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A Note on Erceg's Pseudo-Metric and Pointwise Pseudo-Metric

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Abstract In [6], it is proved that Erceg's pseudo-metric and pointwise pseudo-metric presented by Shi are equivalent. In this paper, it is proved that Theorem 1 in [6] is wrong. Further it is proved that Erceg's pseudo-metric and pointwise pseudo-metric are not equivalent in the sense of topology.

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Since Q-neighborhoods[9] of "point" and soon more general R-neighborhoods[14] were introduced, the topological theories on lattices have made great developments. Specially, the convergence theories of Moore-Smith[9,14] and the embedded theorems[7] make a mark that the pointwise topological theories have achieved a comparative success.

Metric space is a type of the most important topological space. Thus before many concepts in topology theory are put forward, they are thought over in metric space. Certainly, in more general topological theory on lattices, metric theory similarly possesses very important position. Therefore, the research of metric problems in a sense will determine its developmental orientation in the field.

B. Hutton[3] and M. A. Erceg[2] are the precursors in the research of metric uniformities and metric theories. Their works are regarded as excellent achievements of "no point group" on the fuzzy metrization. Nevertheless, there are some deficiencies in their researches, which cannot directly reflect the characteristics of pointwise L-topology, that is to say, the relationship between a fuzzy point and its Q-neighborhoods or a fuzzy point and R-neighborhoods cannot be reflected. With an aim to character it, Liang[5] and Peng[8] respectively gave the pointwise description of Erceg's pseudo-metric, and obtained many important conclusions, but they still do not solve the problem that r-spheres of a fuzzy point are its Q-neighborhoods. In [8], Peng gave the following simplification of Erceg's pseudo-metric.

Definition 1[8] *An Erceg pseudo-metric on L is a map $p : M(L) \times M(L) \to [0, +\infty]$ satisfying*

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(B1) $\forall a, b \in M(L)$, *if* $a \geq b$, *then* $p(a,b) = 0$;

(B2) $\forall a, b, c \in M(L)$, $p(a,c) \leq p(a,b) + p(b,c)$;

(B3) $\forall a, b \in M(L)$, $p(a,b) = \bigvee_{y \ll b} \bigwedge_{x \ll a} p(x,y)$;

(B4) $\forall a, b \in M(L)$, $\exists y \not\leq b'$ *such that* $p(a,y) < r \Leftrightarrow \exists x \not\leq a'$ *such that* $p(b,x) < r$.

In 1996, Shi presented another theory of quasi-uniformity and pseudo-quasi-metric in fuzzy setting[10], which are claimed to be of pointwise quasi-uniformity and pointwise pseudo-quasi-metric respectively in order to distinguish them from the works of B. Hutton and M. A. Erceg[2,3]. Then Shi again put forward pointwise quasi-uniformity and pointwise pseudo-quasi-metric on fuzzy lattices in a simple way[11]. The relationship between a fuzzy point and its Q-neighborhoods can directly be reflected in the theory of Shi, and its L-topology is Q-C_I. Besides, in [11], Shi still proved that the topology satisfying C_{II} on fuzzy lattices can be metrizable if and only if it is regular. Recently, in [13], by means of $\triangle$-maps and $*$-maps, which are elements of $(L^{M(L)})$, Shi presented some brief characterizations of a pointwise uniformities on lattices, and a few pointwise metrization theorems were proved. A pointwise pseudo-metric is defined as follows.

Definition 2[11] *A pointwise pseudo-metric (or A Shi pseudo-metric) on* L *is a map* $d : M(L) \times M(L) \to [0, +\infty)$ *satisfying*

(M1) $\forall a \in M(L)$, $d(a,a) = 0$;

(M2) $\forall a, b, c \in M(L), d(a,c) \leq d(a,b) + d(b,c)$;

(M3) $\forall a, b \in M(L), d(a,b) = \bigwedge_{c \ll b} d(a,c)$;

(M4) $\forall a, b \in M(L)$, $\exists x \not\leq a'$ *such that* $d(x,b) < r \Leftrightarrow \exists y \not\leq b'$ *such that* $d(y,a) < r$.

Remark 3 (1) In Definition 2, (M1) can be replaced by the following

(M1)* $\forall a, b \in M(L), d(a,b) = 0$ if $a \leq b$.

Because by (M3), $d(a,b) = \bigwedge_{c \ll b} d(a,c) \leq \bigwedge_{c \ll a} d(a,c) = d(a,a) = 0$.

(2) The original definition of Shi's pseudo-metric[10] includes the following condition:

(M5)* $\forall a, b \in M(L), a \leq b \Rightarrow d(a,c) \leq d(b,c)$.

But it is unnecessary. Because $\forall a, b \in M(L)$ with $a \leq b$, it follows that $d(a,c) \leq d(a,b) + d(b,c) = 0 + d(b,c) = d(b,c)$.

In 1984, Liang ever gave the following example to explain that Erceg's pseudo-metric is not Q$-C_I$ in [4], but she did not give a detailed proof.

Example 4 Let X and Y be two nonempty sets, $|Y| > \aleph_0$ ($\aleph_0$ is numerable set), and $L = P(Y)$, i.e., the set of all subsets of Y. Then $(L, \cup, \cap, ')$ is a completely distributive lattice with an order reversing involution "$'$", where "$'$" is complement operation of sets. $\forall r > 0$, define $D_r : L \to L$ as $D_r(A) = A$ for each $A \in L$. It is obvious that $\{D_r | r > 0\}$ satisfies $(D1) - (D5)$, which is equivalent to Erceg's pseudo-metric. It is easy to verify that its induced topology η_D is not Q-C_I.

In order to explain that Erceg's pseudo-metric is not equivalent to pointwise pseudo-metric in the sense of topology, Shi also cited Example 4 (see Example 6.6 in [12]), which is pointed out to be wrong by D. X. Zhang in *the Zentralblatt Math.*, MSC 2000: * 54A40 Fuzzy topology 54E35 Metric spaces, metrizability 03E72 Fuzzy sets (logic), i.e., in Example 4 its topology η_D

induced by Erceg's pseudo-metric is Q-C_I. Hence, whether Erceg's pseudo-metric is equivalent to pointwise pseudo-metric still does not be solved.

In addition, in [6], Liang proved that a pseudo-metric can be obtained from an Erceg's pseudo-metric. The conclusion is as follows:

Theorem 5 *Suppose that p is an Erceg pseudo-metric on L. Let $p^*(a,b) = p(b,a)$. Then p^* is a pointwise pseudo-metric on L.*

In fact, the conclusion is wrong, which can be shown by the following example.

Example 6 $L = [0,1]$ is the real unit interval. We define a map $p : (0,1] \times (0,1] \to [0,+\infty)$ as follows:

$$p(a,b) = \begin{cases} 0, & \text{if } a \geq b; \\ 1, & \text{if } a < b. \end{cases}$$

Firstly, it can be shown that p is an Erceg pseudo-metric on $[0,1]$, i.e., p satisfies (B1), (B2), (B3) and (B4).

(B1) and (B2) are trivial.

(B3) If $a, b \in (0,1]$ and $a \geq b$, then $p(a,b) = 0$. Because $\bigvee_{y<b} \bigwedge_{x<a} p(x,y) \leq \bigvee_{y<b} p(y,y) = 0$, we can obtain $p(a,b) = \bigvee_{y<b} \bigwedge_{x<a} p(x,y)$. Similarly, when $a < b$, we have $p(a,b) = 1$. Because $\bigvee_{y<b} \bigwedge_{x<a} p(x,y) \geq \bigwedge_{x<a} p(x,a) = 1$, we still can obtain $p(a,b) = \bigvee_{y<b} \bigwedge_{x<a} p(x,y)$. Thus p satisfies $(B3)$.

(B4) We only need prove $\bigwedge_{y>1-b} p(a,y) = \bigwedge_{x>1-a} p(b,x)$. It can be obtained from these implications: $\bigwedge_{y>1-b} p(a,y) = 1 \Leftrightarrow y > 1-b$ implies $y > a \Leftrightarrow 1-b \geq a \Leftrightarrow x > 1-a$ implies $x > b \Leftrightarrow \bigwedge_{x>1-a} p(b,x) = 1$.

Secondly, let $p^*(b,a) = p(a,b)$. We prove that p^* is not a Shi pseudo-metric. In fact, $\forall a \in (0,1]$, $p^*(a,a) = 0$, but $\bigwedge_{c<a} p^*(a,c) = 1$. Thus $p^*(a,b) \neq \bigwedge_{c<b} p^*(a,c)$.

Therefore, the proposition holds.

Example 6 indicates that by the way in Theorem 5, we cannot obtain a Shi pseudo-metric from an Erceg pseudo-metric, but the converse is true, as formulated in the following theorem.

Theorem 7 *Let d be a Shi pseudo-metric on L, and $p(b,a) = d(a,b)$. Then p is an Erceg pseudo-metric.*

In order to prove p is an Erceg pseudo-metric, we only need prove the following lemma.

Lemma 8 *If d is a Shi pseudo-metric on L, then (M3)* $\forall a, b \in M(L), d(a,b) = \bigvee_{c \ll a} d(c,b)$.*

Proof From (M1) and (M2) we have $d(a,b) \geq \bigvee_{c \ll a} d(c,b)$. Next, we prove $d(a,b) \leq \bigvee_{c \ll a} d(c,b)$. Suppose that

$$d(a,b) \neq \bigvee_{c \ll a} d(c,b).$$

Then there exist $s, r > 0$ such that

$$d(a,b) > r > s > \bigvee_{c \ll a} d(c,b).$$

But $\forall c \ll a$, we have

$$r < d(a,b) \leq d(a,c) + d(c,b) \leq d(a,c) + s.$$

This shows $\forall c \ll a, d(a,c) \geq r - s$. Thus we obtain

$$0 = d(a,a) = \bigwedge_{c \ll a} d(a,c) \geq r - s > 0.$$

It is a contradiction. So (M3)* holds.

Then, are Erceg's pseudo-metric and Shi's pseudo-metric equivalent? It is known that whether two metrics are equivalent depends on whether their induced topologies are the same in general topology (see the definition of metric equivalence in chapter 4, [1]), i.e., two pseudo-metrics are equivalent if their induced topologies are the same, or else it is claimed that they are not equivalent. Thus in the sense of L-topology, it can be examined whether Erceg's pseudo-metric and pointwise pseudo-metric are equivalent, based on the fact whether their induced topologies are the same or not. In [12], it is pointed out that its topology induced by a pointwise pseudo-metric is Q-C_I. Then, is its topology induced by an Erceg's pseudo-metric Q-C_I? The following example just shows that in general the topology induced by an Erceg's pseudo-metric is not Q-C_I.

Example 9 Let ω_1 denote the first uncountable ordinal number and $[0, \omega_1]$ denote the set of all ordinal numbers from 0 to ω_1. Put $L = [-\omega_1, 0] \cup [0, \omega_1]$. Then L is a completely distributive lattice with an order-reversing involution. Let $X = \{x\}$ be a single point set. For each real numbers $r > 0$ and each $A \in L$, define $D_r(A) = A$, i.e., $D_r : L \to L$ is an identity map. Then $\{D_r | r > 0\}$ is the L-fuzzy point x_{ω_1} and the family of R-neighborhood maps of x_{ω_1} is $\{C_a | a \in [-\omega_1, \omega_1)\}$, where C_a denotes constant L-fuzzy set taking value a. Obviously, there is no countable subset $\mathcal{A}$ of $\{C_a | a \in [-\omega_1, \omega_1)\}$ such that $\mathcal{A}$ forms a local closed R-neighborhood basis of x_{ω_1}. Therefore this Erceg's metric topology is not C_I.

As mentioned above, Shi's pointwise pseudo-metric certainly is Erceg's pseudo-metric, but the converse is not true. Therefore, pointwise pseudo-metric and Erceg's pseudo-metric are not equivalent in the sense of L-topology.

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