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# A New Class of Minimally Spectrally Arbitrary Sign Patterns

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**Abstract** If every monic real polynomial of degree $n$ can be achieved as the characteristic polynomial of some matrix $B \in Q(A)$, then sign pattern $A$ of order $n$ is a spectrally arbitrary pattern. A sign pattern $A$ is minimally spectrally arbitrary if it is spectrally arbitrary but is not spectrally arbitrary if any nonzero entry (or entries) of $A$ is replaced by zero. In this article, we give some new sign patterns which are minimally spectrally arbitrary for order $n \geq 9$.

**Keywords** sign pattern; potentially nilpotent; spectrally arbitrary pattern.

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## 1. Introduction

A sign pattern $A$ is a matrix whose entries come from $\{+, -, 0\}$. The sign pattern class of $A$, denoted $Q(A)$, is

$$Q(A) = \{B = [b_{ij}] \in M_n(R) | \text{sign} b_{ij} = a_{ij} \quad \text{for all} \quad i, j\}.$$

A sign pattern $\breve{A} = [\breve{a}_{ij}]$ is a superpattern of a sign pattern $A = [a_{ij}]$ if $\breve{a}_{ij} = a_{ij}$ whenever $a_{ij} \neq 0$. If $\breve{A}$ is a superpattern of $A$, then $A$ is a subpattern of $\breve{A}$. A subpattern of $\breve{A}$ which is not $\breve{A}$ itself is a proper subpattern of $\breve{A}$.

A sign pattern $A$ is sign nonsingular if every matrix $B \in Q(A)$ is nonsingular, and $A$ is sign singular if every matrix $B \in Q(A)$ is singular. A sign pattern $A$ is a spectrally arbitrary pattern (SAP) if for any given real monic polynomial $g(x)$ of degree $n$, there is a real matrix $B \in Q(A)$ with characteristic polynomial $g(x)$. If sign pattern $A$ is a SAP and no proper subpattern of $A$ is a SAP, then $A$ is a minimally spectrally arbitrary pattern (MSAP). If there is a real matrix $B \in Q(A)$ with characteristic polynomial $g(x) = x^n$, then $A$ is potentially nilpotent (PN). Note that each SAP must be PN.

The question of the existence of a SAP arose in [1]. The first SAP of order $n$ for each $n \geq 2$ was provided in [5]. Later, some papers[2−4] introduce some sign patterns which are SAPs for order $n \geq 2$. In this paper, we introduce some new sign patterns which are MSAPs for order

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$n \geq 9$. Much of this work is motivated by the inertia and spectral problems considered in [1].

## 2. Some preliminaries

**Lemma 2.1**[1] *Let $A$ be a sign pattern of order $n$, and suppose that there exists some nilpotent realization $B \in Q(A)$ with at least $n$ nonzero entries, say $b_{i_1j_1}, \ldots, b_{i_nj_n}$. Let $X$ be the matrix obtained by replacing these entries in $B$ by variables $x_1, \ldots, x_n$. If the Jacobian of the coefficients of the characteristic polynomial of $X$ with respect to the variables $x_1, \ldots, x_n$ is nonzero at $(x_1, \ldots, x_n) = (b_{i_1j_1}, \ldots, b_{i_nj_n})$, then every superpattern of $A$ is spectrally arbitrary.*

Let $n \geq 9$, and $A$ be a sign pattern of order $n$ as follows.

$$A = \begin{pmatrix} - & + & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \eta \\ \beta_1 & 0 & + & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \beta_2 & 0 & 0 & + & \ddots & & & & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & & & & \vdots \\ \beta_{n-7} & 0 & \vdots & & \ddots & + & \ddots & & & & \vdots \\ \beta_{n-6} & \beta_{n-4} & \vdots & & & \ddots & + & \ddots & & & \vdots \\ 0 & \beta_{n-5} & 0 & \cdots & \cdots & \cdots & 0 & + & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \alpha & + & \ddots & \vdots \\ 0 & \gamma_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & + & 0 \\ 0 & \gamma_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & + \\ \gamma_3 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}, \qquad (2.1)$$

where the entries $\alpha,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \eta \in \{+,-\}$ and $\beta_i \in \{+,-\}$ for $i = 1, 2, \ldots, n-4$.

We shall demonstrate six patterns of form (2.1) are MSAPs, and the other sign patterns of form (2.1) are not SAPs. For convenience, suppose that $B = [b_{ij}] \in Q(A)$ has been scaled such that $b_{11} = -1$, $b_{i,i+1} = 1$ for $i = 1, 2, \ldots, n-1$ (otherwise they can be adjusted to be 1 by suitable similarities), and has the following form.

$$B = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b \\ d_1 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ d_2 & 0 & 0 & 1 & \ddots & & & & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & & & & \vdots \\ d_{n-7} & 0 & \vdots & & \ddots & 1 & \ddots & & & & \vdots \\ d_{n-6} & d_{n-4} & \vdots & & & \ddots & 1 & \ddots & & & \vdots \\ 0 & d_{n-5} & 0 & \cdots & \cdots & \cdots & 0 & 1 & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & a & 1 & \ddots & \vdots \\ 0 & c_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 \\ 0 & c_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ c_3 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}. \qquad (2.2)$$

**Lemma 2.2** *Let* $f_B(\lambda)=\det(\lambda I-B)=\lambda^n+f_1\lambda^{n-1}+f_2\lambda^{n-2}+\cdots+f_{n-1}\lambda+f_n$. *Then*

*(1)* $f_1=1-a$,

$f_2=-a-bc_3-d_1$,

$f_3=abc_3+ad_1-d_2$ *(If* $n=9$, *then* $f_3=abc_3+ad_1-d_2-d_5$*)*,

$f_i=ad_{i-2}-d_{i-1}$, *for* $i=4,5,\ldots,n-7$ $(n\geq 11)$,

$f_{n-6}=ad_{n-8}-d_{n-7}-d_{n-4}$ $(n\geq 10)$,

$f_{n-5}=-d_{n-4}+ad_{n-4}+ad_{n-7}-d_{n-6}-d_{n-5}$,

$f_{n-4}=-d_{n-5}+ad_{n-6}+ad_{n-5}+ad_{n-4}+bc_3d_{n-4}$,

$f_{n-3}=bc_3d_{n-5}-abc_3d_{n-4}+ad_{n-5}-c_1$,

$f_{n-2}=-c_1-abc_3d_{n-5}-c_2$,

$f_{n-1}=-c_2+bc_1c_3$,

$f_n=bc_2c_3-c_3$.

*(2) For arbitrary given* $d_1$,

$$\frac{\partial(f_1,f_2,\ldots,f_{n-1},f_n)}{\partial(a,b,c_1,c_2,c_3,d_2,d_3,\ldots,d_{n-4})}=b^2c_3^3(a+bc_3+1).$$

**Proof** (1)

$$f_B(\lambda)=\begin{vmatrix}
\lambda+1 & -1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -b\\
-d_1 & \lambda & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
-d_2 & 0 & \lambda & -1 & \ddots & & & & & & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & & & & \vdots\\
-d_{n-7} & 0 & \cdots & 0 & \ddots & -1 & \ddots & & & & \vdots\\
-d_{n-6} & -d_{n-4} & 0 & \cdots & 0 & \lambda & -1 & \ddots & & & \vdots\\
0 & -d_{n-5} & 0 & \cdots & \cdots & 0 & \lambda & -1 & \ddots & & \vdots\\
0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \lambda-a & -1 & \ddots & \vdots\\
0 & -c_1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \lambda & -1 & 0\\
0 & -c_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \lambda & -1\\
-c_3 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \lambda
\end{vmatrix}_n.$$

By adding $\lambda$ times of the ith row to the $(i+1)st$ row, for $i=1,2,\ldots,n-5$, and expanding along the third column in order, we have

$$f_B(\lambda)=\begin{vmatrix}
\lambda+1 & -1 & 0 & 0 & 0 & -b\\
g(\lambda) & -d_{n-5}-d_{n-4}\lambda & -1 & 0 & 0 & -b\lambda^{n-5}\\
0 & 0 & \lambda-a & -1 & 0 & 0\\
0 & -c_1 & 0 & \lambda & -1 & 0\\
0 & -c_2 & 0 & 0 & \lambda & -1\\
-c_3 & 0 & 0 & 0 & 0 & \lambda
\end{vmatrix},$$

where $g(\lambda)=\lambda^{n-5}(\lambda+1)-\sum_{i=1}^{n-6}d_i\lambda^{n-5-i}$. Thus

$$f_B(\lambda)=(-c_3)[1+b\lambda^{n-3}(\lambda-a)]+[\lambda(\lambda+1)-bc_3][\lambda^2(-d_{n-5}-d_{n-4}\lambda)(\lambda-a)-c_1\lambda-c_2]+\lambda^3g(\lambda)(\lambda-a)$$

$$
\begin{aligned}
=&\lambda^n-(a-1)\lambda^{n-1}-(a+bc_3+d_1)\lambda^{n-2}+(abc_3+ad_1-d_2)\lambda^{n-3}+a\sum_{i=2}^{n-9}d_i\lambda^{n-2-i}-\\
&\sum_{i=3}^{n-8}d_i\lambda^{n-1-i}+(ad_{n-8}-d_{n-4}-d_{n-7})\lambda^6+(ad_{n-7}-d_{n-6}-d_{n-5}+ad_{n-4}-d_{n-4})\lambda^5+\\
&(ad_{n-6}+ad_{n-5}-d_{n-5}+ad_{n-4}+bc_3d_{n-4})\lambda^4+(bc_3d_{n-5}-abc_3d_{n-4}+ad_{n-5}-c_1)\lambda^3-\\
&(c_1+abc_3d_{n-5}+c_2)\lambda^2+(bc_1c_3-c_2)\lambda+bc_2c_3-c_3.
\end{aligned}
$$

So result (1) is right.

(2) For arbitrary given $d_1$, we have

$$
\frac{\partial(f_1,f_2,\ldots,f_{n-1},f_n)}{\partial(a,b,c_1,c_2,c_3,d_2,d_3,\ldots,d_{n-4})}
$$

$$
=\left|\begin{array}{ccccc}
-1 & 0 & 0 & 0 & 0\\
-1 & -c_3 & 0 & 0 & -b\\
bc_3+d_1 & ac_3 & 0 & 0 & ab\\
d_2 & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
d_{n-9} & 0 & 0 & 0 & 0\\
d_{n-8} & 0 & 0 & 0 & 0\\
d_{n-7}+d_{n-4} & 0 & 0 & 0 & 0\\
d_{n-6}+d_{n-5}+d_{n-4} & c_3d_{n-4} & 0 & 0 & bd_{n-4}\\
-bc_3d_{n-4}+d_{n-5} & c_3d_{n-5}-ac_3d_{n-4} & -1 & 0 & bd_{n-5}-abd_{n-4}\\
-bc_3d_{n-5} & -ac_3d_{n-5} & -1 & -1 & -abd_{n-5}\\
0 & c_1c_3 & bc_3 & -1 & bc_1\\
0 & c_2c_3 & 0 & bc_3 & bc_2-1
\end{array}\right.
$$

$$
\left.\begin{array}{cccccccc}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
0 & & & & & & & \vdots\\
-1 & \ddots & & & & & & \vdots\\
a & -1 & \ddots & & & & & \vdots\\
0 & \ddots & \ddots & \ddots & & & & \vdots\\
\vdots & \ddots & a & -1 & 0 & 0 & 0 & 0\\
\vdots & & \ddots & a & -1 & 0 & 0 & -1\\
\vdots & & & \ddots & a & -1 & -1 & a-1\\
0 & \cdots & \cdots & \cdots & 0 & a & a-1 & bc_3+a\\
0 & \cdots & \cdots & \cdots & \cdots & 0 & bc_3+a & -abc_3\\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -abc_3 & 0\\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0\\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0
\end{array}\right|.
$$

By expanding along the first row, adding $a$ times of the ith row to the $(i+1)st$ row, for

$i=1,2,\ldots,n-3$, and expanding along the fifth column, we have

$$\frac{\partial(f_1,f_2,\ldots,f_{n-1},f_n)}{\partial(a,b,c_1,c_2,c_3,d_2,d_3,\ldots,d_{n-4})}$$

$$=-\begin{vmatrix} -c_3 & 0 & 0 & -b & 0 & 0 \\ c_3d_{n-4} & 0 & 0 & bd_{n-4} & -1 & bc_3 \\ c_3d_{n-5} & -1 & 0 & bd_{n-5} & bc_3 & 0 \\ 0 & -1-a & -1 & 0 & 0 & 0 \\ c_1c_3 & bc_3 & -1 & bc_1 & 0 & 0 \\ c_2c_3 & 0 & bc_3 & bc_2-1 & 0 & 0 \end{vmatrix} = b^2c_3^3(a+bc_3+1).$$

Thus result (2) follows.

**Lemma 2.3**[3] *For $n\geq 2$, an irreducible spectrally arbitrary sign pattern of order $n$ has at least $2n-1$ nonzero entries.*

**Lemma 2.4** *Suppose $A$ is a sign pattern which has the form* (2.1). *If $A$ is a SAP, then $A$ is a MSAP.*

**Proof** Let $T=[t_{ij}]$ be a subpattern of $A$ and $T$ be a SAP.

(1) $t_{n-3,n-3}\neq 0$. Otherwise, the trace of $T$ is negative.

(2) $t_{n,1}\neq 0$ and $t_{i,i+1}\neq 0$, for $i=2,3,\ldots,n-2$. Otherwise, $T$ is sign singular.

(3) $t_{1,2}\neq 0$, $t_{1,n}\neq 0$, $t_{n-1,2}\neq 0$, and $t_{n-1,n}\neq 0$. Otherwise, $T$ is sign nonsingular or sign singular.

(4) $T$ is a SAP, so there is a real matrix $B\in Q(T)$ which is nilpotent. Without loss of generality, suppose that $B$ has the form (2.2). From $f_1=f_2=\cdots=f_n=0$ as in Lemma 2.2, we can conclude that $a=1$, $bc_3=-1-d_1$, $d_i=-1$, for $i=2,3,\ldots,n-8$, $f_{n-6}=-1-d_{n-7}-d_{n-4}=0$, $f_{n-5}=d_{n-7}-d_{n-6}-d_{n-5}=0$, $f_{n-4}=d_{n-6}-d_1d_{n-4}=0$, $f_{n-3}=-d_1d_{n-5}+d_{n-4}+d_1d_{n-4}-c_1=0$, $f_{n-2}=-c_1-c_2+d_{n-5}+d_1d_{n-5}=0$, $f_{n-1}=-c_2-c_1-c_1d_1=0$, and $f_n=-c_3+bc_2c_3=0$.

(4a) Clearly $d_i\neq 0$, for $i=2,3,\ldots,n-8$.

(4b) $d_1\neq 0$. Otherwise, $f_{n-1}=-c_2-c_1=0$ and $f_{n-4}=d_{n-6}=0$, so $f_{n-2}=d_{n-5}=0$. Then the number of nonzero entries of $T$ is less than $2n-1$, and we know $T$ is not a SAP by Lemma 2.3.

(4c) $c_1\neq 0$. Otherwise, $f_{n-1}=-c_2=0$, which is contrary to $t_{n-1,2}\neq 0$ in Case (3).

(4d) $d_{n-5}\neq 0$. Otherwise, $f_{n-2}=-c_1-c_2=0$, so $f_{n-1}=-c_1d_1=0$, which is contrary to Case (4b) and (4c).

(4e) $d_{n-7}\neq 0$. Otherwise, from $f_{n-6}=-1-d_{n-4}=0$, $f_{n-5}=-d_{n-5}-d_{n-6}=0$, and $f_{n-4}=d_{n-6}-d_1d_{n-4}=0$, we have $d_{n-5}=d_1$. Then $f_{n-2}-f_{n-1}=d_1^2+d_1+c_1d_1=0$ and $f_{n-3}=-d_1^2-d_1-c_1-1=0$, thus $d_1=0$, which is contrary to Case (4b).

(4f) $d_{n-6}\neq 0$. Otherwise, $f_{n-4}=-d_1d_{n-4}=0$. From $d_1\neq 0$ in Case (4b), we have $d_{n-4}=0$. But the resultant sign pattern cannot be spectrally arbitrary.

(4g) $d_{n-4} \neq 0$. Otherwise, $f_{n-4} = d_{n-6} = 0$, which is contrary to Case (4f).
Thus, the result is right. □

## 3. Main results

Let $A_1, A_2, A_3, A_4, A_5, A_6$ be sign patterns of form (2.1) as follows.

(1) $\alpha = +$, $\beta_1 = +$, $\beta_i = -$, for $i = 2, 3, \ldots, n-8$, $\beta_{n-7} = \beta_{n-5} = +$, $\beta_{n-6} = \beta_{n-4} = -$, $\gamma_1 = \gamma_3 = -$, $\gamma_2 = \eta = +$, denoted by $A_1$.

(2) $\alpha = +$, $\beta_1 = +$, $\beta_i = -$, for $i = 2, 3, \ldots, n-7$, $\beta_{n-6} = \beta_{n-4} = +$, $\beta_{n-5} = -$, $\gamma_2 = \eta = -$, $\gamma_1 = \gamma_3 = +$, denoted by $A_2$.

(3) $\alpha = +$, $\beta_1 = +$, $\beta_i = -$, for $i = 2, 3, \ldots, n-4$, $\gamma_2 = \eta = -$, $\gamma_1 = \gamma_3 = +$, denoted by $A_3$.

(4) $\alpha = +$, $\beta_i = -$, for $i = 1, 2, \ldots, n-7$, $\beta_{n-6} = +$, $\beta_{n-5} = \beta_{n-4} = -$, $\gamma_1 = \gamma_3 = -$, $\gamma_2 = \eta = +$, denoted by $A_4$.

(5) $\alpha = +$, $\beta_i = -$, for $i = 1, 2, \ldots, n-5$, $\beta_{n-4} = +$, $\gamma_1 = \gamma_3 = -$, $\gamma_2 = \eta = +$, denoted by $A_5$.

(6) $\alpha = +$, $\beta_i = -$, for $i = 1, 2, \ldots, n-6$, $\beta_{n-5} = \beta_{n-4} = +$, $\gamma_1 = \gamma_2 = \gamma_3 = \eta = -$, denoted by $A_6$.

We shall prove that sign patterns $A_1, A_2, \ldots, A_6$ are MSAPs, and the other sign patterns of form (2.1) are not SAPs.

**Theorem 3.1** *Let $A$ have form (2.1). Then $A$ is a SAP if and only if $A$ is one of the sign patterns $A_1, A_2, \ldots, A_6$.*

**Proof** Sufficiency. Let $B$ be a matrix of form (2.2) and denote $J = b^2c_3^3(a + bc_3 + 1)$. If

$$(a, b, c_1, c_2, c_3, d_1, d_2, d_3, \ldots, d_{n-4}) = (1, \frac{1}{3}, -2, 3, -\frac{9}{2}, \frac{1}{2}, d_2, \ldots, d_{n-8}, \frac{1}{9}, -\frac{5}{9}, \frac{2}{3}, -\frac{10}{9})$$

with $d_i = -1$, for $i = 2, \ldots, n-8$, then $B \in Q(A_1)$ is nilpotent, and $J = -\frac{81}{16} \neq 0$. If

$$(a, b, c_1, c_2, c_3, d_1, d_2, d_3, \ldots, d_{n-4}) = (1, -\frac{1}{5}, 2, -5, \frac{25}{2}, \frac{3}{2}, d_2, \ldots, d_{n-8}, -\frac{27}{25}, \frac{3}{25}, -\frac{6}{5}, \frac{2}{25})$$

with $d_i = -1$, for $i = 2, \ldots, n-8$, then $B \in Q(A_2)$ is nilpotent, and $J = -(\frac{25}{4})^2 \neq 0$. If

$$(a, b, c_1, c_2, c_3, d_1, d_2, d_3, \ldots, d_{n-4}) = (1, -\frac{1}{3}, 1, -3, 9, 2, d_2, \ldots, d_{n-8}, -\frac{8}{9}, -\frac{2}{9}, -\frac{2}{3}, -\frac{1}{9})$$

with $d_i = -1$, for $i = 2, \ldots, n-8$, then $B \in Q(A_3)$ is nilpotent, and $J = -81 \neq 0$. If

$$(a, b, c_1, c_2, c_3, d_1, d_2, d_3, \ldots, d_{n-4}) = (1, 3, -\frac{2}{3}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{2}, d_2, \ldots, d_{n-8}, -\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, -\frac{2}{3})$$

with $d_i = -1$, for $i = 2, \ldots, n-8$, then $B \in Q(A_4)$ is nilpotent, and $J = -\frac{1}{16} \neq 0$. If

$$(a, b, c_1, c_2, c_3, d_1, d_2, d_3, \ldots, d_{n-4}) = (1, 7, -\frac{4}{7}, \frac{1}{7}, -\frac{1}{28}, -\frac{3}{4}, d_2, \ldots, d_{n-8}, -\frac{27}{7}, -\frac{15}{7}, -\frac{12}{7}, \frac{20}{7})$$

with $d_i = -1$, for $i = 2, \ldots, n-8$, then $B \in Q(A_5)$ is nilpotent, and $J = -\frac{1}{16^2} \neq 0$. If

$$(a, b, c_1, c_2, c_3, d_1, d_2, d_3, \ldots, d_{n-4}) = (1, -3, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -2, d_2, \ldots, d_{n-8}, -\frac{8}{3}, -\frac{10}{3}, \frac{2}{3}, \frac{5}{3})$$

with $d_i=-1$, for $i=2,\dots,n-8$, then $B\in Q(A_6)$ is nilpotent, and $J=-1\neq 0$. By Lemma 2.1, we know that $A_1,A_2,\dots,A_6$ are SAPs.

Necessity. Suppose sign pattern $A$ of form (2.1) is spectrally arbitrary. Then there is a real matrix $B\in Q(A)$ which is nilpotent. Without loss of generality, suppose that $B$ has the form (2.2). From $f_1=f_2=\cdots=f_n=0$ as in Lemma 2.2, and by the fact that there are $n$ equations and $n+1$ unknowns, we can express the other $n$ unknowns by $d_1$. So we can conclude that $a=1$, $d_i=-1$, for $i=2,3,\dots,n-8$, $d_{n-4}=\frac{d_1^2-d_1-1}{(d_1+1)^2(1-d_1)}$, $c_2=\frac{d_1+1}{1-d_1}$, $c_1=\frac{1}{d_1-1}$, $d_{n-5}=\frac{d_1}{(d_1+1)(1-d_1)}$, $d_{n-6}=\frac{d_1(d_1^2-d_1-1)}{(d_1+1)^2(1-d_1)}$, $d_{n-7}=\frac{d_1^3}{(d_1+1)^2(1-d_1)}$, $c_3=\frac{(d_1+1)^2}{d_1-1}$, and $b=\frac{1-d_1}{d_1+1}$. From the value of $d_1$, we can conclude the signs of the other $n$ unknowns. Thus $A$ must be one of the sign patterns $A_i$ $(i=1,2,\dots,6)$.

**Theorem 3.2** $A_i$ $(1\le i\le 6)$ *are MSAPs, and every superpattern of them is a SAP.*

**Proof** By Theorem 3.1, Lemmas 2.1 and 2.4, the result is clear. □

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