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The Analytic Function in the Right Half Plane Defined by Laplace-Stieltjes Transforms

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Abstract In this paper, the growth of analytic function defined by L-S transforms convergent in the right half plane is studied and some properties on the L-S transform F(s) and its relative transforms f(s) are obtained.

Keywords Laplace-Stieltjes transform; convergent half plane; Newton polygon; order.

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Some problems on the growth and the value distribution of analytic functions defined by Dirichlet series have been studied for a long time and lots of important results were obtained in [1], [2] and [3], but the correlative researches of L-S Transforms are seldom discussed. From [4] and [5], Dirichlet series was regarded as a special example of L-S transforms and some properties of Dirichlet series may be the same with L-S transforms [6]. Yu[1,4] first studied the growth of Dirichlet series which was uniformly convergent in the complex plane, and obtained some properties on its implicative series, then he extended the results to L-S transforms. In this paper, we continue those studies on the L-S transforms which are convergent in the right half plane and obtained some new results on its relative transforms.

Consider L-S transforms[5]

$$F(s)=\int_0^{+\infty} e^{-sx}\mathrm{d}\alpha(x), \quad s=\sigma+it, \tag{1}$$

where $\alpha(x)$ is a bounded variation on any interval $[0,X]$ $(0<X<+\infty)$, and σ and t are real variables. We choose a sequence $\{\lambda_n\}$:

$$0=\lambda_1<\lambda_2<\lambda_3<\cdots<\lambda_n\uparrow+\infty, \tag{2}$$

which satisfies the following conditions:

$$\overline{\lim_{n\to\infty}}(\lambda_{n+1}-\lambda_n)<+\infty \tag{3}$$

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and

$$\overline{\lim_{n\to\infty}}\frac{n}{\lambda_n}=D<+\infty,\quad \overline{\lim_{n\to\infty}}\frac{\ln A_n^*}{\lambda_n}=0, \tag{4}$$

where

$$A_n^*=\sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^{x}e^{-ity}\mathrm{d}\alpha(y)\right|.$$

According to Valiron-Knopp-Bohr formula[5] and the conditions of (3) and (4), it follows that $\sigma_u^F=0$, where σ_u^F is the uniformly convergent abscissa of (1). Then the transform $F(s)$ is analytic in the right half-plane. We set

$$M_u(\sigma,F)=\sup_{0<x<+\infty,-\infty<t<+\infty}\left|\int_0^x e^{-(\sigma+it)y}\mathrm{d}\alpha(y)\right|,\quad \sigma>0,$$

$$\mu(\sigma,F)=\max_{1\le n<+\infty}\{A_n^*e^{-\lambda_n\sigma}\},\quad \sigma>0.$$

By the second formula of (4), for any σ,

$$\overline{\lim_{n\to\infty}}\frac{\ln A_n^*-\lambda_n\sigma}{\lambda_n}=-\sigma<0\quad\text{or}\quad\overline{\lim_{n\to\infty}}\ln A_n^*e^{-\lambda_n\sigma}=-\infty.$$

We can see that $\mu(\sigma,F)$ exists.

Let $\{P_n\}=\{(\lambda_n,-\ln A_n^*)\}$ $(n=1,2,\ldots)$ be a sequence on the xOy plane. Make a convex Newton polygon $\Pi(F)$ from $\{P_n\}$ such that its vertices are in $\{P_n\}$ and the other points are on or above the edge of it. For any $\sigma>0$, we draw a line over P_n with the slope $-\sigma$:

$$y+\ln A_n^*=-\sigma(x-\lambda_n).$$

The ordinate of the crossover point between the line and the y-axis is $-\ln A_n^*e^{-\lambda_n\sigma}$. Therefore $-\ln\mu(\sigma,F)=\min_{1\le n<+\infty}\{-\ln A_n^*e^{-\lambda_n\sigma}\}$. Let $n(\sigma)=\max\{n;\mu(\sigma,F)=A_n^*e^{-\lambda_n\sigma}\}$ denote the maximum term index of (1). Then $\mu(\sigma,F)=A_{n(\sigma)}^*e^{-\lambda_{n(\sigma)}\sigma}$.

Suppose that $G_n=-\ln A_n^*$. Using the similar method to [1] gives:

$$\ln\mu(\sigma,F)=\begin{cases}-G_1, & -\sigma<\frac{G_2-G_1}{\lambda_2-\lambda_1},\\ -G_1-\int_{-\frac{G_2-G_1}{\lambda_2-\lambda_1}}^{\sigma}\lambda_{n(x)}\mathrm{d}x, & 0>-\sigma\ge\frac{G_2-G_1}{\lambda_2-\lambda_1}.\end{cases}$$

It is obvious that $\ln\mu(\sigma,F)$ is a decreasing convex function in $(0,+\infty)$. We can also give the definition of the order τ_u and the order τ_μ as follows

$$\tau_u=\overline{\lim_{\sigma\to0^+}}\frac{\ln^+\ln^+M_u(\sigma,F)}{-\ln\sigma},\quad \tau_\mu=\overline{\lim_{\sigma\to0^+}}\frac{\ln^+\ln^+\mu(\sigma,F)}{-\ln\sigma}.$$

Next, we will investigate the relation between the maximum modulus $M_u(\sigma,F)$ and the maximum term $\mu(\sigma,F)$ of $F(s)$ defined by (1) convergent in the right half plane $\{s|\mathrm{Re}s=\sigma>0\}$.

Theorem 1 *Suppose that* $\sigma_u^F=0$, *and the sequence* (2) *satisfies the conditions of* (3) *and* (4). *Then* $\forall\varepsilon\in(0,1)$, *when* σ *is sufficiently close to* 0^+, $\tau_u=\tau_\mu$ *holds.*

Proof Firstly, let

$$I(x;\sigma+it)=\int_0^x e^{-(\sigma+it)y}\mathrm{d}\alpha(y).$$

From (3), there exists $K>0$ satisfying $0<\lambda_{n+1}-\lambda_n\leq K\quad (n=1,2,3,\ldots)$. As $\sigma(>0)$ sufficiently reaches 0, it follows $e^{K\sigma}<\frac{3}{2}$. When $x>\lambda_n$, we have

$$\begin{aligned}\int_{\lambda_n}^{x} e^{-ity}\mathrm{d}\alpha(y) &= \int_{\lambda_n}^{x} e^{\sigma y} d_y I(y;\sigma+it)\\ &= I(y;\sigma+it)e^{\sigma y}|_{\lambda_n}^{x} - \sigma\int_{\lambda_n}^{x} e^{\sigma y} I(y;\sigma+it)\mathrm{d}y.\end{aligned}$$

For any $\sigma>0$, and any $x\in(\lambda_n,\lambda_{n+1}]$, it follows that

$$\begin{aligned}|\int_{\lambda_n}^{x} e^{-ity}\mathrm{d}\alpha(y)| &\leq M_u(\sigma,F)[|e^{\sigma x}+e^{\sigma\lambda_n}|+|e^{\sigma x}-e^{\sigma\lambda_n}|]\\ &\leq 2M_u(\sigma,F)e^{(\lambda_n+K)\sigma}\leq 3M_u(\sigma,F)e^{\lambda_n\sigma},\end{aligned}$$

then $\frac{1}{3}\mu(\sigma,F)\leq M_u(\sigma,F)$ and $\overline{\lim}_{\sigma\to 0^+}\frac{\ln^+\ln^+\mu(\sigma,F)}{-\ln\sigma}\leq\overline{\lim}_{\sigma\to 0^+}\frac{\ln^+\ln^+M_u(\sigma,F)}{-\ln\sigma}$.

Secondly, for any $x>0$, there exists $n\in N, \lambda_n<x\leq\lambda_{n+1}$, such that

$$\int_0^x e^{-(\sigma+it)y}\mathrm{d}\alpha(y)=\sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} e^{-(\sigma+it)y}\mathrm{d}\alpha(y)+\int_{\lambda_n}^{x} e^{-(\sigma+it)y}\mathrm{d}\alpha(y).$$

Let

$$I_k(x;it)=\int_{\lambda_k}^{x} e^{-ity}\mathrm{d}\alpha(y),\quad \lambda_k\leq\ x\leq\lambda_{k+1}.$$

For any $t\in R$, we have

$$|I_k(x;it)|\leq A_k^*\leq\mu(\sigma,F)e^{\lambda_k\sigma},\quad \sigma>0. \tag{5}$$

Hence for any $x\in(\lambda_k,\lambda_{k+1}]$ and $\sigma>0$, we have

$$\begin{aligned}\int_0^x e^{-(\sigma+it)y}\mathrm{d}\alpha(y)=&\sum_{k=1}^{n-1}[e^{-\lambda_{k+1}\sigma}I_k(\lambda_{k+1};it)+\sigma\int_{\lambda_k}^{\lambda_{k+1}} e^{-\sigma y}I_k(y;it)\mathrm{d}y]+\\ & e^{-\sigma x}I_n(x;it)+\sigma\int_{\lambda_n}^{x} e^{-\sigma y}I_n(y;it)\mathrm{d}y.\end{aligned}$$

From (5) and $\forall\varepsilon\in(0,1)$, we obtain $|I_k(x;it)|\leq\mu((1-\varepsilon)\sigma,F)e^{\lambda_k(1-\varepsilon)\sigma}$. Then

$$\begin{aligned}|\int_0^x e^{-(\sigma+it)y}\mathrm{d}\alpha(y)|\leq&\sum_{k=1}^{n-1}\mu((1-\varepsilon)\sigma,F)e^{\lambda_k(1-\varepsilon)\sigma}(e^{-\lambda_{k+1}\sigma}+|e^{-\lambda_{k+1}\sigma}-e^{-\lambda_k\sigma}|)+\\ &\mu((1-\varepsilon)\sigma,F)e^{\lambda_n(1-\varepsilon)\sigma}(e^{-\sigma x}+|e^{-\sigma x}-e^{-\lambda_n\sigma}|)\\ =&\sum_{k=1}^{n}\mu((1-\varepsilon)\sigma,F)e^{\lambda_k(1-\varepsilon)\sigma}e^{-\lambda_k\sigma}\\ \leq&\mu((1-\varepsilon)\sigma,F)\sum_{k=1}^{+\infty}e^{-\lambda_k\varepsilon\sigma}.\end{aligned}$$

From the first formula of (4), it follows that for the above $\varepsilon>0$, there exists $N(\varepsilon)>0$, for any $n>N(\varepsilon)$, we have $\lambda_n>\frac{n}{D+\varepsilon}$, such that

$$\sum_{k=1}^{+\infty}e^{-\lambda_k\varepsilon\sigma}\leq\sum_{k=1}^{N(\varepsilon)}e^{-\lambda_k\varepsilon\sigma}+\sum_{k=N(\varepsilon)+1}^{+\infty}e^{-k\frac{\sigma\varepsilon}{D+\varepsilon}}<K(\varepsilon)\frac{1}{\sigma},\quad \sigma\to+0,$$

where $K(\varepsilon)$ is a constant dependent on ε and (3). So for any $\varepsilon \in (0,1)$ and $t \in R$, it follows that $M_u(\sigma, F) \leq K(\varepsilon)\mu((1-\varepsilon)\sigma, F) \cdot \frac{1}{\sigma}$.

Consequently, we have

$$\overline{\lim_{\sigma\to 0^+}}\frac{\ln^+\ln^+\mu(\sigma,F)}{-\ln\sigma} \geq \overline{\lim_{\sigma\to 0^+}}\frac{\ln^+\ln^+ M_u(\sigma,F)}{-\ln\sigma}.$$

The proof is completed.

Theorem 2 *Suppose that L-S transforms* (1) *of the order* $\tau_\mu \in (0,+\infty)$ *satisfy* (2),(3) *and* (4). *Then there will be only two situations on* $\ln^+\mu(\sigma,F)$*:*

1) For $\forall\eta_n \downarrow 0 (\eta_n < \tau_\mu)$*, there exists* $\{\xi_n\} \in (0,1)$*, such that for* $\forall\sigma < \xi_n, \forall n \in N_+$*, it follows that* $\ln^+\mu(\sigma,F) > \sigma^{-(\tau_\mu-\eta_n)}$*;*

2) Otherwise there exists $\eta_n \downarrow 0 (\eta_n < \tau_\mu)$ *and* $\sigma_n \downarrow 0^+$*, then* $\ln^+\mu(\sigma_n,F) = \sigma_n^{-(\tau_\mu-\eta_n)}$.

Proof 1) is possible for $\tau_\mu > 0$, we only need to prove 2). Suppose that 1) is untrue. Take $\varepsilon_n \downarrow 0 (\varepsilon_n < \tau_\mu)$. Then there exists positive numbers $\sigma_1' < 1$ and $k_1 \in N_+$, such that

$$\ln^+\mu(\sigma_1',F) \leq {\sigma_1'}^{-(\tau_\mu-\varepsilon_{k_1})}.$$

Since $\tau_\mu > 0$, there exists positive number $\sigma_1^* < \sigma_1'$ so that $\ln^+\mu(\sigma_1^*,F) > {\sigma_1^*}^{-(\tau_\mu-\varepsilon_{k_1})}$ and then $\exists\sigma_1 \in (\sigma_1^*, \sigma_1')$ satisfying $\ln^+\mu(\sigma_1,F) = {\sigma_1}^{-(\tau_\mu-\varepsilon_{k_1})}$.

Take the other sequence $\{\varepsilon_k\}(k > k_1)$. Since 1) is untrue, there exists positive numbers $\sigma_2' < \sigma_1^*$ and $k_2 > k_1$, satisfying $\ln^+\mu(\sigma_2',F) \leq {\sigma_2'}^{-(\tau_\mu-\varepsilon_{k_2})}$. For $f(s)$ of the order $\tau_\mu > 0$, there exists positive number $\sigma_2^* < \sigma_2'$ such that

$$\ln^+\mu(\sigma_2^*,F) > {\sigma_2^*}^{-(\tau_\mu-\varepsilon_{k_2})}.$$

Therefore there exists $\sigma_2 \in (\sigma_2^*, \sigma_2')$ satisfying $\ln^+\mu(\sigma_2,F) = {\sigma_2}^{-(\tau_\mu-\varepsilon_{k_2})}$. The rest may be deduced analogously. So there exists $\varepsilon_{k_n} \downarrow 0$, and $\sigma_n \downarrow 0^+$ which satisfy

$$\ln^+\mu(\sigma_n,F) = {\sigma_n}^{-(\tau_\mu-\varepsilon_{k_n})}.$$

Set $\eta_n = \varepsilon_{k_n}$, we can obtain 2). Theorem 2 is proved.

Suppose that $F(s)$ meets the conditions of (2), (3) and (4). Then it is analytic in the right half-plane. In the following text, we only discuss the situation of $\tau_\mu > 0$. For any $\sigma \in (0,1]$, we define the function $V(\sigma)$ as follows:

1^0. When $0 < \tau_\mu < +\infty$:

Case 1) of Theorem 2, we set $V(\sigma) = \ln^+\mu(\sigma,F)$.

Case 2) of Theorem 2, for $\sigma_{n+1} < \sigma < \sigma_n$, we set

$$V(\sigma) = \max\{\ln^+\mu(\sigma,F), \sigma^{-(\tau_\mu-\eta_n)}\}, \quad n = 1,2,\ldots$$

2^0. When $\tau_\mu = +\infty$, set $\frac{\ln V(\sigma)}{-\ln\sigma} = \max_{\sigma\leq x\leq 1}\frac{\ln^+\ln^+\mu(x,F)}{-\ln x}$.

Under case 1^0, $V(\sigma)$ is a decreasing convex function on σ. Under case 2^0, $V(\sigma)$ is a continuous function on σ. Under both cases 1^0 and 2^0, we always have:

$$V(\sigma) \geq \ln^+\mu(\sigma,F), \quad \sigma \in (0,1] \tag{6}$$

and there exists a decreasing positive sequence $\{\sigma_n'\} \searrow 0$ such that

$$V(\sigma_n') = \ln \mu(\sigma_n', F). \tag{7}$$

So we have the following results:

(i) When $\sigma \to 0^+$, we see that $\lim\limits_{\sigma\to 0^+} \frac{\ln V(\sigma)}{-\ln\sigma} = \tau_\mu$, then $V(\sigma) > (\ln\sigma)^2$.

(ii) Suppose that $W(\sigma) = e^{V(\sigma)}$. Then $W(\sigma)$ is a decreasing function on σ and for any $h \in (0, +\infty)$, it satisfies $\int_0^1 \frac{W(\sigma+h)}{W(\sigma)}\mathrm{d}\sigma < \infty$.

Lemma 1 Suppose $\Omega(x) = \int_0^1 \frac{e^{x(1-\sigma)}}{W(\sigma)}\mathrm{d}\sigma$ *is an increasing function of* $x(> 0)$ *and we set*

$$\nu(x) = \max_{0<\sigma\le 1} \frac{e^{x(1-\sigma)}}{W(\sigma)} = \frac{e^{x(1-\sigma_x)}}{W(\sigma_x)}. \tag{8}$$

Then $\frac{\nu(x)}{2x} \le \Omega(x) \le \nu(x)$, *when* x *is sufficiently large.*

Proof When x is sufficiently large, it follows that

$$\begin{aligned}\Omega(x) &\ge \int_{\sigma_x}^1 \frac{e^{x(1-\sigma)}}{W(\sigma)}\mathrm{d}\sigma \ge \frac{1}{W(\sigma_x)}\int_{\sigma_x}^1 e^{x(1-\sigma)}\mathrm{d}\sigma = \frac{e^{x(1-\sigma_x)}-1}{xW(\sigma_x)}\\ &> \frac{e^{x(1-\sigma_x)}}{2xW(\sigma_x)} > \frac{\nu(x)}{2x}.\end{aligned}$$

In addition, we can easily get $\Omega(x) \le \nu(x)\int_0^1 \mathrm{d}\sigma = \nu(x)$. The proof is completed.

Theorem 3 *Suppose that* $F(s)$ *satisfies* $\sigma_\mu^F = 0$. *Then it is analytic in the right half-plane. We set a new relative transform of* $F(s)$ *as follows*

$$f(s) = \int_0^{+\infty} e^{-sx}\Omega(x)\mathrm{d}\alpha(x).$$

Then its associated abscissas of uniform convergence $\sigma_\mu^f = 1$, *and when* $\mathrm{Re}s > 1$, *we have*

$$f(s) = \int_0^1 \frac{F(s+x-1)}{W(x)}\mathrm{d}x. \tag{9}$$

Proof Choose a sequence λ_n which satisfy (2), (3) and (4). Then we set

$$B_n^* = \sup_{\lambda_n<x<\lambda_{n+1}, -\infty<t<+\infty} |\int_{\lambda_n}^x e^{-ity}\Omega(y)\mathrm{d}\alpha(y)|.$$

By Theorem 3.1 of the paper [5], we have

$$A_n^*\Omega(\lambda_n) \le B_n^* \le 2A_n^*\Omega(\lambda_{n+1}).$$

Let K be a constant dependent of (3). Combining this with Lemma 1, we obtain

$$\ln A_n^* + \ln\nu(\lambda_n) - \ln 2\lambda_n \le \ln B_n^* \le \ln A_n^* + \ln\nu(\lambda_n + K) + \ln 2. \tag{10}$$

On the one hand, from (6) and (8), it follows that

$$\ln\nu(\lambda_n + K) \le (\lambda_n + K)(1 - \sigma_{\lambda_n+K}) - \ln^+\mu(\sigma_{\lambda_n+K}, F) \le K - K\sigma_{\lambda_n+K} + \lambda_n - \ln A_n^*.$$

On the other hand, from (7), there exists $\{\sigma_n\} \downarrow 0^+$ satisfying

$$\ln \nu(\lambda_n) \geq \lambda_n(1-\sigma_n) - V(\sigma_n) = \lambda_n(1-\sigma_n) - \log \mu(\sigma_n, F).$$

We choose $\{k_n\}$, so that $\mu(\sigma_n, F) = A^*_{k_n} e^{-\lambda_{k_n}\sigma_n}$. Then

$$\ln \nu(\lambda_{k_n}) \geq \lambda_{k_n}(1-\sigma_n) - \ln A^*_{k_n} + \lambda_{k_n}\sigma_n = \lambda_{k_n} - \ln A^*_{k_n}.$$

Consequently, from (10) and the above results, it follows that

$$\begin{aligned} \overline{\lim_{n\to\infty}} \frac{\ln B^*_n}{\lambda_n} &\geq \overline{\lim_{n\to\infty}} \frac{\ln A^*_{k_n} + \ln \nu(\lambda_{k_n}) - \ln 2\lambda_{k_n}}{\lambda_{k_n}} \\ &\geq \overline{\lim_{n\to\infty}} \frac{\lambda_{k_n} - \ln 2\lambda_{k_n}}{\lambda_{k_n}} = 1, \\ \overline{\lim_{n\to\infty}} \frac{\ln B^*_n}{\lambda_n} &\leq \overline{\lim_{n\to\infty}} \frac{\ln A^*_n + \ln \nu(\lambda_n + K) + \ln 2}{\lambda_n} \\ &\leq \overline{\lim_{n\to\infty}} \frac{K - K\sigma_{\lambda_n+K} + \lambda_n + \ln 2}{\lambda_n} = 1. \end{aligned}$$

So, from the first formula of (4) and Theorem 1.3 of [5], we obtain $\sigma^f_u = 1$. When $\mathrm{Re}s > 1$, we can change the integral order of $f(s)$ since it is uniformly convergent in $\mathrm{Re}s > 1$,

$$\int_0^{+\infty} \int_0^1 \frac{e^{-x(\sigma+s-1)}}{W(\sigma)} \mathrm{d}\sigma \mathrm{d}\alpha(x) = \int_0^1 \int_0^{+\infty} \frac{e^{-x(\sigma+s-1)}}{W(\sigma)} \mathrm{d}\alpha(x) \mathrm{d}\sigma$$

and then we have the result (9).

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