ge p^2$, and we can find distinct maximal subgroups $H_1, H_2$ of $F$ in which $L$ is contained. Since $H_i/L < F/L$, $H_i/L$ is not normal in $G/L$, and so $H_i$ is not normal in $G$ for any $i = 1, 2$. Thus $H_1, H_2$ are abelian. Observe that $|F : H_1 \cap H_2| = p^2$, and $C_F(H_1 \cap H_2) \ge \langle H_1, H_2\rangle = F$. We conclude that $H_1 \cap H_2 = Z(F) = L$ and $|F : L| = p^2$.

Now we conclude that $F = P \times Q$, where $P$ is the nonabelian Sylow $p$-subgroup of $F$ with $|P : Z(P)| = p^2$, and $Q = O_{p'}(F) = O_{p'}(Z(G))$. To finish the proof of the part "only if", we need to show that $P$ is an extraspecial group of order $p^3$. To this end, we can assume by Lemma 2.1 that

$$Q = 1, \text{ i.e., } F = P.$$

Since $|P : Z(P)| = p^2$, $Z(G) = Z(P) \ge \Phi(P) \ge P'$. Suppose that $P' < Z(P)$. Let $K \cong G/P$ be a cyclic $p'$-subgroup. Then $P/P' = C_{P/P'}(K) \times [P/P', K]$ by Lemma 2.2. Since $K$ acts trivially on $C_{P/P'}(K)$ and $P'$, $C_{P/P'}(K) = Z(P)/P'$. Now $P/P' = Z(P)/P' \times V/P'$, where $V/P' = [P/P', K] \cong P/Z(P)$. Write $Z(P)/P' = \langle a_1\rangle \times \cdots \times \langle a_t\rangle$, and assume that $a_1$ of order $p^{k+1}$ for some integer $k \ge 0$. Since $P/Z(P)$ cannot be cyclic, we have $V/P' = \langle b_1\rangle \times \langle b_2\rangle$. Then

$$P/P' = \langle a_1\rangle \times \cdots \times \langle a_t\rangle \times \langle b_1\rangle \times \langle b_2\rangle = \langle a_1\rangle \times \cdots \times \langle a_t\rangle \times \langle a_1^{p^k} b_1\rangle \times \langle b_2\rangle.$$

Set $V_1/P' = \langle a_1^{p^k} b_1\rangle \times \langle b_2\rangle$. Let us investigate $V, V_1$. Assume that $V$ (or $V_1$) is abelian. Then $C_P(b_2) \ge \langle V, Z(P)\rangle = P$, $b_2 \in Z(P)$, a contradiction. Assume that neither $V$ nor $V_1$ is abelian. Then $V, V_1$ are both normal in $G$. This implies that

$$\langle b_2\rangle = V_1/P' \cap V/P' \lhd G/P'.$$

which contradicts the fact that $V/P' \cong P/Z(P)$ is a chief factor of $G$. Thus $P' = Z(P)$, and so $P' = Z(P) = \Phi(P)$. Now applying [5, Lemma 6] we conclude that $|P'| = p$, and so $P$ is an extraspecial $p$-group of order $p^3$.

Furthermore, suppose $p = 2$. Then $P$ is of order 8. Note that if $P \cong D_8$, then $P/Z(P)$ cannot be a chief factor of $G$ because $P$ has unique cyclic subgroup of order 4. So $P \cong Q_8$. Also since the group $G/P$ is isomorphic to a subgroup of $\mathrm{Aut}(P/Z(P)) = S_3$, we have $|G/P| = 3$, and $G \cong \mathrm{SL}(2, 3)$.

$\Longleftarrow$. Let $G/Z(G)$ be a $(*)$-Frobenius group satisfying the condition of the theorem. Write $F = F(G)$. Then $F/Z(G) = F(G/Z(G))$ is the Frobenius kernel of $G/Z(G)$. Assume that $F/Z(G)$ is a $p$-group for some prime $p$, and let $P$ be a Sylow $p$-subgroup and $A$ be a $p'$-Hall subgroup, respectively, of $G$. Clearly, $P \le F$ is a normal sylow $p$-subgroup of $G$. Let $D = O_{p'}(Z(G))$. Then $F = P \times D$, $G/F \cong A/D$ is cyclic. Since $D \le Z(A)$, it follows that $A$ is abelian by [3, Ch4, Theorem 5.8(1)]. Let $G_1$ be any subgroup of $G$. Clearly, we may write

$G_1 = P_1A_1$, where $P_1 \leq P$ is normal in $G_1$, and $A_1$ can be assumed to be a subgroup of the abelian $p'$-group $A$. In what follows, we shall prove that $G_1$ is either abelian or normal in $G$.

**Case 1.** $F = F(G)$ is abelian.

Since $P = C_P(A) \times [P, A]$ by Lemma 2.2, we have $G = A[P] = A[C_P(A) \times [P, A]] = C_P(A) \times ([P, A]A)$. Set $H = [P, A]A$. Then $G = C_P(A) \times H$, $C_P(A) = P \cap Z(G)$, $Z(H) = A \cap Z(G)$, and $[P, A]$ is the normal Sylow $p$-subgroup of $H$.

Assume that $A_1 = 1$. Then $G_1 = P_1$ is abelian. Assume that $P_1 = P$. Then $G_1 = PA_1 \geq P_1 \geq G'$ is normal in $G$. Assume that $P_1 \leq Z(G)$. Then $G_1 = P_1 \times A_1$ is abelian.

In the rest of this case, we assume that $P_1 < P, P_1 \not\leq Z(G)$, and $A_1 > 1$.

**Case 1.1.** Suppose that $P_1 \leq [P, A]$. Clearly, $A_1Z(H)/Z(H)$ is a nontrivial $p'$-subgroup and $P_1Z(H)/Z(H)$ is a nontrivial $p$-subgroup of $H/Z(H)$. Since $H/Z(H) \cong G/Z(G)$ is a $(*)$- Frobenius group, $P_1Z(H)/Z(H)$ is the Frobenius kernel of $H/Z(H)$. Thus

$$P_1 \cong P_1Z(H)/Z(H) \cong F/Z(G) \cong P/(P \cap Z(G)) = P/C_P(A) \cong [P, A],$$

and then $P_1 = [P, A]$. Now we have

$$G_1 = A_1P_1 = A_1[P, A] \geq [P, A] \geq G',$$

so $G_1$ is normal in $G$.

**Case 1.2.** Suppose that $P_1 \not\leq [P, A]$. Arguing as in the above paragraph, we conclude that $P_1 > [P, A]$. This also implies that $G_1 > [P, A] \geq G'$ is normal in $G$.

**Case 2.** $F = F(G) = P \times D$ is not abelian, where $P$ is an extraspecial group of order $p^3$, $D = O_{p'}(Z(G)) = O_{p'}(F)$.

Assume that $P_1 = P$. Then $G_1 = A_1P \geq P \geq G'$, and so $G_1$ is normal in $G$.

Assume that $P_1 \leq Z(P)$. Then $Z(P) = O_p(Z(G))$ implies that $G_1 = A_1[P_1] = A_1 \times P_1$ is abelian.

Assume that $Z(P) < P_1 < P$ and $A_1 = 1$. Since $P$ is of order $p^3$, $G_1 = P_1$ is abelian of order $p^2$.

Assume finally that $Z(P) < P_1 < P$. Then $P_1$ is abelian and of order $p^2$. Suppose that $A_1 \not\leq Z(G)$. It is clear that $A_1Z(G)/Z(G)$ is a proper $p'$-subgroup of $G/Z(G)$, and that $P_1Z(G)/Z(G)$ is a proper $p$-subgroup of $G/Z(G)$. Since $G/Z(G)$ is a $(*)$-Frobenius group, it forces $P_1Z(G)/Z(G)$ to be $F/Z(G)$, and then $P_1Z(P) = P$, a contradiction. Therefore, $A_1 \leq Z(G)$, and then $G_1 = A_1[P_1] = A_1 \times P_1$ is abelian. Our proof is now completed. □

**Proof of Theorem 1.2** Let $P$ be a nonabelian Sylow $p$-subgroup of $G$. Suppose that there is another nonabelian Sylow $q$-subgroup $Q$ of $G$, where $p \neq q$. We may assume $p$ is odd. Assume that all subgroups of $P$ are normal in $P$. Then $P$ is abelian by the structure of Hamilton group, a contradiction. Assume that there is a subgroup $P_1$ of $P$ which is not normal in $P$. Then $P_1Q$ is not normal in $G$, and so $P_1Q$ is abelian, thus $Q$ is abelian, a contradiction. Therefore, $P$ is the unique nonabelian Sylow subgroup of $G$, and we may write $G = P \times A$, where $A$ is an abelian

$p'$-Hall subgroup of $G$.

Let $B$ be an abelian subgroup of $P$ with maximal order.

**Case 1.** Suppose that $B$ is normal in $P$.

Set $N = B$. For any proper subgroup $K/N$ of $P/N$, $K$ is not abelian by the choice of $B$, and hence $K/N$ is normal in $P/N$.

**Case 2.** $B$ is not normal in $P$.

There exists a subgroup $M$ of $P$ such that $B < M$ and $|M : B| = p$. Then $M$ is not abelian, and so $M$ is normal in $P$. Since $B$ is not normal in $P$, there exists an element $x \in P$ such that $B^x \neq B$. Clearly, $B$ and $B^x$ are contained in $M^x = M$. Since $B$ and $B^x$ are distinct maximal subgroups of $M$, it follows that $BB^x = M$. Thus $M/B^x = BB^x/B^x \cong B/(B \cap B^x)$. Therefore, $|M : B \cap B^x| = |M : B| \cdot |M : B^x| = p^2$. Since $C_M(B \cap B^x) \geq \langle B, B^x \rangle = M$, we have $B \cap B^x \leq Z(M)$. Now it is easy to see that $B \cap B^x = Z(M) \lhd P$. By the property of chief series of $p$-groups, there exists a normal subgroup $N$ of $P$ such that $B \cap B^x < N < M$. Arguing as in Case 1, we conclude that every subgroup of $P/N$ is normal in $P/N$. Our proof is completed now. □

**Corollary 2.1** *Let $G$ be a finite $p$-group of odd order, and suppose that every subgroup of $G$ is abelian or normal. Then the derived length of $G$ is at most 2.*

**Proof** By Theorem 1.2, there exists an abelian normal subgroup $N$ of $G$ such that every subgroup of $G/N$ is normal. Thus $G/N$ is abelian due to $p \geq 3$ and the structure of Hamilton groups. Therefore, $G' \leq N$ is abelian.

**Corollary 2.2** *Let $G$ be a finite 2-group, and suppose that every subgroup of $G$ is abelian or normal. Then there exists an abelian normal subgroup $N$ of $G$ such that $G/N$ is abelian or a direct product of $Q_8$ and an elementary abelian 2-group.*

**Proof** By the proof of Theorem 1.2, there exists an abelian normal subgroup $N$ of $G$ such that every subgroup of $G/N$ is normal. Therefore, there exists an abelian normal subgroup $N$ of $G$ such that $G/N$ is abelian or direct product of $Q_8$ and an elementary abelian 2-group.

## References

[1] MILLER G A, MORENO H C. *Non-abelian groups in which every subgroup is abelian* [J]. Trans. Amer. Math. Soc., 1903, **4**(4): 398–404.
[2] ROBINSION D J S. *A Course in the Theory of Groups* [M]. Springer-Verlag, New York, 1993.
[3] XU Mingyao. et al. *An Introduction to Finite Groups* [M]. Beijing: Science Press, 2001. (in Chinese)
[4] HUPPERT B. *Endlich Gruppen I* [M]. Berlian Heidelberg New York: Springer-Verlag, 1979.
[5] QIAN Guohua. *On automorphism groups of some finite groups* [J]. Sci. China Ser. A, 2003, **46**(4): 450–458.