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The Existence of Solutions of Initial Value Problems for Nonlinear Second Order Impulsive Integro-Differential Equations of Mixed Type in Banach Spaces

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Abstract By the use of Mönch fixed point theorem and a new comparison result, the solutions of initial value problems for nonlinear second order impulsive integro-differential equations of mixed type in Banach spaces are investigated and the existence theorem is obtained.

Keywords Banach spaces; impulsive integro-differential equations; initial value problems; measure of noncompactness; cone.

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1. Introduction

In this paper we consider the following initial value problems (IVP) for nonlinear second order impulsive integro-differential equations of mixed type in a Banach space E:

$$\begin{cases} u'' = f(t,u,u',Tu,Su), \quad t \in J, t \neq t_k, \\ \triangle u \mid_{t=t_k} = I_k(u(t_k), u'(t_k)), \\ \triangle u' \mid_{t=t_k} = H_k(u(t_k), u'(t_k)), \quad k = 1,2,\ldots,m, \\ u(0) = x_0, u'(0) = x_1, \end{cases} \tag{1}$$

where $J = [0,a](a>0)$, $f \in C[J \times E \times E \times E \times E, E]$, $I_k, H_k \in C[E \times E, E]$ $(k = 1,2,\ldots,m)$, $0 < t_1 < t_2 < \cdots < t_m < a$, $x_0, x_1 \in E$, and

$$(Tu)(t) = \int_0^t k(t,s)u(s)\mathrm{d}s, \quad (Su)(t) = \int_0^a h(t,s)u(s)\mathrm{d}s. \tag{2}$$

In (2), $k \in C[D, R^+], h \in C[J \times J, R^+]$, where $R^+ = [0,+\infty)$, $D = \{(t,s) \in J \times J : t \geq s\}$, $\triangle u \mid_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e.,

$$\triangle u \mid_{t=t_k} = u(t_k^+) - u(t_k^-),$$

where $u(t_k^+)$ and $u(t_k^-)$ represent the right-hand limit and left-hand limit of $u(t)$ at $t = t_k$, respectively. $\triangle u' \mid_{t=t_k}$ has a similar meaning for $u'(t)$.

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Initial value problems for nonlinear integro-differential equations arise from many nonlinear problems in science. Over the last couple of decades, many attempts have been made to study the existence of solutions for first-order or second-order initial value problems with or without impulses in Banach spaces. In particular, for the special case where f does not include u', Tu and Su, Lakshmikantham and Leela[1] discussed the unique solution of IVP(1) by means of the strongly minimal and maximal solutions conditions, Lipschitz condition and Kuratowski measure of noncompactness. Removing the Lipschitz condition, Hao and Liu[2] obtained the existence of solutions of IVP(1) for the case in which f does not include u'. And in another special case where f does not contain u', in [3], Liu, Wu and Hao studied the global solutions of IVP(1). Recently, Su, Liu and et al. investigated the global solutions of IVP(1) where f does not contain impulses.

In this paper, by using Mönch fixed point theorem and a new comparison, we establish the theorem of existence of solutions of IVP(1). The result presented in this paper is new.

2. Preliminaries

In this paper, we always suppose that $(E, \|\cdot\|)$ is a real Banach space and P is a normal cone in E. Let $PC[J,E] = \{x : x$ is a map from J into E such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $x(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$, and $PC^1[J,E]$=$\{x : x$ is a map from J into E such that $x(t)$ is continuously differentiable at $t \neq t_k$, left continuous at $t = t_k$, and $x(t_k^+), x'(t_k^-), x'(t_k^+)$ exist, $k = 1, 2, \ldots, m\}$. Evidently, $PC[J,E]$ is a Banach space with norm

$$\|x\|_{PC} = \sup_{t\in J} \|x(t)\|.$$

For $x \in PC^1[J,E]$, by virtue of mean value theorem

$$x(t_k) - x(t_k - h) \in h\overline{\text{co}}\{x'(t) : t_k - h < t < t_k\}(h > 0),$$

it is easy to see that the left derivative $x'_-(t_k)$ exists and

$$x'_-(t_k) = \lim_{h\to 0^+} h^{-1}[x(t_k) - x(t_k - h)] = x'(t_k^-).$$

In IVP (1) and in the following, $x'(t_k)$ is understood as $x'_-(t_k)$. Evidently, $\mathrm{PC}^1[J,E]$ is also a Banach space with norm

$$\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\},$$

where $\|x\|_{PC}$ is defined above and $\|x'\|_{PC} = \sup_{t\in J} \|x'(t)\|$. Let $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, $J_0 = [0, t_1], J_1 = (t_1,t_2], \ldots, J_m = (t_{m-1}, t_m], J_m = (t_m, a]$, and α denotes the Kuratowski measure of noncompactness in E. $u \in PC^1[J,E] \bigcap C^2[J', E]$ is called a solution of IVP(1) if it satisfies (1).

Let $k_0 = \max\{k(t,s) : (t,s) \in D\}$, $h_0 = \max\{h(t,s) : (t,s) \in J \times J\}$. For $v_0, \omega_0 \in PC^1[J,E], v_0 \leq \omega_0$, we write $[v_0, \omega_0] = \{u \in PC^1[J,E] : v_0(t) \leq u(t) \leq \omega_0(t), v_0'(t) \leq u'(t) \leq \omega_0'(t), t \in J\}$. For $B \subset PC^1[J,E]$, we write $B' = \{x' : x \in B\} \subset \mathrm{PC}[J,E], B_k = \{x |_{J_k} : x \in B\}, B(t) = \{x(t) : x \in B\} \subset E(t \in J)$, and $(TB)(t) = \{(Tx)(t) : x \in B\} \subset E$. $B_k', B'(t), (SB)(t), (TB)'(t)$, $(SB)'(t)$ have the similar meanings.

At the end of this section, we state some lemmas which will be used in Section 3.

Lemma 1 *Assume that E is a real Banach space, and $p \in PC^1[J,E]\bigcap C^2[J',E]$ satisfies*

$$\begin{cases} p''(t) \le -a(t)p(t) - b(t)p'(t) - c(t)(Tp)(t), \quad \forall t \in J, \quad t \ne t_k, \\ \triangle p \mid_{t=t_k} = L_k p'(t_k), \\ \triangle p' \mid_{t=t_k} \le -L'_k p'(t_k), \quad k = 1, 2, \ldots, m, \\ p'(0) \le p(0) \le \theta, \end{cases} \tag{3}$$

where a, b, c are bounded integrable nonnegative functions on J and $L_k, L'_k (k = 1, 2, \cdots, m)$ are nonnegative constants, and provided the following conditions hold

$$\int_0^a (1+t+\sum_{k=1}^m L_k)a(t)\mathrm{d}t + \int_0^a b(t)\mathrm{d}t + \int_0^a c(t)\mathrm{d}t \cdot \int_0^t (1+s+\sum_{k=1}^m L_k)k(t,s)\mathrm{d}s + \sum_{k=1}^m L'_k \le 1. \tag{4}$$

Then $p(t) \le \theta$, $p'(t) \le \theta$, $\forall t \in J$.

Proof Let $p_1(t) = p'(t)(t \in J)$. Then $p_1 \in PC[J,E]\bigcap C^1[J',E]$. By (3) and Lemma 1 in [2], we have

$$\begin{aligned} p(t) &= p(0) + \int_0^t p_1(s)\mathrm{d}s + \sum_{0<t_k<t} [p(t_k^+) - p(t_k)] \\ &= p(0) + \int_0^t p_1(s)\mathrm{d}s + \sum_{0<t_k<t} L_k p_1(t_k), \quad \forall t \in J. \end{aligned} \tag{5}$$

Therefore,

$$(Tp)(t) = \int_0^t k(t,s)[p(0) + \sum_{0<t_k<s} L_k p_1(t_k)]\mathrm{d}s + \int_0^t p_1(r)\mathrm{d}r \int_r^t k(t,s)\mathrm{d}s, \quad \forall t \in J. \tag{6}$$

Substituting (5) and (6) into (3), we get

$$\begin{cases} p'_1(t) \le -b(t)p_1(t) - a_1(t)p(0) - \displaystyle\int_0^t k_1(t,s)p_1(s)\mathrm{d}s - a(t)\sum_{0<t_k<t} L_k p_1(t_k) - \\ \qquad c(t)\displaystyle\int_0^t k(t,s)[\sum_{0<t_k<s} L_k p_1(t_k)]\mathrm{d}s, \quad \forall t \in J, \quad t \ne t_k, \\ \triangle p_1 \mid_{t=t_k} \le -L'_k p_1(t_k), \quad k = 1, 2, \ldots, m, \\ p_1(0) \le p(0) \le \theta, \end{cases} \tag{7}$$

where

$$a_1(t) = a(t) + c(t)\int_0^t k(t,s)\mathrm{d}s, \quad \forall t \in J, \tag{8}$$

$$k_1(t,s) = a(t) + c(t)\int_s^t k(t,r)\mathrm{d}r, \quad \forall (t,s) \in D. \tag{9}$$

For any given $g \in P^*(P^*$ denotes the dual cone of $P)$, let $v(t) = g(p_1(t))$. Then $v \in$

$PC[J,R^1]\bigcap C^1[J',R^1]$ and $v'(t)=g(p_1'(t)),\ \forall t\in J,\ t\neq t_k, k=1,2,\ldots,m$. From (7), we know

$$\begin{cases} v'(t)\leq -b(t)v(t)-a_1(t)g(p(0))-\displaystyle\int_0^t k_1(t,s)v(s)\mathrm{d}s-a(t)\sum_{0<t_k<t}L_kv(t_k)- \\ \qquad c(t)\displaystyle\int_0^t k(t,s)[\sum_{0<t_k<s}L_kv(t_k)]\mathrm{d}s,\quad \forall t\in J,\quad t\neq t_k, \\ \triangle v\mid_{t=t_k}\leq -L_k'v(t_k),\quad k=1,2,\ldots,m, \\ v(0)\leq g(p(0))\leq 0. \end{cases} \tag{10}$$

We shall show that

$$v(t)\leq 0,\quad \forall t\in J. \tag{11}$$

On the contrary, if we suppose (11) is not true, i.e., there exists a $0<t^*\leq a$ such that $v(t^*)>0$. Let $t^*\in J_i$ and $\inf\{v(t):0\leq t\leq t^*\}=-\lambda$. Then $\lambda\geq 0$ and for some $t_*\in J_j(j\leq i)$, $v(t_*)=-\lambda$ or $v(t_j^+)=-\lambda$. We may assume $v(t_*)=-\lambda$ (the proof is similar when $v(t_j^+)=-\lambda$). We have by (10), $g(p(0))\geq -\lambda$, and

$$\begin{cases} v'(t)\leq \lambda b(t)+\lambda a_1(t)+\lambda\int_0^t k_1(t,s)\mathrm{d}s+\lambda a(t)(\sum\limits_{0<t_k<t}L_k)+ \\ \qquad \lambda c(t)\int_0^t k(t,s)(\sum\limits_{0<t_k<s}L_k)\mathrm{d}s,\quad \forall 0\leq t\leq t^*,\quad t\neq t_k, \\ \triangle v\mid_{t=t_k}\leq \lambda L_k',\quad \forall t_k\leq t^*. \end{cases} \tag{12}$$

So, applying formula[12, Lemma 1]

$$v(t^*)=v(t_*)+\int_{t_*}^{t^*}v'(s)\mathrm{d}s+\sum_{k=j+1}^{i}[v(t_k^+)-v(t_k)] \tag{13}$$

to (12), we find

$$\begin{aligned} 0<&v(t^*)\leq -\lambda+\lambda\int_0^a[b(t)+a_1(t)+(\sum_{k=1}^m L_k)a(t)]\mathrm{d}t+ \\ &\lambda\int_0^a\mathrm{d}t\int_0^t k_1(t,s)\mathrm{d}s+\lambda(\sum_{k=1}^m L_k)\int_0^a c(t)\mathrm{d}t\int_0^t k(t,s)\mathrm{d}s+ \\ &\lambda\sum_{k=1}^m L_k', \end{aligned}$$

which implies that $\lambda>0$ and

$$\begin{aligned} &\int_0^a[b(t)+a_1(t)+(\sum_{k=1}^m L_k)a(t)]\mathrm{d}t+\int_0^a\mathrm{d}t\int_0^t k_1(t,s)\mathrm{d}s+ \\ &(\sum_{k=1}^m L_k)\int_0^a c(t)\mathrm{d}t\int_0^t k(t,s)\mathrm{d}s+\sum_{k=1}^m L_k'>1. \end{aligned} \tag{14}$$

It is easy to see by simple calculation of (8), (9) and (14) that

$$\int_0^a[b(t)+a_1(t)+(\sum_{k=1}^m L_k)a(t)]\mathrm{d}t+\int_0^a\mathrm{d}t\int_0^t k_1(t,s)\mathrm{d}s+$$

$$
\begin{aligned}
&(\sum_{k=1}^{m} L_k)\int_0^a c(t)\mathrm{d}t\int_0^t k(t,s)\mathrm{d}s+\sum_{k=1}^{m} L'_k \\
=&\int_0^a (1+t+\sum_{k=1}^{m} L_k)a(t)\mathrm{d}t+\int_0^a b(t)\mathrm{d}t+\int_0^a c(t)\mathrm{d}t\times \\
&\int_0^t (1+s+\sum_{k=1}^{m} L_k)k(t,s)\mathrm{d}s+\sum_{k=1}^{m} L'_k>1,
\end{aligned}
$$

which contradicts (4). Consequently (11) holds.

Since $g\in P^*$ is arbitrary, we get from (11) that $p_1(t)\le\theta$ for $t\in J$, namely, $p'(t)\le\theta$ for $t\in J$. Thus, the function $p(t)$ is nondecreasing on $J_k (k=0,1,2,\dots,m)$. And from (3),

$$\triangle p\,|_{t=t_k}=L_k p'(t_k)\le\theta,\quad k=1,2,\dots,m.$$

We know $p(t)$ is nondecreasing on J. Therefore, $p(t)\le p(0)\le\theta$ for $t\in J$. The lemma is proved. □

Lemma 2[5] *Let $B\subset PC^1[J,E]$ be bounded and equicontinuous on each $J_k (k=0,1,2,\dots,m)$. Then $\alpha(\{x(t):x\in B_k\})$ is continuous on $t\in J_k$ and*

$$\alpha(\{\int_J x(t)\mathrm{d}t : x\in B\})\le\int_J \alpha(\{x(t):x\in B\})\mathrm{d}t.$$

Lemma 3[5] *Assume that $m\in C[J_i,R^+]$ $(i=0,1,2,\dots,m)$ satisfies*

$$m(t)\le M\int_0^t m(s)\mathrm{d}s+N\int_0^a m(s)\mathrm{d}s+\sum_{0<t_k<t} M_k m(t_k),\quad t\in J,$$

where $M>0, N\ge 0, M_k\ge 0$ $(k=1,2,\dots,m)$ are constants. Then $m(t)\equiv 0$ for any $t\in J$, provided one of the following conditions holds

(i) $N[(e^{Mt_1}-1)+(1+M_1)(e^{Mt_2}-e^{Mt_1})+\cdots+\prod_{k=1}^m(1+M_k)(e^{Ma}-e^{Mt_m})]<M$;

(ii) $(M+N)[t_1+(t_2-t_1)(1+M_1)+\cdots+(a-t_m)\prod_{k=1}^m(1+M_k)]<1$.

Lemma 4[7] *Assume that $B\subset PC^1[J,E]$ is bounded, and B' is equicontinuous on each J_k $(k=0,1,2,\dots,m)$. Then*

$$\alpha(B)=\max\{\sup_{t\in J}\alpha(B(t)),\sup_{t\in J}\alpha(B'(t))\}.$$

Lemma 5[8] *Let $B=\{x_n\}\subset L[J,E]$, and suppose that there exists a $g\in L[J,R^+]$ such that $\|x_n(t)\|\le g(t)$ for any $t\in J$ and $x_n\in B$. Then $\alpha(B(t))\in L[J,R^+]$ and*

$$\alpha(\{\int_0^t x_n(s)\mathrm{d}s : n\in N\})\le 2\int_0^t \alpha(B(s))\mathrm{d}s,\quad \forall t\in J.$$

Lemma 6[9] *Let E be a Banach space, $K\subset E$ closed and convex and $F:K\to K$ continuous with the further property that for $x\in K$, we have $B\subset K$ countable, $\overline{B}=\overline{\mathrm{co}}(\{x\}\cup F(B))\Rightarrow B$ is relatively compact. Then F has a fixed point in K.*

3. Main result

We are now in a position to prove our existence results. Let us list the following assumptions for convenience.

(H1) There exist $v_0, \omega_0 \in PC^1[J,E]\bigcap C^2[J',E]$ such that $v_0(t) \le \omega_0(t)$, $v_0'(t) \le \omega_0'(t), \forall t \in J$ and bounded integrable nonnegative functions $a(t), b(t), c(t)$ and nonnegative constants $L_k, L_k'(k = 1,2,\ldots,m)$ which satisfy (4), for any $h \in [v_0, \omega_0]$,

$$\begin{cases} v_0'' \le f(t,h,h',Th,Sh) - a(t)(v_0 - h) - b(t)(v_0' - h') - c(t)(Tv_0 - Th), \\ \qquad \forall t \in J, \quad t \ne t_k, \\ \triangle v_0 \mid_{t=t_k} = I_k(h(t_k), h'(t_k)) + L_k(v_0'(t_k) - h'(t_k)), \\ \triangle v_0' \mid_{t=t_k} \le H_k(h(t_k), h'(t_k)) - L_k'(v_0'(t_k) - h'(t_k)), \quad k = 1,2,\ldots,m, \\ v_0(0) \le x_0, \quad v_0'(0) - v_0(0) \le x_1 - x_0, \end{cases}$$

$$\begin{cases} \omega_0'' \ge f(t,h,h',Th,Sh) - a(t)(\omega_0 - h) - b(t)(\omega_0' - h') - c(t)(T\omega_0 - Th), \\ \qquad \forall t \in J, \quad t \ne t_k, \\ \triangle \omega_0 \mid_{t=t_k} = I_k(h(t_k), h'(t_k)) + L_k(\omega_0'(t_k) - h'(t_k)), \\ \triangle \omega_0' \mid_{t=t_k} \ge H_k(h(t_k), h'(t_k)) - L_k'(\omega_0'(t_k) - h'(t_k)), \quad k = 1,2,\ldots,m, \\ \omega_0(0) \ge x_0, \quad \omega_0'(0) - \omega_0(0) \ge x_1 - x_0. \end{cases}$$

(H2) For any countable bounded equicontinuous set $B = \{u_n\} \subset [v_0, \omega_0]$ and $t \in J$,

$$\begin{aligned} \alpha(f(t,B(t),B'(t),(TB)(t),(SB)(t))) \le & k_1\alpha(B(t)) + k_2\alpha(B'(t)) + \\ & k_3\alpha((TB)(t)) + k_4\alpha((SB)(t)), \end{aligned}$$

where k_i $(i = 1,2,3,4)$ are constants satisfying one of the following two conditions

(i) $ak_4h_0(e^{Ma} - 1) < k_1 + k_2 + 2a^* + 2b^* + ak_0k_3 + 2ac^*k_0$,

(ii) $2(k_1 + k_2 + 2a^* + 2b^* + ak_0k_3 + 2ac^*k_0 + ak_4h_0)\max\{a,1\}a < 1$,

where $M = \max\{2a(k_1+k_2+2a^*+2b^*+ak_0k_3+2ak_0c^*), 2(k_1+k_2+2a^*+2b^*+ak_0k_3+2ak_0c^*)\}$, $a^* = \sup\{a(t) : t \in J\}$, $b^* = \sup\{b(t) : t \in J\}$, $c^* = \sup\{c(t) : t \in J\}$.

Theorem 1 *Let E be a real Banach space and P be a normal cone in E. Assume that conditions (H1) and (H2) hold. Then IVP(1) has a solution u^* in $[v_0, \omega_0]$.*

Proof First, for any $h \in [v_0, \omega_0]$, we consider the following initial value problems for linear second order integro-differential equation (LIVP) in E

$$\begin{cases} u''(t) = g(t) - a(t)u(t) - b(t)u'(t) - c(t)(Tu)(t), \quad \forall t \in J, \quad t \ne t_k, \\ \triangle u \mid_{t=t_k} = I_k(h(t_k), h'(t_k)) + L_k(u'(t_k) - h'(t_k)), \\ \triangle u' \mid_{t=t_k} = H_k(h(t_k), h'(t_k)) - L_k'(u'(t_k) - h'(t_k)), \quad k = 1,2,\ldots,m, \\ u(0) = x_0, \quad u'(0) = x_1, \end{cases} \tag{15}$$

where

$$g(t) = f(t,h(t),h'(t),(Th)(t),(Sh)(t)) + a(t)h(t) + b(t)h'(t) + c(t)(Th)(t), \quad \forall t \in J.$$

It is easy to check that $u \in PC^1[J,E]\bigcap C^2[J',E]$ is a solution of LIVP(15) if and only if

$u \in PC[J,E] \bigcap C^1[J',E]$ is a unique solution of the following integrable equation

$$\begin{aligned} u(t) =& x_0 + tx_1 + \int_0^t (t-s)[g(s) - a(s)u(s) - b(s)u'(s) - c(s)(Tu)(s)]\mathrm{d}s + \\ & \sum_{0<t_k<t} \{[I_k(h(t_k), h'(t_k)) + L_k(u'(t_k) - h'(t_k))] + \\ & (t-t_k)[H_k(h(t_k), h'(t_k)) - L'_k(u'(t_k) - h'(t_k))]\}, \quad \forall t \in J,\ t \neq t_k. \end{aligned} \tag{16}$$

We can define an operator

$$Ah = u,$$

where u, h satisfy (16). Then

$$\begin{aligned} (Ah)'(t) =& u'(t) \\ =& x_1 + \int_0^t [g(s) - a(s)u(s) - b(s)u'(s) - c(s)(Tu)(s)]\mathrm{d}s + \\ & \sum_{0<t_k<t} [H_k(h(t_k), h'(t_k)) - L'_k(u'(t_k) - h'(t_k))], \quad \forall t \in J,\ t \neq t_k. \end{aligned} \tag{17}$$

We can easily find $u \in PC^1[J,E] \bigcap C^2[J',E]$ is a solution of IVP(1) if and only if $u \in PC[J,E] \bigcap C^1[J',E]$ is a fixed point of A.

In the following, we will show that A has a fixed point in $PC^1[J,E] \bigcap C^2[J',E]$. We will divide the proof into three steps.

(i) We will show that the operator A: $[v_0, \omega_0] \to [v_0, \omega_0]$.

In fact, for any $h \in [v_0, \omega_0]$, let $u = Ah$. All we need to do is to prove $v_0 \le u \le \omega_0,\ v'_0 \le u' \le \omega'_0$. Let $p = u - \omega_0$. By (15) and (H1), we know

$$\begin{cases} p'' = u'' - \omega''_0 \\ \quad \le f(t, h, h', Th, Sh) + a(t)(h-u) + b(t)(h'-u') + c(t)(Th - Tu) - \\ \quad\quad f(t, h, h', Th, Sh) + a(t)(\omega_0 - h) + b(t)(\omega'_0 - h') + c(t)(T\omega_0 - Th) \\ \quad = -a(t)p(t) - b(t)p'(t) - c(t)(Tp)(t), \quad \forall t \in J, \quad t \neq t_k, \\ \triangle p\mid_{t=t_k} = I_k(h(t_k), h'(t_k)) + L_k(u'(t_k) - h'(t_k)) - I_k(h(t_k), h'(t_k)) - \\ \quad\quad L_k(\omega'_0(t_k) - h'(t_k)) = L_k p'(t_k), \\ \triangle p'\mid_{t=t_k} \le H_k(h(t_k), h'(t_k)) - L'_k(u'(t_k) - h'(t_k)) - H_k(h(t_k), h'(t_k)) + \\ \quad\quad L'_k(\omega'_0(t_k) - h'(t_k)) = -L'_k p'(t_k), \quad k = 1, 2, \ldots, m, \\ p'(0) = u'(0) - \omega'_0(0) = x_1 - \omega'_0(0) \le x_0 - \omega_0(0) = u(0) - \omega_0(0) = p(0) \le \theta. \end{cases}$$

From Lemma 1, we get $p(t) \le 0, p'(t) \le 0$. Therefore $u \le \omega_0,\ u' \le \omega'_0$. By similar method we can obtain $v_0 \le u,\ v'_0 \le u'$.

(ii) We now prove that A: $[v_0, \omega_0] \to [v_0, \omega_0]$ is continuous. Let $A = A_1 + A_2$, where

$$\begin{aligned} (A_1h)(t) =& x_0 + tx_1 + \int_0^t (t-s)[g(s) - a(s)u(s) - b(s)u'(s) - c(s)(Tu)(s)]\mathrm{d}s, \\ (A_2h)(t) =& \sum_{0<t_k<t} \{[I_k(h(t_k), h'(t_k)) + L_k(u'(t_k) - h'(t_k))] + \\ & (t-t_k)[H_k(h(t_k), h'(t_k)) - L'_k(u'(t_k) - h'(t_k))]\}, \quad \forall t \in J,\ t \neq t_k. \end{aligned}$$

The proof of (ii) is similar to that of [10].

(iii) In the end we will show A has a fixed point in $[v_0, \omega_0]$. For $x \in [v_0, \omega_0]$, $B = \{u_n\} \subset [v_0, \omega_0]$ satisfying

$$\overline{B} = \overline{\text{co}}(\{x\} \cup (AB)), \tag{18}$$

we shall prove that B is relatively compact.

From (H1), we get

$$\begin{aligned} v_0'' + a(t)v_0 + b(t)v_0' + c(t)(Tv_0) \leq & f(t, u_n, u_n', Tu_n, Su_n) + a(t)u_n + \\ & b(t)u_n' + c(t)(Tu_n) \\ \leq & \omega_0'' + a(t)\omega_0 + b(t)\omega_0' + c(t)(T\omega_0), \end{aligned}$$

$$\triangle v_0 \mid_{t=t_k} - L_k v_0'(t_k) \leq I_k(u_n(t_k), u_n'(t_k)) - L_k u_n'(t_k) \leq \triangle \omega_0 \mid_{t=t_k} - L_k \omega_0'(t_k),$$

$$\triangle v_0' \mid_{t=t_k} + L_k' v_0'(t_k) \leq H_k(u_n(t_k), u_n'(t_k)) + L_k' u_n'(t_k) \leq \triangle \omega_0 \mid_{t=t_k} + L_k' \omega_0'(t_k).$$

Therefore, $\{f(t, u_n, u_n', Tu_n, Su_n) + a(t)u_n + b(t)u_n' + c(t)(Tu_n) : u_n \in B\}$ are bounded in $PC^1[J, E]$ and $\{I_k(u_n(t_k), u_n'(t_k)) - L_k u_n'(t_k) : k = 1, 2, \ldots, m\}$, $\{H_k(u_n(t_k), u_n'(t_k)) + L_k' u_n'(t_k) : k = 1, 2, \ldots, m\}$ are bounded in E. Together with (16) and (17) we can easily get $(AB)(t)$, $(AB)'(t)$ are bounded and equicontinuous on J_i $(i = 0, 1, 2, \ldots, m)$ and from (18) we know $B(t)$, $B'(t)$ are bounded and equicontinuous on J_i $(i = 0, 1, 2, \ldots, m)$. Hence, by Lemma 4, we have $\alpha(B) = \max\{\sup_{t \in J} \alpha(B(t)), \sup_{t \in J} \alpha(B'(t))\}$. Let $m(t) = \max\{\alpha(B(t)), \alpha(B'(t))\}$. Then, from Lemma 2, we can obtain $m \in C[J_i, R^+](i = 0, 1, 2, \ldots, m)$.

For $t \in J_0 = [0, t_1]$, from (18), Lemma 2, 5, the definition of A and the nature of the measure of noncompactness, we can get

$$\begin{aligned} \alpha(B(t)) = & \alpha(\overline{B}(t)) = \alpha((AB)(t)) \\ = & \alpha\Big(\int_0^t (t-s)[g(s) - a(s)u(s) - b(s)u'(s) - c(s)(Tu)(s)]\mathrm{d}s\Big) \\ \leq & 2a \int_0^t \alpha(f(s, B(s), B'(s), (TB)(s), (SB)(s)))\mathrm{d}s + \\ & 4aa^* \int_0^t \alpha(B(s))\mathrm{d}s + 4ab^* \int_0^t \alpha(B'(s))\mathrm{d}s + 4ac^* \int_0^t \alpha((TB)(s))\mathrm{d}s \\ \leq & (2ak_1 + 4aa^*) \int_0^t \alpha(B(s))\mathrm{d}s + (2ak_2 + 4ab^*) \int_0^t \alpha(B'(s))\mathrm{d}s + \\ & (2ak_3 + 4ac^*) \int_0^t \alpha((TB)(s))\mathrm{d}s + 2ak_4 \int_0^t \alpha((SB)(s))\mathrm{d}s \\ \leq & 2a(k_1 + k_2 + 2a^* + 2b^*) \int_0^t m(s)\mathrm{d}s + 2a(k_3 + 2c^*)k_0 t \int_0^t m(s)\mathrm{d}s + \\ & 2ak_4 h_0 t \int_0^a m(s)\mathrm{d}s \\ \leq & 2a(k_1 + k_2 + 2a^* + 2b^* + ak_0 k_3 + 2ac^* k_0) \int_0^t m(s)\mathrm{d}s + \\ & 2a^2 k_4 h_0 \int_0^a m(s)\mathrm{d}s, \end{aligned} \tag{19}$$

$$\begin{aligned}
\alpha(B'(t)) =&\alpha(\overline{B'}(t)) = \alpha((AB)'(t))\\
=&\alpha(\int_0^t [g(s) - a(s)u(s) - b(s)u'(s) - c(s)(Tu)(s)]\mathrm{d}s)\\
\leq&2\int_0^t \alpha(f(s, B(s), B'(s), (TB)(s), (SB)(s)))\mathrm{d}s+\\
&4a^*\int_0^t \alpha(B(s))\mathrm{d}s + 4b^*\int_0^t \alpha(B'(s))\mathrm{d}s + 4c^*\int_0^t \alpha((TB)(s))\mathrm{d}s\\
\leq&2(k_1 + k_2 + 2a^* + 2b^* + ak_0k_3 + 2ac^*k_0)\int_0^t m(s)\mathrm{d}s+\\
&2ak_4h_0\int_0^a m(s)\mathrm{d}s. \qquad (20)
\end{aligned}$$

From (19) and (20), we have

$$m(s) \leq M\int_0^t m(s)\mathrm{d}s + N\int_0^a m(s)\mathrm{d}s, \quad \forall t \in J_0,$$

where

$$\begin{aligned}
M =&\max\{2a(k_1 + k_2 + 2a^* + 2b^* + ak_0k_3 + 2ac^*k_0), 2(k_1 + k_2 + 2a^*+\\
&2b^* + ak_0k_3 + 2ac^*k_0)\},\\
N =&\max\{2a^2k_4h_0, 2ak_4h_0\}. \qquad (21)
\end{aligned}$$

Therefore, from (H1) and Lemma 3, $m(t) \equiv 0, \forall t \in J_0$. Especially,

$$\alpha(B(t_1)) = \alpha(B'(t_1)) = 0. \qquad (22)$$

Observing that $I_1, H_1 \in C[E \times E, E]$, we have

$$\alpha(I_1(B(t_1), B'(t_1))) = 0, \quad \alpha(H_1(B(t_1), B'(t_1))) = 0. \qquad (23)$$

Using the similar method, for $t \in (t_1, t_2]$, we get

$$\begin{aligned}
\alpha(B(t)) \leq&2a(k_1 + k_2 + 2a^* + 2b^* + ak_0k_3 + 2ac^*k_0)\int_0^t m(s)\mathrm{d}s+\\
&2a^2k_4h_0\int_0^a m(s)\mathrm{d}s + \alpha(I_1(B(t_1), B'(t_1)))+\\
&2L_1\alpha(B'(t_1)) + a\alpha(H_1(B(t_1), B'(t_1))) + 2aL_1'\alpha(B'(t_1)).
\end{aligned}$$

By (22) and (23), we know

$$\begin{aligned}
\alpha(B(t)) \leq&2a(k_1 + k_2 + 2a^* + 2b^* + ak_0k_3 + 2ac^*k_0)\int_0^t m(s)\mathrm{d}s+\\
&2a^2k_4h_0\int_0^a m(s)\mathrm{d}s. \qquad (24)
\end{aligned}$$

Similarly, we can obtain

$$\alpha(B'(t)) \leq 2(k_1 + k_2 + 2a^* + 2b^* + ak_0k_3 + 2ac^*k_0)\int_0^t m(s)\mathrm{d}s+$$

$$2ak_4h_0\int_0^a m(s)\mathrm{d}s. \tag{25}$$

Together with (24) and (25), we get

$$m(s) \le M\int_0^t m(s)\mathrm{d}s + N\int_0^a m(s)\mathrm{d}s, \quad \forall t \in J_1,$$

where M, N are defined by (21). Thus, from Lemma 3, we have $m(t) \equiv 0, \forall t \in J_1$. And so,

$$\alpha(B(t_2)) = \alpha(B'(t_2)) = 0.$$

By the continuity of I_2, H_2, we obtain

$$\alpha(I_2(B(t_2), B'(t_2))) = 0, \quad \alpha(H_2(B(t_2), B'(t_2))) = 0.$$

Similarly to above, we can easily verify that $\alpha(B(t)) = 0, \alpha(B'(t)) = 0, t \in J_i$ $(i = 2, 3, \ldots, m)$. Hence, $\alpha(B) = 0, t \in J$, which implies B is a relatively compact set in $PC^1[J, E]$. From Lemma 6, A has a fixed point u^* in $[v_0, \omega_0]$, i.e., IVP(1) has a solution in $PC^1[J, E] \bigcap C^2[J', E]$. The proof is completed. □

Remark 1 In this paper, we discussed the initial value problems for nonlinear second order impulsive integro-differential equations of mixed type which contain impulses, therefore, the conditions for the comparison result are different from those in [4].

Remark 2 We can let $k_4 = 0$ where the IVP(1) does not include impulses and f does not include Su, and the assumptions of (H2) hold for any $k_1 \ge 0, k_2 \ge 0, k_3 \ge 0$.

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