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Morita Duality of Semigroup Graded Rings

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Abstract This paper studies Morita duality of semigroup-graded rings, and discusses an equivalence between duality functors of graded module category and bigraded bimodules. An important result is obtained: A semigroup bigraded R-A-bimodule Q defines a semigroup graded Morita duality if and only if Q is gr-faithfully balanced and $\mathrm{Ref}({}_RQ)$, $\mathrm{Ref}(Q_A)$ is closed under graded submodules and graded quotients.

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The concept of Morita duality plays a central role in module theory and ring theory. In [1]–[4] authors studied Morita duality of associative rings[5] and group graded rings[4]. The aim of this paper is to investigate the semigroup graded version of this concept and extends some corresponding results.

1. Semigroup graded rings

Let S be a semigroup with identity e. For every $x, y \in S$, we define

$$[xy^{-1}] = \{t \in S | ty = x\}.$$

For each pair $t, y \in S$ we have $t \in [(ty)y^{-1}]$, so that for fixed y the collection $\{[xy^{-1}] | x \in S, [xy^{-1}] \neq \emptyset\}$ is a partition of S. Similarly, for the semigroup Ω and for every $\sigma, \tau \in \Omega$, we define

$$[\sigma^{-1}\tau] = \{\omega \in \Omega | \sigma\omega = \tau\},$$

and for fixed σ the collection $\{[\sigma^{-1}\tau] | \tau \in \Omega, [\sigma^{-1}\tau] \neq \emptyset\}$ is a partition of Ω.

A subset $I \subseteq \Omega$ is called a right ideal if for any $\sigma \in I$, $\tau \in \Omega$ we have $\sigma\tau \in I$.

Let S be a semigroup with identity e. A unital ring R is called an S-graded ring (or graded by S) if there is a family $\{R_s | s \in S\}$ of additive subgroups of R such that $R = \bigoplus_{s \in S} R_s$, and for

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each pair $s, t \in S$ we have $R_s R_t \subseteq R_{st}$. A left R-module M is called S-graded if there is a family $\{M_s | \, s \in S\}$ of additive subgroups of M such that $M = \bigoplus_{s\in S} M_s$, and for each pair $s, t \in S$ we have $R_s M_t \subseteq M_{st}$. If M is a graded left R-module, the elements of $h(M) = \bigcup_{s\in S} M_s$ are called homogeneous elements of M.

In this paper, let R (resp. A) be an S-graded (Ω-graded) ring, where S and Ω are semigroups with identity. (R, S)-gr (resp. gr-(A, Ω)) will denote the category of graded left R-modules (resp. right A-modules).

Let A be an Ω-graded ring and R an S-graded ring. For each pair $\sigma, \tau \in \Omega$ and $a \in A$, we set $A_{\sigma^{-1}\tau} = \bigoplus_{\substack{\omega\in\Omega \\ \sigma\omega=\tau}} A_\omega$, $a_{\sigma^{-1}\tau} = \sum_{\substack{\omega\in\Omega \\ \sigma\omega=\tau}} a_\omega$. For each pair $x, y \in S$ and $r \in R$, we set $R_{xy^{-1}} = \bigoplus_{\substack{t\in S \\ tx=y}} R_t$, $r_{xy^{-1}} = \sum_{\substack{t\in S \\ tx=y}} r_t$.

Proposition 1.1 *Let A be an Ω-graded ring. Suppose $N \in$ gr-(A, Ω) and set $\sigma, \tau, \omega \in \Omega$. Then*

(1) $N_\sigma A_{\sigma^{-1}\tau} \subseteq N_\tau$;

(2) $(na)_\sigma = \Sigma_{\tau\in\Omega} n_\tau a_{\tau^{-1}\sigma}$ for all $a \in A, n \in N$;

(3) If $n \in N_\sigma, a \in A$, then $(na)_\tau = n_\sigma a_{\sigma^{-1}\tau}$ for each $\tau \in \Omega$;

(4) If $n \in N_\sigma$, then $n = n1_{\sigma^{-1}\sigma}$, while $n1_{\sigma^{-1}\tau} = 0$ for each $\sigma \neq \tau$;

(5) $N_{\sigma^{-1}\tau} A_\omega \subseteq N_{\sigma^{-1}\tau\omega}$;

(6) If I is a right ideal of Ω, then $N_I = N_I A$ is a graded A-submodule of M, where $N_I = \sum_{\sigma\in I} n_\sigma$.

Proof (1) is obvious and (2) follows directly from $(na)_\sigma = \Sigma_{\omega\in\Omega} \Sigma_{\substack{\tau\in\Omega \\ \tau\omega=\sigma}} n_\tau a_\omega = \Sigma_{\tau\in\Omega} n_\tau a_{\tau^{-1}\sigma}$. Statement (3) is a special case of (2).

For each $n \in N_\sigma$, note that $n = n_\sigma$ and that $n_\sigma = (n1)_\sigma = n_\sigma 1_{\sigma^{-1}\sigma}$, we have $n = n1_{\sigma^{-1}\sigma}$, while $0 = (n_\sigma)_\tau = (n1)_\tau = n_\sigma 1_{\sigma^{-1}\tau} = n1_{\sigma^{-1}\tau}$, so (4) holds.

Let $\upsilon \in [\sigma^{-1}\tau]$. Then $\sigma\upsilon = \tau$, and thus $\sigma\upsilon\omega = \tau\omega \Rightarrow \upsilon\omega \in [\sigma^{-1}\tau\omega]$, which implies that $N_{\sigma^{-1}\tau} A_\omega \subseteq N_{\sigma^{-1}\tau\omega}$, so (5) holds.

Finally for (6), if I is a right ideal of Ω and $\tau \in I$, then for each $\sigma \in \Omega, n_\tau \in N_\tau$ and $a_\sigma \in A_\sigma$ we have $\tau\sigma \in I, n_\tau a_\sigma \in N_{\tau\sigma} \subseteq N_I$, so that $N_I A \subseteq N_I$. Conversely because A has the identity, the relation $N_I \subseteq N_I A$ obviously holds and the result follows. $\square$

For $M \in (R, S)$-gr and $s \in S$, set $M(s)_t = M_{ts^{-1}}$ for all $t \in S$, then $M(s) = \bigoplus_{t\in S} M_{ts^{-1}}$ is an S-graded left R-module. For each $s \in S$, we define ${}^s\mathcal{P} = R1_{ss^{-1}}$, which is a graded submodule of $R(s)$ by setting $({}^s\mathcal{P})_t = R_{ts^{-1}} 1_{ss^{-1}}$ for all $t \in S$.

Similarly, for $N \in$ gr-(A, Ω) and $\sigma \in \Omega$, $(\sigma)N = \bigoplus_{\tau\in\Omega} N_{\sigma^{-1}\tau}$ is an Ω-graded right A-module. For each $\sigma \in \Omega$, we define $\mathcal{Q}^\sigma = 1_{\sigma^{-1}\sigma} A$, which is a graded submodule of $(\sigma)A$ by setting $(\mathcal{Q}^\sigma)_\tau = 1_{\sigma^{-1}\sigma} A_{\sigma^{-1}\tau}$ for all $\tau \in \Omega$.

Proposition 1.2[7] *The collection $\{{}^s\mathcal{P}\}_{s\in S}$ defined above forms a system of finitely generated projective generators of (R, S)-gr.*

Proposition 1.3 *Let A be a unital ring graded by the semigroup Ω and let $\sigma \in \Omega$.*

(1) For $N \in \text{gr-}(A,\Omega)$, the map $\varphi^\sigma(N) : \mathrm{Hom}_{\text{gr-}(A,\Omega)}(\mathcal{Q}^\sigma, N) \longrightarrow N_\sigma$ given by

$$\varphi^\sigma(N) : f \mapsto f(1_{\sigma^{-1}\sigma})$$

is an isomorphism of abelian groups.

(2) The functor $\mathrm{Hom}_{\text{gr-}(A,\Omega)}(\mathcal{Q}^\sigma, -)$ is isomorphic to the functor $(-)_\sigma$.

Proposition 1.4 *The collection $\{\mathcal{Q}^\sigma\}_{\sigma\in\Omega}$ defined above forms a system of finitely generated projective generators of* $\text{gr-}(A,\Omega)$.

2. Bigraded bimodules

In this section, we denote by S and Ω two semigroups and $R = \bigoplus_{s\in S} R_s$ and $A = \bigoplus_{\sigma\in\Omega} A_\sigma$ two rings graded by S and Ω, respectively.

An R-A-bimodule ${}_RQ_A$ is said to be a bigraded R-A-bimodule if there is a family $\{\, {}_sQ_\sigma \mid s \in S, \sigma \in \Omega \,\}$ of additive subgroups of Q such that $Q = \bigoplus_{\substack{s\in S\\ \sigma\in\Omega}} {}_sQ_\sigma$ and for every $s, t \in S$, $\sigma, \tau \in \Omega$, we have $R_s \cdot {}_tQ_\sigma \cdot A_\tau \subseteq {}_{st}Q_{\sigma\tau}$.

For every $s \in S, \sigma \in \Omega$, set ${}_sQ = \bigoplus\limits_{\sigma\in\Omega} {}_sQ_\sigma$ and $Q_\sigma = \bigoplus_{s\in S} {}_sQ_\sigma$, then ${}_sQ$ is a graded right A-submodule of Q_A and Q_σ is a graded left R-submodule of ${}_RQ$. Thus for every $M \in (R,S)\text{-gr}$

$$\mathrm{Hom}_{(R,S)\text{-gr}}(M,Q) = \{f \in \mathrm{Hom}(M,Q) \mid f(M_s) \subseteq {}_sQ, \forall\, s \in S\},$$

has a natural structure of right A-module. For every $\sigma \in \Omega$ set

$$M^*_\sigma = \{\, f \in \mathrm{Hom}_{(R,S)\text{-gr}}(M,Q) | \mathrm{Im} f \subseteq Q_\sigma \}.$$

Thus $M^* = \sum_{\sigma\in\Omega} M^*_\sigma$, which is an A-submodule of $\mathrm{Hom}_{(R,S)\text{-gr}}(M,Q)$, can be considered as a graded right A-module by setting $(M^*)_\sigma = M^*_\sigma$.

Now let $M_1, M_2 \in (R,S)\text{-gr}$ and $f \in \mathrm{Hom}_{(R,S)\text{-gr}}(M_1,M_2)$. Then for every $\sigma \in \Omega$ and $\alpha \in (M_2^*)_\sigma$, $\mathrm{Im}(f\alpha) \subseteq \mathrm{Im}(\alpha) \subseteq Q_\sigma$ so that $f\alpha \in (M_1^*)_\sigma$. Thus the transpose of f induces a morphism $f^* : M_2^* \to M_1^*$ of graded right A-modules and we have the following two duality functors:

$$H_R(-,Q) : (R,S)\text{-gr} \longrightarrow \text{gr-}(A,\Omega), \quad M \mapsto M^*, \quad f \mapsto f^*$$

$$H_A(-,Q) : \text{gr-}(A,\Omega) \longrightarrow (R,S)\text{-gr}, \quad N \mapsto N^*, \quad g \mapsto g^*.$$

Proposition 2.1 *Let ${}_RQ_A = \bigoplus_{\substack{s\in S\\ \sigma\in\Omega}} {}_sQ_\sigma$ be a bigraded R-A-bimodule and let $M \in (R,S)\text{-gr}, N \in \text{gr-}(A,\Omega)$. Then there is an isomorphism*

$$\eta : \mathrm{Hom}_R(M, \mathrm{Hom}_A(N,Q)) \xrightarrow{\sim} \mathrm{Hom}_A(N, \mathrm{Hom}_R(M,Q)).$$

Moreover, it can induce an isomorphism

$$\overline{\eta} : \mathrm{Hom}_{(R,S)\text{-gr}}(M, \mathrm{Hom}_A(N,Q)) \xrightarrow{\sim} \mathrm{Hom}_{\text{gr-}(A,\Omega)}(N, \mathrm{Hom}_R(M,Q))$$

natural in each of the three variables.

Let ${}_RQ_A$ be a bigraded R-A-bimodule. We define the functors by

$$\omega = \omega^1 : 1_{(R,S)\text{-gr}} \to (-)^{**} = H_A(-,Q) \circ H_R(-,Q)$$

$$\omega = \omega^2 : 1_{\text{gr-}(A,\Omega)} \to (-)^{**} = H_R(-,Q) \circ H_A(-,Q).$$

For every $M \in (R,S)\text{-gr}, m \in M, f \in M^*$, $[(m)\omega^1_M](f) = (m)f$ and for every $N \in \text{gr-}(A,\Omega), n \in N, g \in N^*$, $(g)[\omega^2_N(n)] = g(n)$.

Proposition 2.2 *For every $M \in (R,S)\text{-gr}$, $(\omega_M)^* \circ \omega_{M^*}$ is the identity map 1_{M^*} and for every $N \in \text{gr-}(A,\Omega)$, $(\omega_N)^* \circ \omega_{N^*}$ is the identity map 1_{N^*}.*

Proof By the definition $(\omega_M)^* : M^{***} \to M^*$, $\omega_{M^*} : M^* \to M^{***}$. Let $\gamma \in M^*$. Then for every $m \in M$

$$(m)[(\omega_M)^* \circ (\omega_{M^*}(\gamma))] = (m)[\omega_{m^*}(\gamma) \circ \omega_M] = [(m)\omega_M](\gamma) = (m)\gamma,$$

which implies that $(\omega_M)^* \circ \omega_{M^*} = 1_{M^*}$. □

Definition 2.3 *Let ${}_RQ_A$ be a bigraded R-A-bimodule and let $M \in (R,S)\text{-gr}$ (resp. $N \in \text{gr-}(A,\Omega)$). M (resp. N) is called Q-reflexive if ω_M (resp. ω_N) is an isomorphism.*

We will denote by $\text{Ref}({}_RQ)$ (resp. $\text{Ref}(Q_A)$ the full subcategory of R-gr (resp. gr-A) consisting of Q-reflexive graded left R-modules (resp. right A-modules). Obviously, if M is Q-reflexive, so is M^*.

Definition 2.4 *Let $\Re$ be a family of graded left R-modules. An graded left R-module $M \in (R,S)\text{-gr}$ is called gr-cogenerated by $\Re$ if $\Re$ cogenerates M in $(R,S)\text{-gr}$, i.e., if there is an embedding, in $(R,S)\text{-gr}$, of M into a direct product, in $(R,S)\text{-gr}$, of elements of $\Re$.*

Lemma 2.5 *Let $\Re$ be a family of graded left R-modules. Then*

(1) $M \in (R,S)\text{-gr}$ is gr-cogenerated by $\Re$ iff for every $0 \neq x \in h(M)$ there exists a $C \in \Re$ and an $f \in \text{Hom}_{(R,S)\text{-gr}}(M,C)$ such that $(x)f \neq 0$.

(2) $\Re$ is a set of cogenerators in $(R,S)\text{-gr}$ iff $\Re$ gr-cogenerates any graded left R-module.

Lemma 2.6 *Let ${}_RQ_A$ be a bigraded R-A-bimodule and let $M \in (R,S)\text{-gr}$. Then the following statements are equivalent:*

(1) ω_M is a monomorphism.

(2) M is gr-cogenerated by the family $\{Q_\sigma | \, \sigma \in \Omega\}$.

Let $Q = \bigoplus_{\substack{s\in S\\ \sigma\in\Omega}} {}_sQ_\sigma$ be a bigraded R-A-bimodule. Note that $R_{ts^{-1}} \cdot 1_{ss^{-1}} \cdot {}_sQ = R_{ts^{-1}} \cdot {}_sQ \subseteq {}_tQ$ and that $Q_\sigma \cdot 1_{\sigma^{-1}\sigma} \cdot A_{\sigma^{-1}\tau} = Q_\sigma \cdot A_{\sigma^{-1}\tau} \subseteq Q_\tau$ by [7, Lemma 2.4], for every $s,t \in S$ and every $\sigma,\tau \in \Omega$ there exist canonical group homomorphisms $\lambda_{s,t} : ({}^s\mathcal{P})_t \longrightarrow \text{Hom}_{\text{gr-}(A,\Omega)}({}_sQ, {}_tQ)$ defined by

$$[(r)\lambda_{s,t}](q) = rq, \quad \text{for every} \quad r \in ({}^s\mathcal{P})_t, \quad q \in {}_sQ$$

and $\rho_{\sigma,\tau} : (\mathcal{Q}^\sigma)_\tau \longrightarrow \text{Hom}_{(R,S)\text{-gr}}(Q_\sigma, Q_\tau)$ defined by

$$(p)[\rho_{\sigma,\tau}(a)] = pa, \quad \text{for every} \quad a \in (\mathcal{Q}^\sigma)_\tau, \; p \in Q_\sigma.$$

Definition 2.7 *A bigraded R-A-bimodule ${}_RQ_A$ is said to be gr-faithfully balanced if for every $s,t \in S$ and for every $\sigma,\tau \in \Omega$ the group homomorphisms $\lambda_{s,t}$ and $\rho_{\sigma,\tau}$ are isomorphic.*

Proposition 2.8 *Let ${}_RQ_A = \bigoplus_{\substack{s\in S\\ \sigma\in\Omega}} {}_sQ_\sigma$ be a bigraded R-A-bimodule. The following statements are equivalent:*

(1) ${}_RQ_A$ is gr-faithfully balanced.

(2) For every $s \in S$ and every $\sigma \in \Omega$, ${}^s\mathcal{P}$ and $\mathcal{Q}^\sigma$ are Q-reflexive.

Proof For every $s \in S$, ${}^s\mathcal{P}$ is finite generated by Proposition 1.2, and $({}^s\mathcal{P})^* = \mathrm{Hom}_{(R,S)\text{-gr}}({}^s\mathcal{P}, Q) \cong {}_sQ$ by [7, Proposition 2.11], so $({}^s\mathcal{P})^{**} \cong ({}_sQ)^* = \mathrm{Hom}_{(R,S)\text{-gr}}({}_sQ, Q) = \bigoplus_{t\in S} \mathrm{Hom}_{(R,S)\text{-gr}}({}_sQ, {}_tQ)$; On the other hand, we have ${}^s\mathcal{P} = \bigoplus_{t\in S} {}^s\mathcal{P}_t$. Thus ${}_RQ_A$ is gr-faithfully balanced iff ${}^s\mathcal{P}$ is Q-reflexive. Similarly, for every $\sigma \in \Omega$, $(\mathcal{Q}^\sigma)^* = \mathrm{Hom}_{\text{gr-}(A,\Omega)}(\mathcal{Q}^\sigma, Q) \cong Q_\sigma$ by Propositions 1.3 and 1.4 and ${}_RQ_A$ is gr-faithfully balanced iff $\mathcal{Q}^\sigma$ is Q-reflexive. □

3. Semigroup graded Morita duality

Proposition 3.1 *Let $R = \bigoplus_{s\in S} R_s$ and $A = \bigoplus_{\sigma\in\Omega} A_\sigma$ be two rings graded by S and Ω respectively. Let $\mathfrak{C}$, $\mathfrak{D}$ be full subcategories of (R,S)-gr and gr-(A,Ω) respectively and assume that the contravariant functors $F : \mathfrak{C} \to \mathfrak{D}$ and $F' : \mathfrak{D} \to \mathfrak{C}$ yield a duality between $\mathfrak{C}$ and $\mathfrak{D}$. If for all $s \in S$, ${}^s\mathcal{P} \in \mathfrak{C}$, then there exists a bigraded R-A-bimodule $Q = \bigoplus_{\substack{s\in S\\ \sigma\in\Omega}} {}_sQ_\sigma$ such that $F' \simeq H_A(-, Q)$.*

Proof For every $s \in S$ we set ${}_sQ = F({}^s\mathcal{P}) \in \mathfrak{D}$, which is a graded right A-module, then $Q = \bigoplus_{s\in S} {}_sQ$ is a bigraded R-A-bimodule. Let $N \in$ gr-(A,Ω). Then we have $F'(N)_s \cong \mathrm{Hom}_{(R,S)\text{-gr}}({}^s\mathcal{P}, F'(N))$ by [7, Proposition 2.11] and thus

$$\begin{aligned} F'(N) = &\bigoplus_{s\in S} F'(N)_s \cong \bigoplus_{s\in S} \mathrm{Hom}_{(R,S)\text{-gr}}({}^s\mathcal{P}, F'(N)) \\ \overset{F}{\cong} &\bigoplus_{s\in S} \mathrm{Hom}_{\text{gr-}(A,\Omega)}(FF'(N), F({}^s\mathcal{P})) = \bigoplus_{s\in S} \mathrm{Hom}_{\text{gr-}(A,\Omega)}(N, {}_sQ) \\ \cong &\mathrm{Hom}_{\text{gr-}(A,\Omega)}(N, Q) \cong H_A(N, Q). \end{aligned}$$

Theorem 3.2 *Let $R = \bigoplus_{s\in S} R_s$ and $A = \bigoplus_{\sigma\in\Omega} A_\sigma$ be two rings graded by S and Ω, respectively, $\mathfrak{C}$ and $\mathfrak{D}$ full subcategories of (R,S)-gr and gr-(A,Ω) respectively and assume that for every $s \in S$ and $\sigma \in \Omega$, ${}^s\mathcal{P} \in \mathcal{C}$ and $\mathcal{Q}^\sigma \in \mathfrak{D}$. Let $F : \mathfrak{C} \to \mathfrak{D}$ and $F' : \mathfrak{D} \to \mathfrak{C}$ be a duality. Then there exists a bigraded R-A-bimodule $Q = \bigoplus_{\substack{s\in S\\ \sigma\in\Omega}} {}_sQ_\sigma$ such that:*

(1) For every $s \in S$ and $\sigma \in \Omega$, ${}_sQ \simeq F({}^s\mathcal{P})$, $Q_\sigma \simeq F'(\mathcal{Q}^\sigma)$.

(2) There are natural isomorphisms $F \cong H_R(-, Q)$, $F' \cong H_A(-, Q)$.

(3) $\mathfrak{C} \subseteq \mathrm{Ref}({}_RQ)$ and $\mathfrak{D} \subseteq \mathrm{Ref}(Q_A)$.

(4) ${}_RQ_A$ is gr-faithfully balanced.

Proof Statements (1) and (2) follow by Proposition 3.1. Note that (2) and (3) are equivalent, we only prove that statement (3) is true. For every $s \in S$, we have $({}^s\mathcal{P})^* = \mathrm{Hom}_{(R,S)\text{-gr}}({}^s\mathcal{P}, Q) \cong {}_sQ$, then $({}^s\mathcal{P})^{**} \cong ({}_sQ)^* = (F({}^s\mathcal{P}))^* \cong F'(F({}^s\mathcal{P})) \cong {}^s\mathcal{P}$, which implies that ${}^s\mathcal{P} \in \mathrm{Ref}({}_RQ)$. Similarly, we have $\mathcal{Q}^\sigma \in \mathrm{Ref}(Q_A)$.

Definition 3.3 *Let ${}_RQ_A$ be a bigraded R-A-bimodule. We say that ${}_RQ_A$ defines a semigroup* *graded Morita duality if*

M1) For every $s \in S$ and $\sigma \in \Omega$, ${}^s\mathcal{P}$ and $\mathcal{Q}^\sigma$ are Q-reflexive.

M2) Ref$({}_RQ)$ and Ref(Q_A) *are closed under graded submodules and graded quotients.*

Lemma 3.4 *Let ${}_RQ_A$ be a bigraded R-A-bimodule and let $M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence in* (R,S)-gr. *Then $0 \longrightarrow (M'')^* \xrightarrow{g^*} M^* \xrightarrow{f^*} (M')^*$ is an exact sequence in* gr-(A,Ω).

Recall a graded right A-module $N \in$ gr-(A,Ω) is gr-injective if the functor $H_R(-,N)$ is exact.

Lemma 3.5 *Let ${}_RQ_A$ be a bigraded R-A-bimodule. Then the following assertions are equivalent:*

(1) The contravariant functor $(-)^* = H_R(-,Q)$ is exact.

(2) For every $s \in S$ and for every left ideal I of ${}^s\mathcal{P}$, the dual j^* of the inclusion $j : I \hookrightarrow {}^s\mathcal{P}$ is surjective.

(3) For every $\sigma \in \Omega$ the graded left R-module Q_σ is gr-injective.

Proof Straightforward. □

Proposition 3.6 *Let $\Re$ be a family of gr-injective left R-modules. Then the following assertions are equivalent:*

(1) $\Re$ is a set of cogenerators in (R,S)-gr.

(2) $\Re$ gr-cogenerates every gr-simple left R-module.

Proof (1)⇒(2) is clear.

(2)⇒(1) Let $M \in (R,S)$-gr and let $0 \neq x \in h(M)$. Then $H = Rx$ contains a gr-maximal graded submodule N (see [5], Lemma 1.7.4) and there exists a $C \in \Re$ and $f \in \mathrm{Hom}_{(R,S)\text{-gr}}(H/N, C)$ such that $(x+N)f \neq 0$. Let $p : H \to H/N$ be the canonical projection. Then pf extends to a graded R-morphism $h : M \to C$, as C is gr-injective, and we have $(x)h = (x)pf = (x+N)f \neq 0$. Apply now Lemma 2.5. □

Proposition 3.7 *Let R be an S-graded ring and assume that ${}^s\mathcal{P}$ is defined as above. Then every gr-simple left R-module M is isomorphic to some quotients of ${}^s\mathcal{P}$.*

Proof Let M be a gr-simple left R-module. Then for every $0 \neq m \in h(M)_x$, $Rm = M$ and we have a epimorphism $\alpha : {}^s\mathcal{P} \longrightarrow M$ defined by $\alpha(r) = rm$, which is graded since ${}^s\mathcal{P}_t \cdot M_s \cdot R_{ts^{-1}} \cdot 1_{ss^{-1}} \cdot M_s \subseteq M_t$. Thus there exists an isomorphism ${}^sp/\mathrm{ker}\alpha \cong M$. □

Theorem 3.8 *Let R and A be rings graded by semigroups S and Ω respectively and let ${}_RQ_A = \bigoplus_{\substack{s\in S\\ \sigma\in\Omega}} {}_sQ_\sigma$ be a bigraded R-A-bimodule. Then the following statements are equivalent:*

(1) ${}_RQ_A$ defines a semigroup graded Morita duality.

(2) For every $s \in S$ and $\sigma \in \Omega$ every graded submodules and graded quotient of ${}^s\mathcal{P}, \mathcal{Q}^\sigma, {}_sQ, Q_\sigma$ is Q-reflexive.

(3) ${}_RQ_A$ is gr-faithfully balanced, $\{\, Q_\sigma \mid \sigma \in \Omega \,\}$ is a set of gr-injective cogenerators of (R,S)-gr, *$\{\, {}_sQ \mid s \in S \,\}$ is a set of gr-injective cogenerators of* gr-(A,Ω).

Proof (1)$\Rightarrow$ (2) Assume that ${}_RQ_A$ defines a semigroup graded Morita duality, we have that ${}^s\mathcal{P}$ and $\mathcal{Q}^\sigma$ are Q-reflexive and for every $s \in S$ and $\sigma \in \Omega$ ${}_sQ \cong H_R({}^s\mathcal{P}, Q) = ({}^s\mathcal{P})^*$ and $Q_\sigma \cong H_A(\mathcal{Q}^\sigma, Q) = (\mathcal{Q}^\sigma)^*$ are Q-reflexive too.

(2)$\Rightarrow$ (3) From Proposition 2.8 we deduce that ${}_RQ_A$ is gr-faithfully balanced. Now consider the canonical short exact sequence

$$0 \longrightarrow I \xrightarrow{\ j\ } {}^sp \xrightarrow{\ p\ } {}^sp/I \longrightarrow 0$$

for every graded ideal I of ${}^x\mathcal{P}$.

Let $H = Imj^* \subseteq I^*$ the image of the dual of j. Then H_A is isomorphic to a graded quotient of $({}^s\mathcal{P})^*$ i.e. $({}^s\mathcal{P})^*/\mathrm{Ker}j^* \cong \mathrm{Im}j^* = H$ and hence Q-reflexive from (2).

Let $i : H \hookrightarrow I^*$ the inclusion and $k : ({}^sp)^* \longrightarrow H$ the corestriction of j^*. Then we have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "j"] & {}^sp \arrow[r, "p"] & {}^sp/I \arrow[r] & 0 \\
0 \arrow[r] & H^* \arrow[r, "k^*"] \arrow[u, "i^*\circ\omega_I"] & ({}^sp)^{**} \arrow[r, "k^*"] \arrow[u, "\omega_{({}^sp)}"] & ({}^sp)^{**}/I \arrow[r] \arrow[u, "\omega_{({}^sp/I)}"] & 0
\end{tikzcd}$$

From (2) the maps $\omega_{({}^s\mathcal{P}/I)}$ and $\omega_{({}^s\mathcal{P})}$ are isomorphic, therefore $i^* \circ \omega_I$ is isomorphic and $I \cong H^*$. Because H is Q-reflexive, H^* is Q-reflexive by Proposition 2.2. Thus I and I^* are Q-reflexive and i^* is isomorphic. This implies that i^{**} is always an isomorphism.

Let $L = I^*$. Then ω_L is an isomorphism. Thus $i = \omega_H i^{**}{\omega_l}^{-1}$ is an isomorphism and j^* is epimorphic. By Proposition 3.5 for every $\sigma \in \Omega$ the graded left R-module Q_σ is gr-injective.

Finally, by Proposition 3.7 every gr-simple left R-module is a graded quotient of ${}^s\mathcal{P}$ for some $s \in S$. Let C be a gr-simple left R-module. We have $C \cong {}^s\mathcal{P}/I$ for some graded submodule I of ${}^x\mathcal{P}$, then C is Q-reflexive and ω_C is isomorphic. Thus $\{\, Q_\sigma \mid \sigma \in \Omega \,\}$ gr-cogenerates C by Lemma 2.6 and $\{\, Q_\sigma \mid \sigma \in \Omega \,\}$ is a set of gr-injective cogenerators of (R, S)-gr by Proposition 3.6.

(3)$\Rightarrow$ (1) By Proposition 2.8 for every $s \in S$ and $\sigma \in \Omega$ ${}^s\mathcal{P}$ and $\mathcal{Q}^\sigma$ are Q-reflexive.

Let L be a graded submodule of a Q-reflexive graded left R-module M. Now consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & M \arrow[r, "p"] & M/L \arrow[r] & 0 \\
0 \arrow[r] & L^{**} \arrow[r] \arrow[u, "\omega_L"] & M^{**} \arrow[r] \arrow[u, "\omega_M"] & (M/L)^{**} \arrow[r] \arrow[u, "\omega_{(M/L)}"] & 0
\end{tikzcd}$$

As every $Q_\sigma(\sigma \in \Omega)$ and every ${}_sQ(s \in S)$ are gr-injective the bottom row is exact by Proposition 3.5. As $\{Q_\sigma \mid \sigma \in \Omega\,\}$ is a family of cogenerators of (R, S)-gr and $\{{}_sQ \mid s \in S\,\}$ is a family of cogenerators of gr-(A, Ω) the maps ω_L and $\omega_{(M/L)}$ are monomorphisms by Lemma 2.6. Then by Snake's lemma, they are epimorphisms.

Theorem 3.9 *Let R and A be rings graded by semigroups S and Ω respectively and let ${}_RQ_A = \bigoplus_{\substack{s\in S\\ \sigma\in\Omega}} {}_sQ_\sigma$ be a bigraded R-A-bimodule. Then ${}_RQ_A$ defines a semigroup graded Morita duality if and only if ${}_RQ_A$ is gr-faithfully balanced and* $\mathrm{Ref}({}_RQ)$ *and* $\mathrm{Ref}(Q_A)$ *are closed under* *graded submodules and graded quotients.*

Proof The necessity follows from the definition of semigroup graded Morita duality and Theorem 3.9. For the sufficiency, since ${}_RQ_A$ is gr-faithfully balanced, for every $s \in S$ and $\sigma \in \Omega$ we have ${}^s\mathcal{P}$ and $\mathcal{Q}^\sigma$ are Q-reflexive by Proposition 2.8, thus $Q_\sigma \cong (\mathcal{Q}^\sigma)^*$ and ${}_sQ \cong ({}^s\mathcal{P})^*$ are also Q-reflexive. This completes the proof by Theorem 3.8. $\square$

References

[1] ANDERSON F W, FULLER K R. *Rings and Categories of Modules* [M]. Springer-Verlag, New York-Heidelberg, 1974.
[2] ĂNH P N, MENINI C. *Morita duality for rings with local units* [J]. J. Algebra, 1994, **164**(3): 632–641.
[3] MENINI C. *Finitely graded ring morita duality and self-injectivity* [J]. Comm. Algebra, 1987, **15**(9): 1779–1797.
[4] MULLER B J. *Linear compactness and Morita duality* [J]. J. Algebra, 1970, **16**: 60–66.
[5] NĂSTĂSESCU C, VAN O F. *Graded Ring Theory* [M]. Mathematical Library, North-Holland, Amsterdam, 1982.
[6] MENINI C, ANGEL D R. *Morita duality and graded rings* [J]. Comm. Algebra, 1991, **19**(6): 1765–1794.
[7] ABRAMS G, MENINI C, ANGEL D R. *Realization theorems for categories of graded modules over semigroup-graded rings* [J]. Comm. Algebra, 1994, **22**(13): 5343–5388.
[8] NĂSTĂSESCU C. *Some constructions over graded rings* [J]. J. Algebra, 1989, **120**(1): 119–138.