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On the Set of Common Consequent Indices of a Class of Binary Relations

MA Hong Ping[1], *MIAO Zheng Ke*[1,2]
(1. School of Mathematical Sciences, Xuzhou Normal University, Jiangsu 221116, China;
2. Institute of Applied Mathematics, Academy of Mathematics and System Science, Chinese Academy of Science, Beijing 100080, China)
(E-mail: mahp@xznu.edu.cn; zkmiao@xznu.edu.cn)

Abstract Let $V=\{a_1,a_2,\ldots,a_n\}$ be a finite set with $n\geq 2$ and $P_n(V)$ the set of all primitive binary relations on V. For $Q\in P_n(V)$, denote by $G(Q)$ the directed graph corresponding to Q. For positive integer $d\leq n$, let $P_n(V,d)=\{Q: Q\in P_n(V) \text{ and } G(Q) \text{ contains exactly } d \text{ loops}\}$. In this paper, it is proved that the set of common consequent indices of binary relations in $P_n(V,d)$ is $\{1,2,\ldots,n-\lceil\frac{d}{2}\rceil\}$. Furthermore, the minimal extremal binary relations are described.

Keywords common consequent index; primitive relation; directed graph.

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1. Introduction

Let $V=\{a_1,a_2,\ldots,a_n\}$ be a finite set with $n\geq 2$. A binary relation Q on V is a subset of $V\times V$. Denote by $B_n(V)$ the set of all binary relations on V. Then under the usual multiplication of binary relations $B_n(V)$ becomes a semigroup.

Let M_n denote the set of all $n\times n$ Boolean matrices. Then M_n is a semigroup under the Boolean matrix multiplication. The map

$$Q\to M(Q)=(m_{ij}),$$

where $m_{ij}=1$ if $(a_i,a_j)\in Q$ and $m_{ij}=0$ otherwise, is an isomorphism of $B_n(V)$ onto M_n. If $Q_1,Q_2\in B_n(V)$, then

$$Q_1\cdot Q_2\to M(Q_1)\cdot M(Q_2)=M(Q_1\cdot Q_2),$$

$$Q_1\cup Q_2\to M(Q_1)+M(Q_2)=M(Q_1\cup Q_2).$$

Let $G_n(V)$ be the set of all directed graphs on n vertices $a_1,a_2,\ldots,a_n$ with allowable loops and no multiple arcs.

It is well known that there is a naturally one-to-one correspondence between $B_n(V)$, M_n and $G_n(V)$. So, for a given $Q\in B_n(V)$, there exists only one $M\in M_n$ and $G\in G_n(V)$

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corresponding to Q. Denote M and G by $M(Q)$ and $G(Q)$, respectively. Similarly, for a given $M \in M_n$ ($G \in G_n(V)$, resp.), we have $Q(M)$ and $G(M)$ ($Q(G)$ and $M(G)$, resp.).

If $Q \in B_n(V)$, $a_i \in V$ and K is a non-empty subset of V, we define $a_iQ = \{x \in V : (a_i, x) \in Q\}$ and $KQ = \bigcup_{a_i \in K} a_iQ$.

Definition 1[1] *Let $Q \in B_n(V)$. We say that a pair of vertices (a_i, a_j), $a_i \neq a_j$, has a common consequent if there is an integer $s > 0$ such that*

$$a_iQ^s \cap a_jQ^s \neq \emptyset. \tag{1}$$

The least integer s satisfying (1) is denoted by $L_Q(a_i, a_j)$.

To shorten the terminology, if a_i, a_j has a common consequent, we also say that $L_Q(a_i, a_j)$ exists.

Definition 2[1] *If there is at least one couple (a_i, a_j) for which $L_Q(a_i, a_j)$ exists, we define $L(Q) = \max L_Q(a_i, a_j)$, where (a_i, a_j) runs through all couples for which $L_Q(a_i, a_j)$ exists. If there is no one couple (a_i, a_j) for which $L_Q(a_i, a_j)$ exists, we define $L(Q) = 0$. $L(Q)$ is called the common consequent index of Q.*

A relation $Q \in B_n(V)$ is called primitive if there is an integer $t \geq 1$ such that $Q^t = V \times V$. It is well known that Q is primitive if and only if $G(Q)$ is strongly connected and the greatest common divisor of all the cycle lengths of $G(Q)$ is 1. Denote by $P_n(V)$ the set of all primitive relations in $B_n(V)$. Clearly, if $Q \in P_n(V)$, then $L_Q(a_i, a_j)$ exists for any pair (a_i, a_j) where $a_i \neq a_j$.

Let $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer $\geq x$ and the greatest integer $\leq x$, respectively, and $|K|$ denote the cardinality of a set K.

It is known to Paz[2] that $L(Q) \leq \frac{n(n-1)}{2}$ for $Q \in B_n(V)$. In 1985, Schwarz[1] proved that for any $Q \in B_n(V)$ (or $Q \in P_n(V)$), $L(Q) \leq \lfloor \frac{(n-1)^2}{2} \rfloor + 1$, and the upper bound is sharp.

Let $E_n = \{L(Q) : Q \in B_n(V)\}$. E_n is called the set of common consequent indices of binary relations on V. In 2000, Zhou and Liu[3] disclosed the existence of gaps in E_n.

The index set problem and the upper bound problem are the two main problems in the study of common consequent index. There also have been some results about the two problems for special classes of binary relations on V (for example, reducible, nearly reducible or symmetric binary relations[3,4].)

In this paper, we consider a special class of primitive binary relations on V. In 1985, Schwarz[1] proved that $\max\{L(Q) : Q \in P_n(V)$ and $G(Q)$ contains a loop$\} \leq n - 1$. For positive integer $d \leq n$, let $P_n(V, d) = \{Q : Q \in P_n(V)$ and $G(Q)$ contains exactly d loops$\}$ and $E_n(V, d) = \{L(Q) : Q \in P_n(V, d)\}$. It is proved that $E_n(V, d) = \{1, 2, \ldots, n - \lceil \frac{d}{2} \rceil\}$. Furthermore, the minimal extremal binary relations are described.

2. Some preliminary lemmas

It is easy to see that the following Lemma 1 holds.

Lemma 1 *Let $Q \in P_n(V)$ and K be a non-empty proper subset of V. Then KQ contains at least one element of V which is not contained in K.*

Lemma 2 *Let $Q \in P_n(V)$ and $a \in V$. If $a \in aQ$, then $|aQ^t| \geq t+1$ for any positive integer $t \leq n-1$.*

Proof We apply induction on t. For $t = 1$, by Lemma 1, there exists $x_1 \in V$ such that $x_1 \in aQ$ and $x_1 \neq a$. Since $a \in aQ$, we have $|aQ| \geq 2$. Suppose $|aQ^{l-1}| \geq l$ for any positive integer $l \leq n-2$. Put $V_l = aQ^{l-1}$. If we apply Lemma 1 to V_l, then there exists $x_l \in V$ such that $x_l \in V_lQ$ and $x_l \notin V_l$. Also, because $a \in aQ$, $\{x_l\} \cup V_l \subset aQ^l$, it follows that $|aQ^l| \geq l+1$.

Since there is a walk from b to a of length s in $G(Q)$ if and only if $a \in bQ^s$, we have the following.

Corollary 1 *Let $Q \in P_n(V)$ and $a, b \in V$. If $a \in aQ \cap bQ^l$, then $|bQ^{l+t}| \geq t+1$ for any positive integer $t \leq n-1$.*

3. Maximum index and minimal extremal binary relations

Theorem 1 *For any $Q \in P_n(V,d)$, we have*

$$L(Q) \leq n - \lceil \frac{d}{2} \rceil.$$

Proof Let $Q \in P_n(V,d)$. Put $V_1 = \{a \in V : a \in aQ\}$ and $V_2 = V \backslash V_1$. For any $a, b \in V, a \neq b$, we consider the following cases.

Case 1 $a, b \in V_1$. By Lemma 2, we have $|aQ^{n-\lceil \frac{d}{2} \rceil}| \geq n - \lceil \frac{d}{2} \rceil + 1$ and $|bQ^{n-\lceil \frac{d}{2} \rceil}| \geq n - \lceil \frac{d}{2} \rceil + 1$. Since $|aQ^{n-\lceil \frac{d}{2} \rceil} \cup bQ^{n-\lceil \frac{d}{2} \rceil}| \leq n$, we get

$$|aQ^{n-\lceil \frac{d}{2} \rceil} \cap bQ^{n-\lceil \frac{d}{2} \rceil}| \geq 2(n - \lceil \frac{d}{2} \rceil + 1) - n = n - 2\lceil \frac{d}{2} \rceil + 2 > 0.$$

Hence $L_Q(a,b) \leq n - \lceil \frac{d}{2} \rceil$.

Case 2 $a, b \in V_2$. Let $x_1,\ x_2 \in V_1$ such that $d(a, x_1) = d(a, V_1) = k_1,\ d(b, x_2) = d(b, V_1) = k_2$ (where $d(a,x)$ ($d(a,V_1)$, resp.) denotes the length of a shortest path from a to x (some $x \in V_1$, resp.) in $G(Q)$), and P_1 (P_2, resp.) be a shortest path from a (b, resp.) to x_1 (x_2, resp.). Clearly, $k_1 \leq n-d$, $k_2 \leq n-d$. If $x_1 = x_2$, since $n - d \leq n - \lceil \frac{d}{2} \rceil$, then $x_1 \in aQ^{n-\lceil \frac{d}{2} \rceil} \cap bQ^{n-\lceil \frac{d}{2} \rceil}$. If $P_1 \cap P_2 \neq \emptyset$, assume $v \in P_1 \cap P_2$ ($v \neq x_1, x_2$), then $d(v, x_1) = d(v, x_2)$ by the definition of x_1 and x_2, and we can choose $x_1 = x_2$. So in the following we can assume $P_1 \cap P_2 = \emptyset$. Thus $|V(P_1)| + |V(P_2)| \leq n - d + 2$, and $k_1 + k_2 \leq n - d$. On the other hand, by Corollary 1, we have $|aQ^{n-\lceil \frac{d}{2} \rceil}| \geq n - \lceil \frac{d}{2} \rceil - k_1 + 1$ and $|bQ^{n-\lceil \frac{d}{2} \rceil}| \geq n - \lceil \frac{d}{2} \rceil - k_2 + 1$. Since $|aQ^{n-\lceil \frac{d}{2} \rceil} \cup bQ^{n-\lceil \frac{d}{2} \rceil}| \leq n$, we get

$$\begin{aligned} |aQ^{n-\lceil \frac{d}{2} \rceil} \cap bQ^{n-\lceil \frac{d}{2} \rceil}| &\geq (n - \lceil \frac{d}{2} \rceil - k_1 + 1) + (n - \lceil \frac{d}{2} \rceil - k_2 + 1) - n \\ &= n - 2\lceil \frac{d}{2} \rceil - k_1 - k_2 + 2 \geq n - 2\lceil \frac{d}{2} \rceil - (n-d) + 2 \end{aligned}$$

$$= d - 2\lceil\frac{d}{2}\rceil + 2 > 0.$$

Hence $L_Q(a,b) \leq n - \lceil\frac{d}{2}\rceil$.

Case 3 $a \in V_1, b \in V_2$. Since $G(Q)$ is strongly connected, there is a walk of length $n-d$ in $G(Q)$ from b to some vertex $a_i \in V_1$. By Lemma 2 and Corollary 1, we have $|aQ^{n-\lceil\frac{d}{2}\rceil}| \geq n - \lceil\frac{d}{2}\rceil + 1$ and $|bQ^{n-\lceil\frac{d}{2}\rceil}| \geq d - \lceil\frac{d}{2}\rceil + 1$. Since $|aQ^{n-\lceil\frac{d}{2}\rceil} \cup bQ^{n-\lceil\frac{d}{2}\rceil}| \leq n$, we get

$$|aQ^{n-\lceil\frac{d}{2}\rceil} \cap bQ^{n-\lceil\frac{d}{2}\rceil}| \geq (n - \lceil\frac{d}{2}\rceil + 1) + (d - \lceil\frac{d}{2}\rceil + 1) - n = d - 2\lceil\frac{d}{2}\rceil + 2 > 0.$$

Hence $L_Q(a,b) \leq n - \lceil\frac{d}{2}\rceil$.

Combining the above three cases, it is proved that $L(Q) \leq n - \lceil\frac{d}{2}\rceil$.

For positive integer $d \leq n$, let G_n be the directed graph with vertex set V and arc set

$$E = \{(a_i, a_{i+1}) : 1 \leq i \leq n-1\} \cup \{(a_n, a_1)\} \cup \{(a_i, a_i) : 1 \leq i \leq d\}.$$

For positive integer $d \leq n-2$, integer k with $\lceil\frac{d}{2}\rceil \leq k \leq n-\lceil\frac{d}{2}\rceil$ and integer l with $1 \leq l \leq \lceil\frac{d}{2}\rceil$, let $H_n(k,l)$ be the directed graph with vertex set V and arc set

$$E = \{(a_i, a_{i+1}) : 1 \leq i \leq n-1\} \cup \{(a_n, a_1)\} \cup (a_i, a_i) : 1 \leq i \leq \lceil\frac{d}{2}\rceil\}$$
$$\cup \{(a_i, a_i) : i = k+1, \ldots, k+l-1, k+l+1, \ldots, k+\lceil\frac{d}{2}\rceil\}$$

and $T_n(k)$ the directed graph with vertex set V and arc set

$$E = \{(a_i, a_{i+1}) : 1 \leq i \leq n-1\} \cup \{(a_n, a_1)\} \cup \{(a_i, a_i) : 1 \leq i \leq \lceil\frac{d}{2}\rceil\}$$
$$\cup \{(a_i, a_i) : k+1 \leq i \leq k+\lceil\frac{d}{2}\rceil\}.$$

It is obvious that $Q(G_n) \in P_n(V,d)$, $Q(H_n(k,l)) \in P_n(V,d)$ when d is odd and $Q(T_n(k)) \in P_n(V,d)$ when d is even.

Theorem 2 *For $Q \in P_n(V,d)$ with $G(Q) = G_n$ or $H_n(k,l)$ or $T_n(k)$, we have*

$$L(Q) = n - \lceil\frac{d}{2}\rceil.$$

Proof 1) $G(Q) = G_n$.

If $d = n$, then it is easy to check that $L_Q(a_1, a_{\lceil\frac{d+1}{2}\rceil}) = n - \lceil\frac{d}{2}\rceil$. If $d < n$, then we have $d(a_{d+1}, a_{\lceil\frac{d+1}{2}\rceil}) = n - \lfloor\frac{d+1}{2}\rfloor = n - \lceil\frac{d}{2}\rceil$ and $d(a_{\lceil\frac{d+1}{2}\rceil}, a_1) = n - \lceil\frac{d-1}{2}\rceil$. So $L_Q(a_{d+1}, a_{\lceil\frac{d+1}{2}\rceil}) = \min\{d(a_{d+1}, a_{\lceil\frac{d+1}{2}\rceil}), d(a_{\lceil\frac{d+1}{2}\rceil}, a_1)\} = n - \lceil\frac{d}{2}\rceil$. By Theorem 1, we get $L(Q) = n - \lceil\frac{d}{2}\rceil$.

2) $G(Q) = H_n(k,l)$. Now d is odd.

Case 1 $k + \lceil\frac{d}{2}\rceil < n$. Let $x = a_{\lceil\frac{d}{2}\rceil+1}$, $y = a_{k+\lceil\frac{d}{2}\rceil+1}$. Then it follows that $y \notin yQ$. If $x \notin xQ$, let

$$t = \begin{cases} k+1, & \text{if } a_{k+1} \in a_{k+1}Q; \\ k+2, & \text{otherwise.} \end{cases}$$

Then $a_t \in a_tQ$. It is easy to see that $d(y, a_t) = n-(k+\lceil\frac{d}{2}\rceil+1-t) \geq n-\lceil\frac{d}{2}\rceil$ and $d(x, a_1) = n-\lceil\frac{d}{2}\rceil$. Therefore $L_Q(x,y) = \min\{d(y,a_t), d(x,a_1)\} = n - \lceil\frac{d}{2}\rceil$. If $x \in xQ$, then $k = \lceil\frac{d}{2}\rceil$. It is obvious that $d(y,x)=n-k=n-\lceil\frac{d}{2}\rceil$. So $L_Q(x,y)=\min\{d(y,x),d(x,a_1)\}=n-\lceil\frac{d}{2}\rceil$. By Theorem 1, we get $L(Q)=n-\lceil\frac{d}{2}\rceil$.

Case 2 $k+\lceil\frac{d}{2}\rceil=n$. Let $x=a_{\lceil\frac{d}{2}\rceil+1}$. Since $d\le n-2$, we have $x\notin xQ$. If $a_{k+1}\notin a_{k+1}Q$, then $l=1$, and it is easy to see that $H_n(k,l)\cong G_n$. If $a_n\notin a_nQ$, then $a_{k+1}\in a_{k+1}Q$. It is not difficult to check that $L_Q(a_n,x)=\min\{d(a_n,a_{k+1}),d(x,a_1)\}=\min\{n-\lceil\frac{d}{2}\rceil+1,n-\lceil\frac{d}{2}\rceil\}=n-\lceil\frac{d}{2}\rceil$. If $a_{k+1}\in a_{k+1}Q$ and $a_n\in a_nQ$, then we have $L_Q(a_1,x)=\min\{d(x,a_1),d(a_1,a_{k+1})\}=n-\lceil\frac{d}{2}\rceil$. By Theorem 1, we get $L(Q)=n-\lceil\frac{d}{2}\rceil$.

3) $G(Q)=T_n(k)$. Now d is even.

If $k=\lceil\frac{d}{2}\rceil$ or $k=n-\lceil\frac{d}{2}\rceil$, then $T_n(k)\cong G_n$. So we can assume that $\lceil\frac{d}{2}\rceil<k<n-\lceil\frac{d}{2}\rceil$. Let $x=a_{\lceil\frac{d}{2}\rceil+1}$ and $y=a_{k+\lceil\frac{d}{2}\rceil+1}$. Then $x\notin xQ$ and $y\notin yQ$. It is easy to check that $L_Q(x,y)=\min\{d(x,a_1),d(y,a_{k+1})\}=n-\lceil\frac{d}{2}\rceil$. By Theorem 1, we get $L(Q)=n-\lceil\frac{d}{2}\rceil$.

Theorem 2 shows that the upper bound of the index in Theorem 1 is sharp, i.e., $\max\{L(Q):Q\in P_n(V,d)\}=n-\lceil\frac{d}{2}\rceil$.

Clearly, $|Q|\ge n+d$ for any $Q\in P_n(V,d)$. Also by Theorem 2, there exists $Q\in P_n(V,d)$ such that $L(Q)=n-\lceil\frac{d}{2}\rceil$ and $|Q|=n+d$. Such Q is called the minimal extremal binary relation. In the following, we describe the characterization of the minimal extremal binary relations.

Theorem 3 *Let $Q\in P_n(V,d)$. If $L(Q)=n-\lceil\frac{d}{2}\rceil$ and $|Q|=n+d$, then $G(Q)\cong G_n$ or $H_n(k,l)$ when d is odd and $G(Q)\cong G_n$ or $T_n(k)$ when d is even.*

Proof Since $Q\in P_n(V,d)$ and $|Q|=n+d$, $G(Q)=(V,E)$ is strongly connected and $|E|=n+d$. Also it is easy to see that $|E|=n+d$ if and only if G consists of a Hamilton cycle C_n and d loops. Let $C_n=(a_1,a_2,\ldots,a_n,a_1)$.

It is obvious that $G\cong G_n$ when $d=1$ or $d\ge n-1$. So in the following we assume that $2\le d\le n-2$. For convenience, let $a_{n+k}=a_k$.

Put $V_1=\{a_i\in V:(a_i,a_i)\in E\}$ and $V_2=V\backslash V_1$. Let $a_i,a_j\in V$ such that $L_Q(a_i,a_j)=n-\lceil\frac{d}{2}\rceil$. We claim that $a_i\in V_2$ or $a_j\in V_2$. In fact, if $a_i,a_j\in V_1$, it is easy to check that $d(a_i,a_j)\le n-\lceil\frac{n}{2}\rceil$ or $d(a_j,a_i)\le n-\lceil\frac{n}{2}\rceil$. So $L_Q(a_i,a_j)=\min\{d(a_i,a_j),d(a_j,a_i)\}\le n-\lceil\frac{n}{2}\rceil<n-\lceil\frac{d}{2}\rceil$, a contradiction. Without loss of generality, let $a_i\in V_2$.

1) $a_j\in V_1$. Let $a_t\in V_1$ be the first vertex in the path $(a_{i+1},a_{i+2},\ldots,a_j)$. It is obvious that

$$L_Q(a_i,a_j)=\min\{d(a_i,a_j),d(a_j,a_t)\}=n-\lceil\frac{d}{2}\rceil.$$

If $d(a_i,a_j)\ne d(a_j,a_t)$, then we have $d\le d(a_t,a_i)=d(a_t,a_j)+d(a_j,a_i)\le 2\lceil\frac{d}{2}\rceil-1\le d$. So d is odd and $a_t,a_{t+1},\ldots,a_{i-1}\in V_1$. Hence $G\cong G_n$. If $d(a_i,a_j)=d(a_j,a_t)$, then we have

$$d(a_t,a_i)=d(a_t,a_j)+d(a_j,a_i)=2\lceil\frac{d}{2}\rceil=\begin{cases} d, & \text{when } d \text{ is even;}\\ d+1, & \text{when } d \text{ is odd.}\end{cases}$$

Therefore it is not difficult to check that $G\cong G_n$ when d is even and $G\cong H_n(k,l)$ when d is odd.

2) $a_j\in V_2$. Let $a_t\in V_1$ ($a_s\in V_1$, resp.) be the first vertex in the path $(a_{i+1},a_{i+2},\ldots,a_{i-1})$ ($(a_{j+1},a_{j+2},\ldots,a_{j-1})$, resp.). Then we claim that $t\in\{i+1,i+2,\ldots,j-1\}$ and $s\in\{j+$

$1, j+2, \ldots, i-1\}$. Otherwise, by symmetry, we assume that $t \in \{j+1, j+2, \ldots, i-1\}$. Then $s = t$ by the choice of s. So $n - \lceil \frac{d}{2} \rceil = L_Q(a_i, a_j) = d(a_i, a_t)$, and it follows that $d(a_t, a_i) = \lceil \frac{d}{2} \rceil < d$ $(d \geq 2)$, which is a contradiction with $G(Q)$ containing d loops. Hence $L_Q(a_i, a_j) = \min\{d(a_i, a_s), d(a_j, a_t)\}$. If $d(a_i, a_s) \neq d(a_j, a_t)$, then we have $d \leq d(a_t, a_j) + d(a_s, a_i) \leq 2\lceil \frac{d}{2} \rceil - 1 \leq d$. So d is odd and $a_t, a_{t+1}, \ldots, a_{j-1}, a_s, a_{s+1}, \ldots, a_{i-1} \in V_1$. Hence $G \cong H_n(k, l)$. If $d(a_i, a_s) = d(a_j, a_t)$, then $d(a_t, a_j) = d(a_s, a_i) = \lceil \frac{d}{2} \rceil$. So it is easy to check that $G \cong H_n(k, l)$ when d is odd and $G \cong T_n(k)$ when d is even.

4. The index set

In this section, we determine the set of common consequent indices of binary relations in $P_n(V, d)$.

Theorem 4 $E_n(V, d) = \{L(Q) : Q \in P_n(V, d)\} = \{1, 2, \ldots, n - \lceil \frac{d}{2} \rceil\}$.

Proof For any integer $r \in \{1, 2, \ldots, n - \lceil \frac{d}{2} \rceil\}$, we want to construct a directed graph G such that $Q(G) \in P_n(V, d)$ and $L(Q(G)) = r$.

Case 1 $r = 1$. Let G be the directed graph with vertex set V and arc set

$$E = \{(a_1, a_i) : 2 \leq i \leq n\} \cup \{(a_i, a_1) : 2 \leq i \leq n\} \cup \{(a_i, a_i) : 1 \leq i \leq d\}.$$

Then it is easy to check that $Q(G) \in P_n(V, d)$ and $L(Q(G)) = 1$.

Case 2 $\lfloor \frac{d}{2} \rfloor < r \leq n - \lceil \frac{d}{2} \rceil$. Let $m = r + \lceil \frac{d}{2} \rceil$. Then $d < m \leq n$. Let G be the directed graph with vertex set V and arc set

$$\begin{aligned} E' = &\{(a_i, a_{i+1}) : 1 \leq i \leq m-2\} \cup \{(a_{m-1}, a_j) : m \leq j \leq n\} \\ &\cup \{(a_j, a_1) : m \leq j \leq n\} \cup \{(a_i, a_i) : 1 \leq i \leq d\}. \end{aligned}$$

Clearly, $Q(G) \in P_n(V, d)$. Put $V_1 = \{a_1, a_2, \ldots, a_m\}$ and $V_2 = V \backslash V_1$. Then for any $x \in V_2$, x is a copy of a_m with respect to adjacency. By the proof of Theorem 2, we have

(i) for any $a_i, a_j \in V_1$, $a_i \neq a_j$, $L_Q(a_i, a_j) \leq L_Q(a_{d+1}, a_{\lceil \frac{d+1}{2} \rceil}) = m - \lceil \frac{d}{2} \rceil = r$;

(ii) for any $a_i \in V_1$, $a_j \in V_2$, $L_Q(a_i, a_j) = L_Q(a_i, a_m) \leq r$;

(iii) for any $a_i, a_j \in V_2$, $a_i \neq a_j$, $L_Q(a_i, a_j) = 1$.

Hence $L(Q) = r$.

Case 3 $2 \leq r \leq \lfloor \frac{d}{2} \rfloor$. Let $m = 2r$ and G be the directed graph with vertex set V and arc set E' (the same as of Case 2). It is obvious that $Q(G) \in P_n(V, d)$. Put $V_1 = \{a_1, a_2, \ldots, a_m\}$, $V_2 = \{a_{m+1}, a_{m+2}, \ldots, a_d\}$ and $V_3 = \{a_{d+1,} a_{d+2}, \ldots, a_n\}$. Then for any $x \in V_2$, x is a copy of a_m.

For any $a_i, a_j \in V$, $a_i \neq a_j$, it follows that

(i) If $a_i, a_j \in V \backslash V_1$, then $L_Q(a_i, a_j) = 1$.

(ii) If $a_i, a_j \in V_1$, then $L_Q(a_i, a_j) \leq L_Q(a_1, a_{\lceil \frac{m+1}{2} \rceil}) = m - \lceil \frac{m}{2} \rceil = r$.

(iii) If $a_i \in V_2, a_j \in V_1$, then $L_Q(a_i, a_j) = L_Q(a_m, a_j) \leq r$.

(iv) If $a_i \in V_3, a_j \in V_1$, then it is not difficult to check that

$$L_Q(a_i, a_j) \le \begin{cases} d(a_i, a_j) \le d(a_i, a_r) = r, & \text{for } 1 \le j \le r; \\ d(a_j, a_1) \le d(a_{r+1}, a_1) = r, & \text{for } r+1 \le j \le m. \end{cases}$$

Hence $L(Q) = r$. □

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