Journal of Mathematical Research & Exposition
May, 2008, Vol. 28, No. 2, pp. 316–322
DOI:10.3770/j.issn:1000-341X.2008.02.010
Http://jmre.dlut.edu.cn

The Best Constants of Hardy Type Inequalities for $p = -1$

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Abstract For $p > 1$, many improved or generalized results of the well-known Hardy's inequality have been established. In this paper, by means of the weight coefficient method, we establish the following Hardy type inequality for $p = -1$:

$$\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i}a_j\right)^{-1} < 2\sum_{i=1}^{n}\left(1-\frac{\pi^2-9}{3i}\right)a_i^{-1},$$

where $a_i > 0, i = 1, 2, \ldots, n$. For any fixed positive integer $n \geq 2$, we study the best constant C_n such that the inequality $\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i}a_j\right)^{-1} \leq C_n\sum_{i=1}^{n}a_i^{-1}$ holds. Moreover, by means of the Mathematica software, we give some examples.

Keywords Hardy type inequalities; weight coefficient; the best constant.

Document code A
MR(2000) Subject Classification 26D15
Chinese Library Classification O178.1

1. Introduction

A well-known Hardy's inequality states[1]:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{p} < \left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}a_n^{p}, \tag{1}$$

and the coefficient $\left(\frac{p}{p-1}\right)^p$ is the best constant in (1), where $p > 1$, $a_n \geq 0$, $0 < \sum_{n=1}^{\infty}a_n^p < \infty$.

In [1–7], there are some strengthened and generalized results of (1). Let a_k be substituted by $a_k^{1/p}$ and let $p \to +\infty$ in (1). Then (1) becomes Carleman inequality[1]:

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k\right)^{\frac{1}{n}} < e\sum_{n=1}^{\infty}a_n. \tag{2}$$

When $p = 2$ in (1), we have[2]

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{2} < 4\sum_{n=1}^{\infty}\left(1-\frac{1}{3\sqrt{n}+5}\right)a_n^{2}. \tag{3}$$

Received date: 2006-03-28; **Accepted date**: 2006-12-12
Foundation item: the National Natural Science Foundation of China (No. 10671136); the Natural Science Foundation of Sichuan Provincial Education Department (No. 2005A201).

When $p=3/2$ in (1), we have[3]

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{\frac{3}{2}}<3^{\frac{3}{2}}\sum_{n=1}^{\infty}\left(1-\frac{1}{5\sqrt[3]{n}+15}\right)a_n{}^{\frac{3}{2}}. \tag{4}$$

The Inequality (2) has the following strengthened results[4−5]:

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k\right)^{\frac{1}{n}}<e\sum_{n=1}^{\infty}\left(1-\frac{1}{2n+2}\right)a_n, \tag{5}$$

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k\right)^{\frac{1}{n}}<e\sum_{n=1}^{\infty}\left(1+\frac{1}{n+1/5}\right)^{-\frac{1}{2}}a_n. \tag{6}$$

For $p>1$, there is a strengthened result of (1) in [6],

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{p}<\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}\left(1-\frac{C_p}{2n^{1-p^{-1}}}\right)a_n^p, \tag{7}$$

where

$$C_p=\begin{cases}1-(1-p^{-1})^{p-1} & , \quad p\geq 2\\ 1-p^{-1} & , \quad 1<p\leq 2.\end{cases}$$

The Inequality (7) is generalized in [7] as follows:

$$\sum_{n=1}^{\infty}\left\{\left[M_n^{[\alpha]}(a)\right]^{1-\lambda_n}\left[M_n^{[\beta]}(a)\right]^{\lambda_n}\right\}^{p}<\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}\left(1-\frac{C_p}{2n^{1-p^{-1}}}\right)a_n^p, \tag{8}$$

where

$$p>1,\quad 0<\alpha<1<\beta,\quad \lambda_n\leq\left[1+\frac{\beta-1}{m(1-\alpha)}\right]^{-1},$$

$$m:=\min\left\{\frac{2+(n-2)t^{\beta}}{2+(n-2)t^{\alpha}}:t>0\right\},\quad M_n^{[r]}(a):=\left(\frac{1}{n}\sum_{k=1}^{n}a_k^r\right)^{1/r}.$$

In this paper, in case $p=-1$, we establish the Hardy type inequalities which are similar to (1) and (7) by the weight coefficients method. In Section 4, by means of the Mathematica software, we give some examples.

2. Theorem 2.1 and its Proof

In this section, in case $p=-1$, we establish the Hardy type inequality which is similar to (7).

Theorem 2.1 *If the real numbers $a_i>0$, $i=1,2,\ldots,n$, then we have*

$$\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i}a_j\right)^{-1}<2\sum_{i=1}^{n}\left(1-\frac{\pi^2-9}{3i}\right)a_i^{-1}. \tag{9}$$

Proof By the weighted power mean inequality[7]

$$\frac{\sum_{j=1}^{n}\lambda_j a_j}{\sum_{j=1}^{n}\lambda_j} \geq \left(\frac{\sum_{j=1}^{n}\lambda_j a_j^{-1}}{\sum_{j=1}^{n}\lambda_j}\right)^{-1}, \lambda_j > 0, a_j > 0, j = 1, 2, \ldots, n,$$

we obtain that

$$\begin{aligned}\frac{\sum_{j=1}^{i} a_j}{i} &= \frac{\sum_{j=1}^{i} j}{i}\frac{\sum_{j=1}^{i} j\frac{a_j}{j}}{\sum_{j=1}^{i} j} \geq \frac{\sum_{j=1}^{i} j}{i}\left[\frac{\sum_{j=1}^{i} j(\frac{a_j}{j})^{-1}}{\sum_{j=1}^{i} j}\right]^{-1}\\ &= \frac{(\sum_{j=1}^{i} j)^2}{i}\left(\sum_{j=1}^{i}\frac{j^2}{a_j}\right)^{-1} = \frac{i(i+1)^2}{4}\left(\sum_{j=1}^{i}\frac{j^2}{a_j}\right)^{-1},\end{aligned}$$

that is,

$$\left(\frac{1}{i}\sum_{j=1}^{i} a_j\right)^{-1} \leq \frac{4}{i(i+1)^2}\sum_{j=1}^{i}\frac{j^2}{a_j}. \tag{10}$$

According to (10), we get

$$\begin{aligned}\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i} a_j\right)^{-1} &\leq \sum_{i=1}^{n}\left[\frac{4}{i(i+1)^2}\sum_{j=1}^{i}\frac{j^2}{a_j}\right] = 4\sum_{i=1}^{n}\left[i^2\sum_{k=i}^{n}\frac{1}{k(k+1)^2}\right]\frac{1}{a_i}\\ &\leq 4\sum_{i=1}^{n}\left[i^2\sum_{k=i}^{\infty}\frac{1}{k(k+1)^2}\right]\frac{1}{a_i} = \sum_{i=1}^{n} f(i)a_i^{-1},\end{aligned}$$

where $f(i) := 4i^2\sum_{k=i}^{\infty}\frac{1}{k(k+1)^2}$. Therefore,

$$\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i} a_j\right)^{-1} \leq \sum_{i=1}^{n} f(i)a_i^{-1}, \tag{11}$$

and equality in (11) holds if and only if $n \to \infty$, $\frac{a_1}{1} = \frac{a_2}{2} = \cdots = \frac{a_i}{i} = \cdots$. By $\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$ and

$$\sum_{k=i}^{\infty}\frac{1}{k(k+1)^2} = \sum_{k=i}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \sum_{k=i}^{\infty}\frac{1}{(k+1)^2} = \frac{1}{i} + \sum_{k=1}^{i}\frac{1}{k^2} - \frac{\pi^2}{6},$$

we see that

$$f(i) := 4i^2\sum_{k=i}^{\infty}\frac{1}{k(k+1)^2} = 4i^2\left(\frac{1}{i} + \sum_{k=1}^{i}\frac{1}{k^2} - \frac{\pi^2}{6}\right). \tag{12}$$

Combining results (11) and (12), we only need to prove

$$f(i) \leq 2\left(1 - \frac{\pi^2 - 9}{3i}\right) \Leftrightarrow \varphi(i) := \frac{1}{i} + \sum_{k=1}^{i}\frac{1}{k^2} - \frac{1}{2i^2} + \frac{\pi^2 - 9}{6i^3} - \frac{\pi^2}{6} \leq 0, \quad \forall i \geq 1, \tag{13}$$

where equality of the latter inequality in (13) holds if and only if $i = 1$ or $i \to \infty$. Notice

$$\begin{aligned}\varphi(i+1) - \varphi(i) &= \frac{1}{i+1} - \frac{1}{i} + \frac{1}{2(i+1)^2} + \frac{1}{2i^2} + \frac{\pi^2 - 9}{6(i+1)^3} - \frac{\pi^2 - 9}{6i^3}\\ &= \frac{1}{2[i(i+1)]^2} + \frac{\pi^2 - 9}{6}\cdot\left[\frac{1}{(i+1)^3} - \frac{1}{i^3}\right]\end{aligned}$$

$$= \frac{3(10-\pi^2)i(i+1)-(\pi^2-9)}{6[i(i+1)]^3},$$

$$\varphi(2)-\varphi(1) = \frac{3(10-\pi^2)\times 2-(\pi^2-9)}{48} = \frac{69-7\pi^2}{48} = \frac{0.0872308076255024\cdots}{48} < 0;$$

when $i \geq 2$, we have

$$\begin{aligned}\varphi(i+1)-\varphi(i) &= \frac{3(10-\pi^2)i(i+1)-(\pi^2-9)}{6[i(i+1)]^3} \geq \frac{3(10-\pi^2)\times 6-(\pi^2-9)}{6[i(i+1)]^3} \\ &= \frac{189-19\pi^2}{6[i(i+1)]^3} = \frac{1.4775163793021875\cdots}{6[i(i+1)]^3} > 0.\end{aligned}$$

Consequently,

$$\varphi(1) > \varphi(2) < \varphi(3) < \cdots < \varphi(i) < \varphi(i+1) < \cdots,$$

$$\varphi(i) \leq \max\{\varphi(1), \varphi(\infty)\} = \max\{0, 0\} = 0.$$

Thus, (13) is proved. By (11) and (13) and $\varphi(i)$ is not identical to 0, we know (9) holds. This completes the proof of Theorem 2.1. □

3. The best property of the coefficients in the Inequality (9)

By the Inequality (9), in case $p = -1$, we have the following Hardy type inequality which is similar to (1):

Theorem 3.1 *If the real numbers $a_i > 0$, $i = 1, 2, \ldots, n$, we have*

$$\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i}a_j\right)^{-1} < 2\sum_{i=1}^{n}a_i^{-1}. \tag{14}$$

Under the weight coefficients $1, 2, \ldots, n$, the coefficient 2 in (14) is the best constant. In other words,

$$\sup\{f(i) : i \geq 1\} = 2, \tag{15}$$

where $f(i)$ is defined in (12).

Proof From the Inequality (9), (14) can be obtained. Next, we will prove that the coefficient 2 is the best constant. We first consider the monotonicity of $f(i)$. By (12), we have

$$\begin{aligned}f(i+1)-f(i) &= 4(i+1)^2\left(\frac{1}{i+1}+\sum_{k=1}^{i+1}\frac{1}{k^2}-\frac{\pi^2}{6}\right) - 4i^2\left(\frac{1}{i}+\sum_{k=1}^{i}\frac{1}{k^2}-\frac{\pi^2}{6}\right) \\ &= 4\left[2+(2i+1)\sum_{k=1}^{i}\frac{1}{k^2}-(2i+1)\frac{\pi^2}{6}\right] \\ &= 4(2i+1)\left(\frac{2}{2i+1}+\sum_{k=1}^{i}\frac{1}{k^2}-\frac{\pi^2}{6}\right) \\ &= 4(2i+1)g(i),\end{aligned}$$

where $g(i) := \frac{2}{2i+1} + \sum_{k=1}^{i} \frac{1}{k^2} - \frac{\pi^2}{6}$. Now we need to consider the monotonicity of $g(i)$. Because

$$\begin{aligned} g(i+1) - g(i) &= \frac{2}{2i+3} + \sum_{k=1}^{i+1} \frac{1}{k^2} - \frac{\pi^2}{6} - \left(\frac{2}{2i+1} + \sum_{k=1}^{i} \frac{1}{k^2} - \frac{\pi^2}{6} \right) \\ &= 2\left(\frac{1}{2i+3} - \frac{1}{2i+1} \right) + \frac{1}{(i+1)^2} \\ &= -\frac{1}{(2i+3)(2i+1)(i+1)^2} < 0, \end{aligned}$$

$g(i)$ is decreasing strictly with respect to i and

$$g(i) > g(\infty) = \lim_{i\to\infty} \left(\frac{2}{2i+1} + \sum_{k=1}^{i} \frac{1}{k^2} - \frac{\pi^2}{6} \right) = 0.$$

It follows that

$$f(i+1) - f(i) = 4(2i+1)g(i) > 0.$$

Hence $f(i)$ is increasing strictly with respect to i. Furthermore, by (13), we see that

$$f(i) < 2, \quad \forall i \geq 1. \tag{16}$$

Thus, $\lim_{i\to\infty} f(i)$ exists and

$$\sup\{f(i) : i \geq 1\} = \lim_{i\to\infty} f(i). \tag{17}$$

While

$$\begin{aligned} f(i) &= 4i^2 \left(\frac{1}{i} + \sum_{k=1}^{i} \frac{1}{k^2} - \frac{\pi^2}{6} \right) = 4i^2 \left[\frac{1}{i} - \sum_{k=i}^{\infty} \frac{1}{(k+1)^2} \right] \\ &> 4i^2 \left[\frac{1}{i} - \sum_{k=i}^{\infty} \frac{1}{(k+1/2)(k+3/2)} \right] \\ &= 4i^2 \left[\frac{1}{i} - \sum_{k=i}^{\infty} \left(\frac{1}{k+1/2} - \frac{1}{k+1+1/2} \right) \right] \\ &= 4i^2 \left(\frac{1}{i} - \frac{1}{i+1/2} \right) = \frac{4i}{2i+1}, \end{aligned}$$

this leads to

$$\frac{4}{2i+1} < f(i), \quad \forall i \geq 1. \tag{18}$$

Combining (16)–(18), we know that $\sup\{f(i) : i \geq 1\} = \lim_{i\to\infty} f(i) = 2$. This completes the proof of Theorem 3.1. □

Theorem 3.2 *Under the weight coefficients $1, 2, \ldots, n$, the coefficient $(\pi^2 - 9)/3$ in (9) is the best constant. In other words, if there exists a constant C $(C > 0)$ that is independent of i and n such that $f(i) \leq 2(1 - C/i)$ $(\forall i \geq 1)$, then we have*

$$C \leq \frac{\pi^2 - 9}{3} = 0.28986813369645265 \cdots.$$

Proof For $f(i) \le 2(1-C/i)$ $(\forall i \ge 1)$, set $i=1$, we obtain that

$$4\left(2-\frac{\pi^2}{6}\right) \le 2(1-C) \Rightarrow C \le \frac{\pi^2-9}{3}.$$

Theorem 3.2 is proved. □

4. The further problem

In the Inequality (14), the best constant 2 is independent of n, which motivates us to consider the following problem.

Problem 4.1 If the real numbers $a_i > 0, i=1,2,\ldots,n$, for any fixed positive integer $n \ge 2$, we try to find a minimum positive constant C_n (the best constant) that only depends on n such that

$$\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i} a_j\right)^{-1} \le C_n \sum_{i=1}^{n} a_i^{-1}. \tag{19}$$

For the above problem, we have the theorem as follows:

Theorem 4.1 *If the real numbers $a_i > 0, i=1,2,\ldots,n, n \ge 2$, then the minimum constant C_n that only depends on n and satisfies (19) is determined by the following conditions:*

$$y_1 = \sqrt{C_n}, y_n = \sqrt{n}, y_{i+1} = \sqrt{C_n} + \sqrt{y_i^2 - i}, \quad i=1,2,\ldots,n-1. \tag{20}$$

Proof According to the proof of Theorem 2.1, for any real number $\lambda_i > 0$ $(i=1,2,\ldots,n)$, we have

$$\left(\frac{1}{i}\sum_{j=1}^{i} a_j\right)^{-1} \le i\left(\sum_{j=1}^{i}\lambda_j\right)^{-2}\sum_{j=1}^{i}\lambda_j^2 a_j^{-1}.$$

It follows that

$$\begin{aligned}\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i} a_j\right)^{-1} &\le \sum_{i=1}^{n}\left[i\left(\sum_{j=1}^{i}\lambda_j\right)^{-2}\sum_{j=1}^{i}\lambda_j^2 a_j^{-1}\right]\\ &= \sum_{i=1}^{n}\left[\lambda_i^2\sum_{k=i}^{n} k\left(\sum_{j=1}^{k}\lambda_j\right)^{-2}\right]a_i^{-1} = \sum_{i=1}^{n} x_i a_i^{-1},\end{aligned} \tag{21}$$

and the equality in (21) holds if and only if $\frac{a_1}{\lambda_1} = \frac{a_2}{\lambda_2} = \cdots = \frac{a_n}{\lambda_n}$, where

$$x_i = \lambda_i^2 \sum_{k=i}^{n} k\left(\sum_{j=1}^{k}\lambda_j\right)^{-2}, \quad i=1,2,\ldots,n. \tag{22}$$

Therefore, if we can choose a constant $\lambda_i > 0$ $(i=1,2,\ldots,n)$ such that

$$x_1 = x_2 = \cdots = x_n = C_n > 0, \tag{23}$$

then C_n is the minimum positive constant such that (19) holds.

From (22), we know that (23) is equivalent to

$$n\lambda_n^2\left(\sum_{j=1}^{n}\lambda_j\right)^{-2}=C_n,\frac{C_n}{\lambda_i^2}-\frac{C_n}{\lambda_{i+1}^2}=i\left(\sum_{j=1}^{i}\lambda_j\right)^{-2},\quad i=1,2,\ldots,n-1. \tag{24}$$

Write $S_i:=\sum_{k=1}^{i}\lambda_k$, $y_i:=\frac{\sqrt{C_n}S_i}{S_i-S_{i-1}}$ $(1\le i\le n)$, $S_0:=0$. Then (24) is equivalent to

$$\begin{aligned}
&n\left(\frac{S_n-S_{n-1}}{S_n}\right)^2=C_n,\\
&C_n\left[\left(\frac{S_i}{S_i-S_{i-1}}\right)^2-\left(\frac{S_i}{S_{i+1}-S_i}\right)^2\right]=i,\quad i=1,2,\ldots,n-1\\
&\quad\Leftrightarrow y_1=\sqrt{C_n},y_n=\sqrt{n},y_i^2-(y_{i+1}-\sqrt{C_n})^2=i,\quad i=1,2,\ldots,n-1\\
&\quad\Leftrightarrow y_1=\sqrt{C_n},y_n=\sqrt{n},y_{i+1}=\sqrt{C_n}+\sqrt{y_i^2-i},\quad i=1,2,\ldots,n-1.
\end{aligned}$$

This completes the proof of Theorem 4.1. □

Example 4.1 By (20), we know that $y_1,y_2,\ldots,y_n$ can be expressed via C_n. Thus, the algebraic equations which C_n satisfies have been obtained. By means of the Mathematica software, we get

$$C_3=1.204692944799795,\ldots,\quad C_4=1.2611004647671393,\ldots,$$

$$C_5=1.3037394436929084,\ldots,\quad C_6=1.337457769812323,\ldots.$$

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