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# Positive Solutions of Sub-Linear Semi-Positone Boundary Value Problem System

*XU Xi An*
(Department of Mathematics, Xuzhou Normal University, Jiangsu 221116, China)
(E-mail: xuxian68@163.com)

**Abstract** In this paper, we study the existence of positive solutions of a sub-linear semi-positone differential boundary value problems system with positive parameter. We prove that the semi-positone differential boundary value problems system has at least one positive solution for the parameter sufficiently large.

**Keywords** semi-positone differential boundary value problems system; the fixed point index; positive solutions.

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## 1. Introduction

Consider the existence of positive solutions of the boundary value problem system

$$\begin{cases} (p_1(t)x')' + \lambda f_1(t, x(t), y(t)) + q_1(t) = 0, & r < t < R, \\ (p_2(t)y')' + \lambda f_2(t, x(t), y(t)) + q_2(t) = 0, & r < t < R, \\ a_1x(r) - b_1p_1(r)x'(r) = 0, c_1x(R) + d_1p_1(R)x'(R) = 0, & \\ a_2y(r) - b_2p_2(r)y'(r) = 0, c_2y(R) + d_2p_2(R)y'(R) = 0, & \end{cases} \tag{1.1$_\lambda$}$$

where $\lambda > 0$ is a parameter, $f_1, f_2 : [r, R] \times R^+ \times R^+ \mapsto R^+$ are continuous, $q_1, q_2 : [r, R] \mapsto R^1$ are continuous, $p_1, p_2 : R^+ \mapsto (0, \infty)$ are differentiable continuous, $a_i,\ b_i,\ c_i,\ d_i\ (i = 1, 2)$ are nonnegative constants, $R^+ = [0, \infty)$.

The system $(1.1_\lambda)$ is a semi-positone system because $q_1, q_2$ are allowed to take negative values. Semi-positone problems arise in many different areas of applied mathematics and physics, such as the buckling of mechanical systems, the design of suspension bridges, chemical reactions, management of natural resources, thermal equilibrium of plasmas and so on. From an application viewpoint, people are usually interested in the existence of positive solutions for semi-positone problems. The study of semi-positone problems was formally introduced by Castro and Shivaji[1]. During the last ten years, finding positive solutions for semi-positone problems has been actively pursued and significant progress on semi-positone problems has been made[1−12]. To establish the existence results for positive solutions of semi-positone problems, people usually employ the

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method of finding fixed points on some special cones with large enough norm, and then obtaining the positive solutions of the semi-positone problems. This method is simple but very effective. Anuradha et al[2] first used this method to study the existence of positive solutions of some semi-positone boundary value problems. They proved in [2] an existence result of at least one positive solution with a super-linear nonlinearity. Later on, many authors employed this method to show the existence of positive solutions of various kinds of semi-positone boundary value problems[3−12]. Xu and O'Regan[11] extended this method to an abstract operator equation and obtained some abstract existence results for positive solutions of the operator equation. We can deduce the main results of [2]–[10] directly by the abstract results of [11].

Semi-positone systems occur naturally in important applications, for example: predator-prey systems with constant effort harvesting[14−17]. The main purpose of this paper is to establish some existence results for positive solutions of the semi-positone systems ($1.1_\lambda$). To do this, we will continue to employ the method of [2]. However, generally speaking, in system $C[r,R]\times C[r,R]$, knowing that the supremum norm of $u=(u_1,\dots,u_m)$ (say) is large does not necessarily mean that the supremum norm of each $u_i$ is large. Thus establishing that each component $u_i$ of the solution is positive is an additional challenge. Moreover, there are few existence results for positive solutions of semi-positone systems yet. In this paper, we will first give two existence results for positive solutions of a semi-positone operator equations system.

## 2. Some abstract existence results for positive solutions

Let $E$ be a real Banach space, and $P$ a total cone of $E$ which induces the ordering "$\le$" in $E$. Consider the operator equations system

$$\begin{cases} x=\lambda K_1F_1(x,y)+e_1, \\ y=\lambda K_2F_2(x,y)+e_2, \end{cases} \tag{$2.1_\lambda$}$$

where $\lambda>0$ is a parameter, $K_1,K_2:P\mapsto P$ are linear completely continuous operators, $F_1,F_2:P\times P\mapsto P$ are continuous and bounded operators, $e_1,e_2\in E$.

**Remark 2.1** Suppose that $r(K_1)>0, r(K_2)>0$ ($r(K_1)$ and $r(K_2)$ denote the spectrum radii of $K_1$ and $K_2$, respectively). By Krein-Rutman Theorem (see [20, Theorem 3.1]), there exist $h_1,h_2\in P^*\backslash\{\theta\}$ such that

$$K_1^*h_1=r(K_1)h_1,\ \ K_2^*h_2=r(K_2)h_2, \tag{2.2}$$

where $P^*$ is the dual cone of the cone $P$, and $K_1^*$ and $K_2^*$ are the conjugate operators of $K_1$ and $K_2$, respectively.

Let $\varphi_1,\varphi_2\in P\backslash\{\theta\}$, $Q_1=\{x\in P|x\ge\|x\|\varphi_1\}$ and $Q_2=\{x\in P|x\ge\|x\|\varphi_2\}$. Then, $Q_1$ and $Q_2$ are two cones of the Banach space $E$. Let $E^\Delta=E\times E$, $P^\Delta=P\times P$ and $Q^\Delta=Q_1\times Q_2$. For any $(x,y)\in E^\Delta$, let

$$\|(x,y)\|=\|x\|+\|y\|.$$

Then $E^\Delta$ is a real Banach space with the norm $\|(\cdot,\cdot)\|$, and $P^\Delta$ and $Q^\Delta$ are two cones of $E^\Delta$.

For convenience, we make the following assumptions.

($H_1$) $K_i : P \mapsto Q_i (i = 1, 2)$ are completely continuous operators, $F_i : P^\Delta \mapsto P(i = 1, 2)$ are continuous and bounded operators.

($H_2$) There exist $\bar{e}_i \in P(i = 1, 2)$ and $c_0 > 0$, such that

$$\bar{e}_i < c_0\varphi_i,\ e_i + \bar{e}_i \in Q_i,\quad i = 1, 2.$$

($H_3$) For $i = 1, 2$, $r(K_i) > 0$, $h_i(\varphi_i) > 0$, and

$$\lim_{(x,y)\in P^\Delta, \|(x,y)\|\to\infty} \frac{h_i(F_i(x, y))}{\|(x, y)\|} = 0.$$

**Theorem 2.1** *Suppose that* $(H_1) - (H_3)$ *hold. Moreover, there exist* $\zeta > 0$ *and* $\eta > 0$ *such that*

$$h_1(F_1(x, y - \bar{e}_2)) \geq \eta,\ \forall y \geq \zeta\varphi_2, x \in P, \tag{2.3}$$

$$h_2(F_2(x - \bar{e}_1, y)) \geq \eta,\ \forall x \geq \zeta\varphi_1, y \in P. \tag{2.4}$$

*Then there exists* $\lambda^* > 0$ *such that, the operator equations system* $(2.1_\lambda)$ *has at least one positive solution for any* $\lambda \in (\lambda^*, \infty)$.

**Proof** Let $D_1 = \{x \in P | x \geq \bar{e}_1\}$ and $D_2 = \{x \in P | x \geq \bar{e}_2\}$. Then $D_1$ and $D_2$ are two closed convex sets of $E$. By Dugundji Theorem (see Lemma 2.3 in Chapter 1, [13]), we see that for $\alpha = 1$, there exist continuous maps $J_1 : E \mapsto D_1$ and $J_2 : E \mapsto D_2$, such that

$$\|x - J_1(x)\| \leq 2\rho(x, D_1) \leq 2\|x - \bar{e}_1\|,\ \forall x \in E \tag{2.5}$$

and

$$\|x - J_2(x)\| \leq 2\rho(x, D_2) \leq 2\|x - \bar{e}_2\|,\ \forall x \in E, \tag{2.6}$$

where $\rho(x, D_1)$ and $\rho(x, D_2)$ denote the distances from $x$ to $D_1$ and $D_2$, respectively. Obviously, $J_1 : E \mapsto D_1$ and $J_2 : E \mapsto D_2$ are continuous and bounded. For any $\lambda \in [1, \infty)$, define $A_{1\lambda} : P^\Delta \mapsto P$, $A_{2\lambda} : P^\Delta \mapsto P$ and $A_\lambda : P^\Delta \mapsto P^\Delta$ by

$$A_{1\lambda}(x, y) = \lambda K_1F_1(J_1(x) - \bar{e}_1, J_2(y) - \bar{e}_2) + e_1 + \bar{e}_1, \forall (x, y) \in P^\Delta,$$

$$A_{2\lambda}(x, y) = \lambda K_2F_2(J_1(x) - \bar{e}_1, J_2(y) - \bar{e}_2) + e_2 + \bar{e}_2, \forall (x, y) \in P^\Delta,$$

$$A_\lambda(x, y) = (A_{1\lambda}(x, y), A_{2\lambda}(x, y)), \forall (x, y) \in P^\Delta. \tag{2.7}$$

From ($H_1$) and ($H_2$), we see that $A_{1\lambda} : P^\Delta \mapsto Q_1$ and $A_{2\lambda} : P^\Delta \mapsto Q_2$ are completely continuous for any $\lambda \in [1, \infty)$. Thus, $A_\lambda : P^\Delta \mapsto Q^\Delta$ is completely continuous for any $\lambda \in [1, \infty)$.

Let $R_0 = 2(c_0 + \zeta)$, and

$$\lambda^* = \max\{\frac{R_0\|h_1\|}{r(K_1)\eta}, \frac{R_0\|h_2\|}{r(K_2)\eta}\} + 1.$$

Take $(\Psi_1, \Psi_2) \in Q^\Delta \backslash \{\theta, \theta\}$. Let $\lambda_0 > \lambda^*$ be fixed at present. Now we show that

$$(x, y) \neq A_{\lambda_0}(x, y) + \mu(\Psi_1, \Psi_2),\ (x, y) \in \partial B_{R_0} \cap Q^\Delta, \mu \geq 0, \tag{2.8}$$

where $B_{R_0} = \{(x,y) \in E^\Delta | \|(x,y)\| < R_0\}$ and $\partial B_{R_0}$ denotes the boundary of $B_{R_0}$ in $E^\Delta$. Assume by contradiction that there exist $(x_0, y_0) \in \partial B_{R_0} \cap Q^\Delta$ and $\mu_0 \geq 0$ such that

$$(x_0, y_0) = A_{\lambda_0}(x_0, y_0) + \mu_0(\Psi_1, \Psi_2). \tag{2.9}$$

Since $\|(x_0, y_0)\| = \|x_0\| + \|y_0\| = 2(c_0+\zeta)$, we have $\|x_0\| \geq c_0+\zeta$ or $\|y_0\| \geq c_0+\zeta$. If $\|x_0\| \geq c_0+\zeta$, then by $(\mathrm{H}_1)$, $(\mathrm{H}_2)$ and (2.9), $x_0 \in Q_1$, and so

$$x_0 - \bar{e}_1 \geq (\|x_0\| - c_0)\varphi_1 \geq \zeta\varphi_1 > \theta. \tag{2.10}$$

Thus, $J_1(x_0) - \bar{e}_1 = x_0 - \bar{e}_1$. From (2.4), (2.9) and (2.10), we have

$$\begin{aligned}
\|y_0\|\|h_2\| &\geq h_2(y_0) = h_2(A_{2\lambda_0}(x_0, y_0) + \mu_0\Psi_2) \\
&\geq h_2(\lambda_0 K_2 F_2(J_1(x_0) - \bar{e}_1, J_2(y_0) - \bar{e}_2) + e_2 + \bar{e}_2) \\
&\geq h_2(\lambda_0 K_2 F_2(J_1(x_0) - \bar{e}_1, J_2(y_0) - \bar{e}_2)) \\
&= \lambda_0(K_2^* h_2)(F_2(J_1(x_0) - \bar{e}_1, J_2(y_0) - \bar{e}_2)) \\
&= \lambda_0 r(K_2) h_2(F_2(x_0 - \bar{e}_1, J_2(y_0) - \bar{e}_2)) \\
&\geq \lambda_0 r(K_2)\eta.
\end{aligned}$$

Thus,

$$\lambda_0 \leq \frac{\|y_0\|\|h_2\|}{r(K_2)\eta} \leq \frac{R_0\|h_2\|}{r(K_2)\eta}. \tag{2.11}$$

Similarly, if $\|y_0\| \geq c_0 + \zeta$, then we have

$$\lambda_0 \leq \frac{\|x_0\|\|h_1\|}{r(K_1)\eta} \leq \frac{R_0\|h_1\|}{r(K_1)\eta}. \tag{2.12}$$

From (2.11) and (2.12), we have

$$\lambda_0 \leq \max\{\frac{R_0\|h_1\|}{r(K_1)\eta}, \frac{R_0\|h_2\|}{r(K_2)\eta}\},$$

which contradicts the definition of $\lambda_0$, and so (2.8) holds. By the properties of the fixed point index, we have

$$i(A_{\lambda_0}, B_{R_0} \cap Q^\Delta, Q^\Delta) = 0. \tag{2.13}$$

Take $b_{\lambda_0} : 0 < b_{\lambda_0} < [3\lambda_0(\frac{r(K_1)}{h_1(\varphi_1)} + \frac{r(K_2)}{h_2(\varphi_2)})]^{-1}$. From $(\mathrm{H}_3)$, we see that there exists $R'_{\lambda_0} > 0$, such that

$$h_1(F_1(x,y)) \leq b_{\lambda_0}\|(x,y)\|, \forall (x,y) \in P^\Delta, \|(x,y)\| \geq R'_{\lambda_0},$$

$$h_2(F_2(x,y)) \leq b_{\lambda_0}\|(x,y)\|, \forall (x,y) \in P^\Delta, \|(x,y)\| \geq R'_{\lambda_0}.$$

Let

$$\bar{M}_1 = \sup\{h_1(F_1(x,y)) | (x,y) \in P^\Delta, \|(x,y)\| \leq R'_{\lambda_0}\},$$

$$\bar{M}_2 = \sup\{h_2(F_2(x,y)) | (x,y) \in P^\Delta, \|(x,y)\| \leq R'_{\lambda_0}\}$$

and $\bar{M} = \max\{\bar{M}_1, \bar{M}_2\}$. Then we have

$$h_1(F_1(x,y)) \leq b_{\lambda_0}\|(x,y)\| + \bar{M}, \forall (x,y) \in P^\Delta, \tag{2.14}$$

$$h_2(F_2(x,y)) \leq b_{\lambda_0}\|(x,y)\| + \bar{M}, \forall (x,y) \in P^{\Delta}. \tag{2.15}$$

Take

$$R_{\lambda_0} > \max\{R_0, \frac{\|e_1+\bar{e}_1\| + \|e_2+\bar{e}_2\| + (\lambda_0\bar{M} + 3\lambda_0 b_{\lambda_0}\|(\bar{e}_1,\bar{e}_2)\|)(\frac{r(K_1)}{h_1(\varphi_1)} + \frac{r(K_2)}{h_2(\varphi_2)})}{1 - 3\lambda_0 b_{\lambda_0}(\frac{r(K_1)}{h_1(\varphi_1)} + \frac{r(K_2)}{h_2(\varphi_2)})}\}.$$

Next we show that

$$(x,y) \neq \mu A_{\lambda_0}(x,y),\ (x,y) \in \partial B_{R_{\lambda_0}} \cap Q^{\Delta}, \mu \in [0,1]. \tag{2.16}$$

Assume by contradiction that there exist $(x_0,y_0) \in \partial B_{R_{\lambda_0}} \cap Q^{\Delta}$ and $\mu_0 \in [0,1]$ such that

$$(x_0,y_0) = \mu_0 A_{\lambda_0}(x_0,y_0).$$

Obviously, $\mu_0 > 0$. Since $K_1 : P \mapsto Q_1$ and $K_2 : P \mapsto Q_2$, we have from (2.14) and (2.15) that

$$\begin{aligned}
\|(x_0,y_0)\| =& \mu_0\|A_{\lambda_0}(x_0,y_0)\| \leq \|A_{\lambda_0}(x_0,y_0)\| \\
=& \|A_{1\lambda_0}(x_0,y_0)\| + \|A_{2\lambda_0}(x_0,y_0)\| \\
\leq& \|e_1+\bar{e}_1\| + \|\lambda_0 K_1 F_1(J_1(x_0)-\bar{e}_1, J_2(y_0)-\bar{e}_2)\| + \\
& \|e_2+\bar{e}_2\| + \|\lambda_0 K_2 F_2(J_1(x_0)-\bar{e}_1, J_2(y_0)-\bar{e}_2)\| \\
\leq& \|e_1+\bar{e}_1\| + \frac{\lambda_0 h_1(K_1F_1(J_1(x_0)-\bar{e}_1, J_2(y_0)-\bar{e}_2))}{h_1(\varphi_1)} + \\
& \|e_2+\bar{e}_2\| + \frac{\lambda_0 h_2(K_2F_2(J_1(x_0)-\bar{e}_1, J_2(y_0)-\bar{e}_2))}{h_2(\varphi_2)} \\
=& \|e_1+\bar{e}_1\| + \frac{\lambda_0 r(K_1) h_1(F_1(J_1(x_0)-\bar{e}_1, J_2(y_0)-\bar{e}_2))}{h_1(\varphi_1)} + \\
& \|e_2+\bar{e}_2\| + \frac{\lambda_0 r(K_2) h_2(F_2(J_1(x_0)-\bar{e}_1, J_2(y_0)-\bar{e}_2))}{h_2(\varphi_2)} \\
\leq& \|e_1+\bar{e}_1\| + \|e_2+\bar{e}_2\| + \\
& \lambda_0(\frac{r(K_1)}{h_1(\varphi_1)} + \frac{r(K_2)}{h_2(\varphi_2)})[b_{\lambda_0}\|(J_1(x_0)-\bar{e}_1, J_2(y_0)-\bar{e}_2)\| + \bar{M}].
\end{aligned} \tag{2.17}$$

From (2.5) and (2.6), we have

$$\|J_1(x_0)-\bar{e}_1\| \leq \|J_1(x_0)-x_0\| + \|x_0-\bar{e}_1\| \leq 3\|x_0-\bar{e}_1\| \leq 3(\|x_0\| + \|\bar{e}_1\|), \tag{2.18}$$

and

$$\|J_2(y_0)-\bar{e}_2\| \leq \|J_2(y_0)-y_0\| + \|y_0-\bar{e}_2\| \leq 3\|y_0-\bar{e}_2\| \leq 3(\|y_0\| + \|\bar{e}_2\|). \tag{2.19}$$

From (2.17)-(2.19), we have

$$\begin{aligned}
\|(x_0,y_0)\| \leq& \|e_1+\bar{e}_1\| + \|e_2+\bar{e}_2\| + \lambda_0\bar{M}(\frac{r(K_1)}{h_1(\varphi_1)} + \frac{r(K_2)}{h_2(\varphi_2)}) + \\
& 3\lambda_0 b_{\lambda_0}(\|\bar{e}_1\| + \|\bar{e}_2\|)(\frac{r(K_1)}{h_1(\varphi_1)} + \frac{r(K_2)}{h_2(\varphi_2)}) + \\
& 3\lambda_0 b_{\lambda_0}(\frac{r(K_1)}{h_1(\varphi_1)} + \frac{r(K_2)}{h_2(\varphi_2)})\|(x_0,y_0)\|
\end{aligned}$$

and so

$$\|(x_0,y_0)\| \le \frac{\|e_1+\bar{e}_1\|+\|e_2+\bar{e}_2\|+(\lambda_0\bar{M}+3\lambda_0 b_{\lambda_0}\|(\bar{e}_1,\bar{e}_2)\|)(\frac{r(K_1)}{h_1(\varphi_1)}+\frac{r(K_2)}{h_2(\varphi_2)})}{1-3\lambda_0 b_{\lambda_0}(\frac{r(K_1)}{h_1(\varphi_1)}+\frac{r(K_2)}{h_2(\varphi_2)})},$$

which is a contradiction, and so (2.16) holds. From the properties of the fixed point index, we have

$$i(A_{\lambda_0}, B_{R_{\lambda_0}} \cap Q^{\Delta}, Q^{\Delta}) = 1. \tag{2.20}$$

From (2.13) and (2.20), we have

$$i(A_{\lambda_0}, (B_{R_{\lambda_0}}\backslash \overline{B}_{R_0}) \cap Q^{\Delta}, Q^{\Delta}) = 1.$$

Thus, $A_{\lambda_0}$ has at least one fixed point $(x_{\lambda_0}, y_{\lambda_0})$ in $(B_{R_{\lambda_0}}\backslash \overline{B}_{R_0}) \cap Q^{\Delta}$. That is

$$\begin{cases} x_{\lambda_0} = \lambda_0 K_1 F_1(J_1(x_{\lambda_0})-\bar{e}_1, J_2(y_{\lambda_0})-\bar{e}_2)+e_1+\bar{e}_1, \\ y_{\lambda_0} = \lambda_0 K_2 F_2(J_1(x_{\lambda_0})-\bar{e}_1, J_2(y_{\lambda_0})-\bar{e}_2)+e_2+\bar{e}_2. \end{cases} \tag{2.21}$$

Let $\bar{x}_{\lambda_0} = x_{\lambda_0}-\bar{e}_1$, $\bar{y}_{\lambda_0} = x_{\lambda_0}-\bar{e}_2$. We claim that $(\bar{x}_{\lambda_0}, \bar{y}_{\lambda_0}) \in P^{\Delta}$. In fact, since $\|(x_{\lambda_0}, y_{\lambda_0})\| \ge R_0 = 2(c_0+\zeta)$, we have $\|x_{\lambda_0}\| \ge c_0+\zeta$ or $\|y_{\lambda_0}\| \ge c_0+\zeta$. If $\|x_{\lambda_0}\| \ge c_0+\zeta$, then from the fact that $x_{\lambda_0} \in Q_1$, we have

$$\bar{x}_{\lambda_0} = x_{\lambda_0}-\bar{e}_1 \ge (\|x_{\lambda_0}\|-c_0)\varphi_1 \ge \zeta\varphi_1 > \theta,$$

and so $J_1(x_{\lambda_0})-\bar{e}_1 = x_{\lambda_0}-\bar{e}_1$. Therefore, by (2.4) and (2.21), we have

$$\begin{aligned} \|h_2\|\|y_{\lambda_0}\| &\ge h_2(y_{\lambda_0}) = h_2(\lambda_0 K_2 F_2(J_1(x_{\lambda_0})-\bar{e}_1, J_2(y_{\lambda_0})-\bar{e}_2)+e_2+\bar{e}_2) \\ &= h_2(\lambda_0 K_2 F_2(x_{\lambda_0}-\bar{e}_1, J_2(y_{\lambda_0})-\bar{e}_2)+e_2+\bar{e}_2) \\ &\ge h_2(\lambda_0 K_2 F_2(x_{\lambda_0}-\bar{e}_1, J_2(y_{\lambda_0})-\bar{e}_2)) \\ &\ge \lambda_0 r(K_2)\eta. \end{aligned}$$

Consequently,

$$\|y_{\lambda_0}\| \ge \frac{\lambda_0 r(K_2)\eta}{\|h_2\|} \ge \frac{\lambda^* r(K_2)\eta}{\|h_2\|} > c_0.$$

Hence,

$$\bar{y}_{\lambda_0} = y_{\lambda_0}-\bar{e}_2 \ge (\|y_{\lambda_0}\|-c_0)\varphi_2 > \theta.$$

Therefore, $J_2(y_{\lambda_0})-\bar{e}_2 = \bar{y}_{\lambda_0}$.

When $\|y_{\lambda_0}\| \ge c_0+\zeta$, in a similar way we can show that $\bar{x}_{\lambda_0} > \theta,\ \bar{y}_{\lambda_0} > \theta$.

From the discussion above, we have

$$J_1(x_{\lambda_0})-\bar{e}_1 = \bar{x}_{\lambda_0},\ J_2(y_{\lambda_0})-\bar{e}_2 = \bar{y}_{\lambda_0}.$$

Therefore, by (2.21) we have

$$\begin{cases} \bar{x}_{\lambda_0} = \lambda_0 K_1 F_1(\bar{x}_{\lambda_0}, \bar{y}_{\lambda_0})+e_1, \\ \bar{y}_{\lambda_0} = \lambda_0 K_2 F_2(\bar{x}_{\lambda_0}, \bar{y}_{\lambda_0})+e_2. \end{cases}$$

This implies that $(\bar{x}_{\lambda_0}, \bar{y}_{\lambda_0})$ is a positive solutions of $(2.1_{\lambda_0})$. Since $\lambda_0 \in (\lambda^*, \infty)$ is arbitrarily given, we see the conclusion holds.

**Remark 2.2** The assumption ($H_3$) is sub-linear condition. To apply Theorem 2.1, we need to compute the spectrum radii of $K_1$ and $K_2$, and to find the functional $h_1, h_2$. However, in some cases these are not easy. To overcome this difficulty, we give the following Theorem 2.2.

**Theorem 2.2** *Suppose that* ($H_1$) *and* ($H_2$) *hold. Moreover, $P$ is a normal cone, $F_1$ and $F_2$ are increasing on $P^\Delta$, $K_1F_1(\theta, c_0\varphi_2) > \theta, K_2F_2(c_0\varphi_1, \theta) > \theta$, and*

$$\lim_{(x,y)\in P^\Delta, \|(x,y)\|\to\infty} \frac{\|K_1F_1(x,y)\|}{\|(x,y)\|} = 0, \tag{2.22}$$

$$\lim_{(x,y)\in P^\Delta, \|(x,y)\|\to\infty} \frac{\|K_2F_2(x,y)\|}{\|(x,y)\|} = 0. \tag{2.23}$$

*Then the operator equations system* $(2.1_\lambda)$ *has at least one positive solution for $\lambda > 0$ large enough.*

**Proof** The proof is similar to that of Theorem 2.1. For completeness, we give a brief proof. For any $\lambda \in [1, \infty)$, let the operator $A_\lambda$ be defined as (2.7). Take $R_0 \geq 4c_0$ and let

$$\lambda^* = \max\{1, \frac{NR_0}{\|K_1F_1(\theta, c_0\varphi_2)\|}, \frac{NR_0}{\|K_2F_2(c_0\varphi_1, \theta)\|}\},$$

where $N$ is the normal constant of the cone $P$. Let $\lambda_0 \in (\lambda^*, \infty)$ be fixed at present. First we prove that (2.8) still holds. Assume by contradiction that (2.9) holds for some $(x_0, y_0) \in \partial B_{R_0} \cap Q^\Delta$ and $\mu_0 > 0$. Since $\|(x_0, y_0)\| \geq 4c_0$, we have $\|x_0\| \geq 2c_0$ or $\|y_0\| \geq 2c_0$. Assume without loss of generality that $\|x_0\| \geq 2c_0$. Then we have

$$J_1(x_0) - \bar{e}_1 = x_0 - \bar{e}_1 \geq (\|x_0\| - c_0)\varphi_1 \geq c_0\varphi_1 > \theta,$$

and so

$$\begin{aligned} y_0 &= \lambda_0 K_2F_2(J_1(x_0) - \bar{e}_1, J_2(y_0) - \bar{e}_2) + \mu_0\Psi_2 \\ &\geq \lambda_0 K_2F_2(x_0 - \bar{e}_1, J_2(y_0) - \bar{e}_2) \\ &\geq \lambda_0 K_2F_2(c_0\varphi_1, \theta). \end{aligned}$$

Therefore,

$$\lambda_0 \leq \frac{N\|y_0\|}{\|K_2F_2(c_0\varphi_1, \theta)\|} \leq \frac{NR_0}{\|K_2F_2(c_0\varphi_1, \theta)\|},$$

which is a contradiction, and so (2.8) holds. Hence, (2.13) still holds. From (2.22) and (2.23), we see that for any $b_{\lambda_0} : 0 < b_{\lambda_0} < \frac{1}{6\lambda_0}$, there exists $R'_{\lambda_0} > 0$ such that

$$\|K_1F_1(x,y)\| \leq b_{\lambda_0}\|(x,y)\|, \|K_2F_2(x,y)\| \leq b_{\lambda_0}\|(x,y)\|$$

for all $(x,y) \in P^\Delta$ with $\|(x,y)\| \geq R'_{\lambda_0}$. Let

$$\bar{M}_1 = \sup\{\|K_1F_1(x,y)\| | (x,y) \in P^\Delta, \|(x,y)\| \leq R'_{\lambda_0}\},$$

$$\bar{M}_2 = \sup\{\|K_2F_2(x,y)\| | (x,y) \in P^\Delta, \|(x,y)\| \leq R'_{\lambda_0}\},$$

and $\bar{M} = \max\{\bar{M}_1, \bar{M}_2\}$. Then for any $(x,y) \in P^\Delta$, we have

$$\|K_1F_1(x,y)\| \leq b_{\lambda_0}\|(x,y)\| + \bar{M}, \|K_2F_2(x,y)\| \leq b_{\lambda_0}\|(x,y)\| + \bar{M}. \tag{2.24}$$

Take

$$R_{\lambda_0} > \max\{R_0, \frac{6\lambda_0 b_{\lambda_0}\|(\bar{e}_1, \bar{e}_2)\| + 2\lambda_0 \bar{M} + \|(e_1+\bar{e}_1, e_2+\bar{e}_2)\|}{1-6\lambda_0 b_{\lambda_0}}\}.$$

Next we will show (2.16). Assume by contradiction that for some $(x_0, y_0) \in \partial B_{R_{\lambda_0}} \cap Q^{\Delta}, \mu_0 \in [0,1]$, we have

$$(x_0, y_0) = \mu_0 A_{\lambda_0}(x_0, y_0).$$

Then, by (2.18), (2.19) and (2.24), we have

$$\begin{aligned}
\|(x_0,y_0)\| \leq & \|A_{\lambda_0}(x_0,y_0)\| \\
\leq & \lambda_0(\|K_1F_1(J_1(x_0)-\bar{e}_1, J_2(y_0)-\bar{e}_2)\| + \|K_2F_2(J_1(x_0)-\bar{e}_1, J_2(y_0)-\bar{e}_2)\|)+ \\
& \|e_1+\bar{e}_1\| + \|e_2+\bar{e}_2\| \\
\leq & 2\lambda_0 b_{\lambda_0}\|(J_1(x_0)-\bar{e}_1, J_2(y_0)-\bar{e}_2)\| + 2\lambda_0\bar{M} + \|e_1+\bar{e}_1\| + \|e_2+\bar{e}_2\| \\
\leq & 2\lambda_0 b_{\lambda_0}(3\|(x_0,y_0)\| + 3\|(\bar{e}_1,\bar{e}_2)\|) + 2\lambda_0\bar{M} + \|(e_1+\bar{e}_1, e_2+\bar{e}_2)\| \\
= & 6\lambda_0 b_{\lambda_0}\|(x_0,y_0)\| + 6\lambda_0 b_{\lambda_0}\|(\bar{e}_1,\bar{e}_2)\| + 2\lambda_0\bar{M} + \|(e_1+\bar{e}_1, e_2+\bar{e}_2)\|
\end{aligned}$$

and so

$$R_{\lambda_0} = \|(x_0,y_0)\| \leq \frac{6\lambda_0 b_{\lambda_0}\|(\bar{e}_1, \bar{e}_2)\| + 2\lambda_0 \bar{M} + \|(e_1+\bar{e}_1, e_2+\bar{e}_2)\|}{1-6\lambda_0 b_{\lambda_0}},$$

which is a contradiction. Thus (2.16) holds. From the properties of the fixed point index, we see that (2.20) holds. Now we have from (2.13) and (2.20) that

$$i(A_{\lambda_0}, (B_{R_{\lambda_0}} \backslash B_{R_0}) \cap Q^{\Delta}, Q^{\Delta}) = 1.$$

Then, $A_{\lambda_0}$ has at least one fixed point $(x_{\lambda_0}, y_{\lambda_0})$ in $(B_{R_{\lambda_0}} \backslash \overline{B}_{R_0}) \cap Q^{\Delta}$.

Let $\bar{x}_{\lambda_0} = x_{\lambda_0} - \bar{e}_1, \bar{y}_{\lambda_0} = y_{\lambda_0} - \bar{e}_2$. Now we will show that $(\bar{x}_{\lambda_0}, \bar{y}_{\lambda_0}) \in P^{\Delta}$. In fact, since $\|(x_{\lambda_0}, y_{\lambda_0})\| \geq 4c_0$, we have $\|x_{\lambda_0}\| \geq 2c_0$ or $\|y_{\lambda_0}\| \geq 2c_0$. Assume without loss of generality that $\|x_{\lambda_0}\| \geq 2c_0$. Then, we have

$$J_1(x_{\lambda_0}) - \bar{e}_1 = \bar{x}_{\lambda_0} = x_{\lambda_0} - \bar{e}_1 \geq (\|x_{\lambda_0}\| - c_0)\varphi_1 \geq c_0\varphi_1 > \theta$$

and so

$$y_{\lambda_0} = \lambda_0 K_2F_2(x_{\lambda_0} - \bar{e}_1, J_2(y_{\lambda_0}) - \bar{e}_2) + e_2 + \bar{e}_2 \geq \lambda_0 K_2F_2(c_0\varphi_1, \theta).$$

Consequently,

$$\|y_{\lambda_0}\| \geq N^{-1}\lambda_0\|K_2F_2(c_0\varphi_1,\theta)\| \geq N^{-1}\lambda^*\|K_2F_2(c_0\varphi_1,\theta)\| \geq 2c_0.$$

Therefore,

$$J_2(y_{\lambda_0}) - \bar{e}_2 = \bar{y}_{\lambda_0} = y_{\lambda_0} - \bar{e}_2 \geq (\|y_{\lambda_0}\| - c_0)\varphi_2 \geq c_0\varphi_2 > \theta.$$

Then, $(\bar{x}_{\lambda_0}, \bar{y}_{\lambda_0})$ is a positive solution of $(2.1_{\lambda_0})$. Since $\lambda_0 \in (\lambda^*, \infty)$ is arbitrarily given, we see that the conclusion holds. □

## 3. Positive solutions of semi-positone differential boundary value problems system

In this section, we will apply Theorem 2.1 to differential boundary value problems systems $(1.1_\lambda)$. For convenience, we make the following assumptions.

$(\mathrm{A}_1)$ For $i=1,2$, $p_i\in C[r,R]$, $p_i(t)>0(t\in[r,R])$, $f_i(t,x,y)\geq 0,(t,x,y)\in[r,R]\times R^+\times R^+$.

$(\mathrm{A}_2)$ For $i=1,2$, $a_i,b_i,c_i,d_i\geq 0$, and $a_ic_i+a_id_i+b_ic_i>0$.

$(\mathrm{A}_3)$

$$\lim_{(x,y)\geq(0,0),x+y\to\infty}\frac{f_1(t,x,y)}{x+y}=\lim_{(x,y)\geq(0,0),x+y\to\infty}\frac{f_2(t,x,y)}{x+y}=0$$

uniformly with $t\in[\alpha,\beta]\subset(r,R)$.

$(\mathrm{A}_4)$ There exists $M_0>0$, such that for $i=1,2$, $|q_i(t)|\leq M_0(t\in[r,R])$.

$(\mathrm{A}_5)$ There exist two constants $\gamma_1>0$ and $\gamma_2>0$, such that

$$f_1(t,x,y)\geq\gamma_1,\forall x\geq 0,y\geq\gamma_2,t\in[\alpha,\beta],$$

and

$$f_2(t,x,y)\geq\gamma_1,\forall y\geq 0,x\geq\gamma_2,t\in[\alpha,\beta].$$

For any $x\in C[r,R]$, let $\|x\|=\max\limits_{t\in[r,R]}|x(t)|$. Then $E=C[r,R]$ is a real Banach space with the norm $\|\cdot\|$. Let $P=\{x\in E|x(t)\geq 0,t\in[r,R]\}$, $Q_1=\{x\in P|x(t)\geq\|x\|\varphi_1(t),t\in[r,R]\}$ and $Q_2=\{x\in P|x(t)\geq\|x\|\varphi_2(t),t\in[r,R]\}$, where

$$\varphi_i(t)=\min\{\frac{b_i+a_i\int_r^t p_i^{-1}}{b_i+a_i\int_r^R p_i^{-1}},\frac{d_i+c_i\int_t^R p_i^{-1}}{d_i+c_i\int_r^R p_i^{-1}}\},t\in[0,1],i=1,2.$$

Obviously, $P$ is a normal solid cone, and so $P$ is a total cone. $Q_1$ and $Q_2$ are also cones of $E$. For $i=1,2$, let us define the operators $K_i:E\mapsto E$ and $F_i:P^{\Delta}\mapsto P$ by

$$(K_ix)(t)=\int_r^R G_i(t,s)x(s)\mathrm{d}s,t\in[r,R],x\in E,i=1,2,$$

$$F_i(x,y)(t)=f_i(t,x(t),y(t)),t\in[r,R],x,\ y\in E,i=1,2,$$

where

$$G_i(t,s)=\begin{cases}\alpha_i^{-1}(b_i+a_i\int_r^s p_i^{-1})(d_i+c_i\int_t^R p_i^{-1}), & s\leq t,\\ \alpha_i^{-1}(b_i+a_i\int_r^t p_i^{-1})(d_i+c_i\int_s^R p_i^{-1}), & s\geq t,\end{cases}$$

and $\alpha_i=a_id_i+a_ic_i\int_r^R p_i^{-1}+b_ic_i$. For $i=1,2$, let us define the functions $e_i(t)$ and $\bar{e}_i(t)$ by

$$e_i(t)=\int_r^R G_i(t,s)q_i(s)\mathrm{d}s,t\in[0,1],i=1,2,$$

$$\bar{e}_i(t)=M_0\int_r^R G_i(t,s)\mathrm{d}s,t\in[0,1],i=1,2.$$

Now to show the existence of positive solution of $(1.1_\lambda)$, we need only to show that of $(2.1_\lambda)$. From Lemmas 2.1 and 2.2 of [2], we have the following Lemmas 3.1 and 3.2.

**Lemma 3.1** *For $i=1,2$, $K_i:P\mapsto Q_i$ are completely continuous.*

**Lemma 3.2** *Let $\omega(t)=1(t\in[r,R])$. Then there exists $c>0$ such that $(K_i\omega)(t)\leq c\varphi_i(t),t\in[r,R],i=1,2$.*

**Theorem 3.1** *Suppose that* $(A_1)-(A_5)$ *hold. Then* $(1.1_\lambda)$ *has at least one positive solution for large enough* $\lambda>0$.

**Proof** To show Theorem 3.1, we will apply Theorem 2.1. Let $c_0=cM_0$. From Lemmas 3.1 and 3.2, we easily see that $(\mathrm{H}_1)$ and $(\mathrm{H}_2)$ hold. Obviously, $r(K_1)>0, r(K_2)>0$. By Krein-Rutman Theorem, there exist $\phi_i\in P\backslash\{\theta\}$ and $h_i\in P^*\backslash\{\theta\}$, such that

$$K_i\phi_i=r(K_i)\phi_i,\ K_i^*h_i=r(K_i)h_i, i=1,2.$$

Obviously, $\phi_i\in Q_i$. The functionals $h_i (i=1,2)$ can be taken by[8]

$$h_i(u)=\int_r^R\phi_i(t)u(t)\mathrm{d}t, u\in E, i=1,2.$$

Then, we have

$$h_i(\varphi_i)=\int_r^R\phi_i(t)\varphi_i(t)\mathrm{d}t\geq\|\phi_i\|\int_r^R[\varphi_i(t)]^2\mathrm{d}t>0,\ \ i=1,2.$$

For any $\varepsilon>0$, we see from $(\mathrm{A}_3)$ that there exists $\bar{R}_0>0$ such that

$$f_1(t,x,y)\leq\varepsilon(x+y), f_2(t,x,y)\leq\varepsilon(x+y)$$

for each $x\geq0, y\geq0$ with $x+y\geq\bar{R}_0$. Let

$$b=\max\{\sup_{(t,x,y)\in D}f_1(t,x,y),\ \sup_{(t,x,y)\in D}f_2(t,x,y)\},$$

where $D=\{(t,x,y)\in[r,R]\times R^+\times R^+|x+y\leq\bar{R}_0\}$. Then for any $(t,x,y)\in[r,R]\times R^+\times R^+$, we have

$$f_1(t,x,y)\leq\varepsilon(x+y)+b, f_1(t,x,y)\leq\varepsilon(x+y)+b.$$

Let

$$R_0>\frac{b\int_r^R[\phi_1(t)+\phi_2(t)]\mathrm{d}s}{\varepsilon}.$$

Then for any $(x,y)\in P^\Delta$, $\|(x,y)\|\geq R_0$ and $i=1,2$, we have

$$\begin{aligned}\frac{h_i(F_i(x,y))}{\|(x,y)\|}&=\frac{\int_r^R\phi_i(t)f_i(t,x(t),y(t))\mathrm{d}t}{\|(x,y)\|}\\&\leq\frac{b\int_r^R(\phi_1(t)+\phi_2(t))\mathrm{d}t}{\|(x,y)\|}+\frac{\varepsilon\int_r^R(\phi_1(t)+\phi_2(t))(x(t)+y(t))\mathrm{d}t}{\|(x,y)\|}\\&\leq(1+\int_r^R(\phi_1(t)+\phi_2(t))\mathrm{d}t)\varepsilon.\end{aligned}$$

This implies that $(\mathrm{H}_3)$ holds.

Take $\zeta_1\geq\gamma_2[\min_{t\in[\alpha,\beta]}\varphi_2(t)]^{-1}+c_0$. Then for any $y\in P, y\geq\zeta_1\varphi_2$, we have

$$y(t)-\bar{e}_2(t)\geq(\zeta_1-c_0)\varphi_2(t)\geq\gamma_2, t\in[\alpha,\beta].$$

Then, we see by $(\mathrm{A}_5)$ that

$$h_1(F_1(x,y-\bar{e}_2))=\int_r^R\phi_1(t)f_1(t,x(t),y(t)-\bar{e}_2(t))\mathrm{d}t$$

$$\geq \int_{\alpha}^{\beta} \phi_1(t) f_1(t, x(t), y(t) - \bar{e}_2(t)) \mathrm{d}t$$
$$\geq \gamma_1 \int_{\alpha}^{\beta} \phi_1(t) \mathrm{d}t \triangleq \eta_1$$

for any $x \in P$ and $y \in P$ with $y \geq \zeta_1 \varphi_2$. Let $\zeta_2 \geq \gamma_2[\min_{t\in[\alpha,\beta]} \varphi_1(t)]^{-1} + c_0$. Then, we have

$$h_2(F_2(x - \bar{e}_1, y)) \geq \gamma_1 \int_{\alpha}^{\beta} \phi_2(t) \mathrm{d}t \triangleq \eta_2$$

for any $y \in P$, and $x \in P$ with $x \geq \zeta_2 \varphi_1$. Let $\zeta = \max\{\zeta_1, \zeta_2\}$ and $\eta = \min\{\eta_1, \eta_2\}$. Then (2.3) and (2.4) hold. Thus all conditions of Theorem 2.1 are satisfied, and so Theorem 3.1 holds by Theorem 2.1. □

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