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Stability of Stochastic Differential Delay Equations with Markovian Switching

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Abstract The main aim of this paper is to investigate the pth moment exponential stability of stochastic differential delay equations with Markovian switching. A specific Lyapunov function is introduced to obtain the required stability, and the almost sure exponential stability for the delay equations is discussed subsequently.

Keywords Lyapunov function; delay equation; generalized Itô's formula; Brownian motion; Markov chain.

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1. Introduction

Stochastic modelling has come to play an important role in many branches of science and industry. An area of particular interest has been the automatic control of stochastic systems, with consequent emphasis being placed on the analysis of stability in stochastic models[1],[4–6],[8–10],[13].

Recently, stochastic differential delay equations with Markovian switching have received a great deal of attention. Moreover, there are quite a number of papers on the stability of the delay equations[2],[3],[14],[15]. In particular, we here highlight Mao's great contribution. The fundamental theory of existence and uniqueness of solutions of such delay equations has been studied in [11], and the exponential stability in mean square of a stochastic differential delay equation with Markovian switching has also been discussed in [12]. The form of the delay equation is as follows:

$$\mathrm{d}x(t) = f(x(t), x(t-\tau_1), t, r(t))\mathrm{d}t + g(x(t), x(t-\tau_2), t, r(t))\mathrm{d}W(t). \tag{1}$$

In this paper, we shall further allow the time delay to be of time dependent instead of constant, and investigate the pth moment exponential stability of a stochastic differential delay equation of the form:

$$\mathrm{d}x(t) = f(x(t), x(t-\tau_1(t)), t, r(t))\mathrm{d}t + g(x(t), x(t-\tau_2(t)), t, r(t))\mathrm{d}W(t). \tag{2}$$

The form of the equation is expatiated in detail in Section 2. In Section 3, we adopt a specific Lyapunov function which is relatively easy to verify, then we apply the generalized Itô's formula

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to obtain the pth moment exponential stability which is also our main result. We also get the almost sure exponential stability for the delay equation in the last section.

2. Stochastic differential delay equations with Markovian switching

Throughout this paper, unless otherwise specified, we let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $W(t) = (W_1(t), W_2(t), \ldots, W_m(t))^{\mathrm{T}}$ be an m-dimensional Brownian motion defined on the probability space. Let $t_0 \in R_+ = [0, \infty)$ and suppose $\tau_i(t) : [0, \tau_i](i = 1, 2)$ are continuous. Let $\tau = \max[\tau_1, \tau_2] > 0$ and $C([-\tau, 0], R^n)$ denote the family of continuous functions φ from $[-\tau, 0]$ to R^n with the norm $\|\varphi\| = \sup_{-\tau\leq s\leq 0} |\varphi(s)|$, where $|\cdot|$ is the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^{T}. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$ (without any confusion with $\|\varphi\|$). Denote by $C^b_{\mathcal{F}_0}([-\tau, 0], R^n)$ the family of all bounded, $\mathcal{F}_0$ measurable, and $C([-\tau, 0], R^n)$-valued random variables. If $x(t)$ is a continuous R^n-valued stochastic process on $t \in [-\tau, \infty)$, we let $x_t = \{x(t + s) : -\tau \leq s \leq 0\}$ for $t \geq 0$ which is regarded as a $C([-\tau, 0], R^n)$-valued stochastic process.

Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t + \Delta) = j|r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j \end{cases},$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{i\neq j} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$. It is known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+.

Consider a stochastic differential delay equation with Markovian switching of the form

$$\mathrm{d}x(t) = f(x(t), x(t - \tau_1(t)), t, r(t))\mathrm{d}t + g(x(t), x(t - \tau_2(t)), t, r(t))\mathrm{d}W(t),$$

on $t \geq 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0], R^n)$, where

$$f : R^n \times R^n \times R_+ \times S \to R^n \quad \text{and} \quad g : R^n \times R^n \times R_+ \times S \to R^{n\times m}.$$

Let $C^{2,1}(R^n \times R_+ \times S, R_+)$ denote the family of all nonnegative functions $V(x, t, i)$ on $R^n \times R_+ \times S$ which are continuously twice differentiable in x and once differentiable in t. If $V \in C^{2,1}(R^n \times R_+ \times S, R_+)$, define an operator LV from $R^n \times R^n \times R^n \times R_+ \times S$ to R by

$$\begin{aligned} LV(x, y, z, t, i) =& V_t(x, t, i) + V_x(x, t, i)f(x, y, t, i)+ \\ & \frac{1}{2}\mathrm{trace}[g^{\mathrm{T}}(x, z, t, i)V_{xx}g(x, z, t, i)] + \sum_{j=1}^{N} \gamma_{ij}V(x, t, j), \end{aligned} \tag{3}$$

where

$$V_t(x,t,i)=\frac{\partial V(x,t,i)}{\partial t},\quad V_x(x,t,i)=\Big(\frac{\partial V(x,t,i)}{\partial x_1},\ldots,\frac{\partial V(x,t,i)}{\partial x_n}\Big),$$

$$V_{xx}(x,t,i)=\Big(\frac{\partial^2 V(x,t,i)}{\partial x_i\partial x_j}\Big)_{n\times n}.$$

The generalized Itô formula reads as follows: if $V\in C^{2,1}(R^n\times R_+\times S,R_+)$, then for any stopping times $0\le\rho_1\le\rho_2<\infty$,

$$\begin{aligned}EV(x(\rho_2),\rho_2,r(\rho_2)) =& EV(x(\rho_1),\rho_1,r(\rho_1))+\\ & E\int_{\rho_1}^{\rho_2} LV(x(s),x(s-\tau_1(s)),x(s-\tau_2(s)),s,r(s))\mathrm{d}s. \end{aligned}\tag{4}$$

3. The pth moment exponential stability

We give Theorem 1 which includes a standing hypothesis in this paper firstly.

Theorem 1[11] *Assume that both f and g satisfy the local Lipschitz condition and the linear growth condition. Then equation (2) has a unique continuous solution on $t\ge-\tau$, which is denoted by $x(t,\xi)$ in this paper. Moreover, for every $p>0$,*

$$E[\sup_{-\tau\le s\le t}|x(s,\xi)|^p]<\infty,\quad \text{on}\ \ t\ge 0.\tag{5}$$

Now we discuss the pth moment exponential stability for equation (2). We impose the following hypotheses:

(H1) For every $i\in S$, there are constants $\alpha_i\in R$ and $\beta_i,\delta_i\ge 0$ such that

$$x^{\mathrm{T}}f(x,x,t,i)\le\alpha_i|x|^2,$$

and

$$|g(x,z,t,i)|^p\le\beta_i|x|^p+\delta_i|z|^p,$$

for all $x,z\in R^n$ and $t\ge 0$.

(H2) There are three nonnegative constants K_1, K_2 and K_3 such that

$$|f(x,x,t,i)-f(x,y,t,i)|^p\le K_1|x-y|^p,$$

and

$$|f(x,y,t,i)|^p\le K_2|x|^p+K_3|y|^p,$$

for all $x,y\in R^n$, $t\ge 0$ and $i\in S$.

It is easy to see from these hypotheses that $f(0,0,t,i)\equiv 0$ and $g(0,0,t,i)\equiv 0$, so equation (2) admits a trivial solution $x(t,0)\equiv 0$. Using the two hypotheses and the conclusion of Theorem 1 we can deduce that

$$\limsup_{t\to\infty}\frac{1}{t}\ln(E|x(t,\xi)|^p)<0,$$

for any initial data $\xi\in C^b_{\mathcal{F}_0}([-\tau,0],R^n)$.

Theorem 2 *Let hypotheses (H1) and (H2) hold, $p \geq 2$, and assume that there are positive constants $q_1, q_2, \ldots, q_N$. Set*

$$\check{q} = \max_{1\leq i\leq N} q_i, \;\; \hat{q} = \min_{1\leq i\leq N} q_i, \;\; \check{\alpha} = \max_{1\leq i\leq N} \alpha_i, \;\; \check{\beta} = \max_{1\leq i\leq N} \beta_i, \;\; \check{\delta} = \max_{1\leq i\leq N} \delta_i. \tag{6}$$

$$\eta = 1 - \sup_{t\geq 0} \tau_i'(t) > 0, \quad i = 1, 2. \tag{7}$$

$$\mu := \max_{1\leq i\leq N} (\varepsilon q_i + \sum_{j=1}^{N} \gamma_{ij} q_j), \quad \varepsilon > 0. \tag{8}$$

If the following inequality holds,

$$\begin{cases} C_3 := 2^{p-1}\tau^{\frac{p}{2}} e^{\varepsilon\tau}\Big[\tau^{\frac{p}{2}}(K_2 + K_3\dfrac{e^{\varepsilon\tau}}{\eta}) + [\dfrac{p(p-1)}{2}]^{\frac{p}{2}}(\check{\beta} + \check{\delta}\dfrac{e^{\varepsilon\tau}}{\eta})\Big] \\ \check{\alpha} \leq -\dfrac{\mu + p\check{q}(K_1 C_3)^{\frac{1}{p}} + \dfrac{p(p-1)}{2}\check{q}(\check{\beta} + \check{\delta}\dfrac{e^{\varepsilon\tau}}{\eta})^{\frac{2}{p}}}{p\hat{q}} < 0 \end{cases}, \tag{9}$$

then the trivial solution of equation (2) is pth moment exponentially stable.

Proof Fix any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0], R^n)$ and write $x(t, \xi) = x(t)$. We give some $\varepsilon > 0$ sufficiently small, and define a specific Lyapunov function

$$V(x, t, i) = q_i e^{\varepsilon t}|x|^p \;\; \text{for} \;\; (x, t, i) \in R^n \times R_+ \times S.$$

Clearly $V \in C^{2,1}(R^n \times R_+ \times S, R_+)$. Moreover, the operator LV from $R^n \times R^n \times R^n \times R_+ \times S$ to R defined by (3) becomes

$$\begin{aligned} LV(x, y, z, t, i) =& e^{\varepsilon t}\{\varepsilon q_i|x|^p + pq_i|x|^{p-2}x^{\mathrm{T}} f(x, y, t, i) + \frac{1}{2}pq_i|x|^{p-2}|g(x, z, t, i)|^2 + \\ & \frac{1}{2}p(p-2)q_i|x|^{p-4}|x^{\mathrm{T}} g(x, z, t, i)|^2 + \sum_{j=1}^{N} \gamma_{ij} q_j|x|^p\}. \end{aligned} \tag{10}$$

Using hypotheses (H1) and (H2), we derive

$$\begin{aligned} & pq_i|x|^{p-2}x^{\mathrm{T}} f(x, y, t, i) \\ & \leq pq_i\alpha_i|x|^p + pq_i|x|^{p-1}|f(x, x, t, i) - f(x, y, t, i)| \\ & \leq pq_i\alpha_i|x|^p + (p-1)q_i\theta^{\frac{p}{p-1}}|x|^p + q_i\theta^{-p}|f(x, x, t, i) - f(x, y, t, i)|^p \\ & \leq [pq_i\alpha_i + (p-1)\check{q}\theta^{\frac{p}{p-1}}]|x|^p + \check{q}\theta^{-p}K_1|x - y|^p, \end{aligned} \tag{11}$$

and

$$\begin{aligned} & \frac{1}{2}pq_i|x|^{p-2}|g(x, z, t, i)|^2 + \frac{1}{2}p(p-2)q_i|x|^{p-4}|x^{\mathrm{T}} g(x, z, t, i)|^2 \\ & \leq \frac{1}{2}p(p-1)q_i|x|^{p-2}|g(x, z, t, i)|^2 \\ & \leq \frac{1}{2}(p-1)(p-2)q_i\sigma^{\frac{p}{p-2}}|x|^p + (p-1)q_i\sigma^{-\frac{p}{2}}|g(x, z, t, i)|^p \\ & \leq [\frac{1}{2}(p-1)(p-2)\check{q}\sigma^{\frac{p}{p-2}} + (p-1)\check{q}\check{\beta}\sigma^{-\frac{p}{2}}]|x|^p + (p-1)\check{q}\check{\delta}\sigma^{-\frac{p}{2}}|z|^p. \end{aligned} \tag{12}$$

We use the elementary inequality $ab \le \frac{\theta^{\frac{p}{p-1}} a^{\frac{p}{p-1}}}{\frac{p}{p-1}} + \frac{b^p}{p\theta^p}$ in (11) and $ab \le \frac{\sigma^{\frac{p}{p-2}} a^{\frac{p}{p-2}}}{\frac{p}{p-2}} + \frac{b^{\frac{p}{2}}}{\frac{p}{2}\sigma^{\frac{p}{2}}}$ in (12). θ and σ are inequality parameters which will be exactly determined later. $\theta > 0$, $\sigma > 0$. Then, substituting (8), (11) and (12) into (10) yields that

$$\begin{aligned} LV(x,y,z,t,i) \le & e^{\varepsilon t}\{[\mu + pq_i\alpha_i + (p-1)\check{q}\theta^{\frac{p}{p-1}} + \frac{1}{2}(p-1)(p-2)\check{q}\sigma^{\frac{p}{p-2}} + \\ & (p-1)\check{q}\check{\beta}\sigma^{-\frac{p}{2}}]|x|^p + (p-1)\check{q}\check{\delta}\sigma^{-\frac{p}{2}}|z|^p + \check{q}\theta^{-p}K_1|x-y|^p\}. \end{aligned} \tag{13}$$

Noting

$$C_1 := EV(x(0), 0, r(0)) \le \check{q}E|x(0)|^p \le \check{q}E\|\xi\|^p,$$

we obtain, by the generalized Itô's formula, that

$$\begin{aligned} & EV(x(t), t, r(t)) \le C_1 + [\mu + pq_i\alpha_i + (p-1)\check{q}\theta^{\frac{p}{p-1}} + \frac{1}{2}(p-1)(p-2)\check{q}\sigma^{\frac{p}{p-2}} + \\ & (p-1)\check{q}\check{\beta}\sigma^{-\frac{p}{2}}]\int_0^t e^{\varepsilon s}E|x(s)|^p \mathrm{d}s + (p-1)\check{q}\check{\delta}\sigma^{-\frac{p}{2}}\int_0^t e^{\varepsilon s}E|x(s-\tau_2(s))|^p \mathrm{d}s + \\ & \check{q}\theta^{-p}K_1\int_0^t e^{\varepsilon s}E|x(s) - x(s-\tau_1(s))|^p \mathrm{d}s. \end{aligned} \tag{14}$$

Now we compute these integrals in (14) respectively, and we shall use (7).

$$\begin{aligned} \int_0^t e^{\varepsilon s}E|x(s-\tau_2(s))|^p \mathrm{d}s \le & \frac{1}{\eta}\int_{-\tau}^t e^{\varepsilon(s+\tau)}E|x(s)|^p \mathrm{d}s \\ \le & \frac{\tau}{\eta}e^{\varepsilon\tau}E\|\xi\|^p + \frac{e^{\varepsilon\tau}}{\eta}\int_0^t e^{\varepsilon s}E|x(s)|^p \mathrm{d}s. \end{aligned} \tag{15}$$

Moreover, applying Hölder's inequality, the moment inequality and hypotheses (H1) and (H2), by equation (2) we have

$$\begin{aligned} & E|x(t) - x(t-\tau_1(t))|^p \\ & \le 2^{p-1}E|\int_{t-\tau_1(t)}^t f(x(s), x(s-\tau_1(s)), s, r(s))\mathrm{d}s|^p + \\ & \quad 2^{p-1}E|\int_{t-\tau_1(t)}^t g(x(s), x(s-\tau_2(s)), s, r(s))\mathrm{d}W(s)|^p \\ & \le (2\tau)^{p-1}E\int_{t-\tau}^t |f(x(s), x(s-\tau_1(s)), s, r(s))|^p \mathrm{d}s + \\ & \quad 2^{p-1}\tau^{\frac{p}{2}-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}}E\int_{t-\tau}^t |g(x(s), x(s-\tau_2(s)), s, r(s))|^p \mathrm{d}s \\ & \le (2\tau)^{p-1}K_2\int_{t-\tau}^t E|x(s)|^p \mathrm{d}s + (2\tau)^{p-1}K_3\int_{t-\tau}^t E|x(s-\tau_1(s))|^p \mathrm{d}s + \\ & \quad 2^{p-1}\tau^{\frac{p}{2}-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}}\check{\beta}\int_{t-\tau}^t E|x(s)|^p \mathrm{d}s + 2^{p-1}\tau^{\frac{p}{2}-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}}\check{\delta}\int_{t-\tau}^t E|x(s-\tau_2(s))|^p \mathrm{d}s \\ & = [(2\tau)^{p-1}K_2 + 2^{p-1}\tau^{\frac{p}{2}-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}}\check{\beta}]\int_{t-\tau}^t E|x(s)|^p \mathrm{d}s + (2\tau)^{p-1}K_3\int_{t-\tau}^t E|x(s-\tau_1(s))|^p \mathrm{d}s + \\ & \quad 2^{p-1}\tau^{\frac{p}{2}-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}}\check{\delta}\int_{t-\tau}^t E|x(s-\tau_2(s))|^p \mathrm{d}s. \end{aligned} \tag{16}$$

Let $t \geq \tau$. Then

$$\begin{aligned}
&\int_0^t e^{\varepsilon s} E|x(s) - x(s-\tau_1(s))|^p \mathrm{d}s \\
&\leq [(2\tau)^{p-1}K_2 + 2^{p-1}\tau^{\frac{p}{2}-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}}\check{\beta}] \int_0^t e^{\varepsilon s}(\int_{s-\tau}^s E|x(u)|^p \mathrm{d}u)\mathrm{d}s + \\
&\quad (2\tau)^{p-1}K_3 \int_0^t e^{\varepsilon s}(\int_{s-\tau}^s E|x(u-\tau_1(u))|^p \mathrm{d}u)\mathrm{d}s + \\
&\quad 2^{p-1}\tau^{\frac{p}{2}-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}}\check{\delta} \int_0^t e^{\varepsilon s}(\int_{s-\tau}^s E|x(u-\tau_2(u))|^p \mathrm{d}u)\mathrm{d}s. \qquad (17)
\end{aligned}$$

By changing the order of integrations, we can show that

$$\begin{aligned}
&\int_0^t e^{\varepsilon s}(\int_{s-\tau}^s E|x(u)|^p \mathrm{d}u)\mathrm{d}s \leq \int_{-\tau}^t E|x(u)|^p (\int_u^{u+\tau} e^{\varepsilon s}\mathrm{d}s)\mathrm{d}u \\
&\leq \tau e^{\varepsilon\tau} \int_{-\tau}^t e^{\varepsilon u} E|x(u)|^p \mathrm{d}u \leq \tau^2 e^{\varepsilon\tau} E\|\xi\|^p + \tau e^{\varepsilon\tau}\int_0^t e^{\varepsilon u} E|x(u)|^p \mathrm{d}u. \qquad (18)
\end{aligned}$$

Using (15) and (18), we have

$$\begin{aligned}
&\int_0^t e^{\varepsilon s}(\int_{s-\tau}^s E|x(u-\tau_1(u))|^p \mathrm{d}u)\mathrm{d}s \\
&\leq \tau e^{\varepsilon\tau}\int_{-\tau}^t e^{\varepsilon u} E|x(u-\tau_1(u))|^p \mathrm{d}u \\
&\leq \tau e^{\varepsilon\tau}[\int_{-\tau}^0 e^{\varepsilon u} E|x(u-\tau_1(u))|^p \mathrm{d}u + \int_0^t e^{\varepsilon u} E|x(u-\tau_1(u))|^p \mathrm{d}u] \\
&\leq (\tau^2 e^{\varepsilon\tau} + \frac{\tau^2}{\eta}e^{2\varepsilon\tau})E\|\xi\|^p + \frac{\tau e^{2\varepsilon\tau}}{\eta}\int_0^t e^{\varepsilon u} E|x(u)|^p \mathrm{d}u. \qquad (19)
\end{aligned}$$

Proceeding with the same argument as in (19), we can get that

$$\begin{aligned}
&\int_0^t e^{\varepsilon s}(\int_{s-\tau}^s E|x(u-\tau_2(u))|^p \mathrm{d}u)\mathrm{d}s \\
&\leq (\tau^2 e^{\varepsilon\tau} + \frac{\tau^2}{\eta}e^{2\varepsilon\tau})E\|\xi\|^p + \frac{\tau e^{2\varepsilon\tau}}{\eta}\int_0^t e^{\varepsilon u} E|x(u)|^p \mathrm{d}u. \qquad (20)
\end{aligned}$$

Substituting (18), (19) and (20) into (17) gives

$$\int_0^t e^{\varepsilon s} E|x(s) - x(s-\tau_1(s))|^p \mathrm{d}s = C_2 + C_3 \int_0^t e^{\varepsilon u} E|x(u)|^p \mathrm{d}u, \qquad (21)$$

where

$$\begin{aligned}
C_2 :=& \{[(2\tau)^{p-1}K_2 + 2^{p-1}\tau^{\frac{p}{2}-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}}\check{\beta}] + [(2\tau)^{p-1}K_3 + \\
&\quad 2^{p-1}\tau^{\frac{p}{2}-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}}\check{\delta}](1 + \frac{e^{\varepsilon\tau}}{\eta})\}\tau^2 e^{\varepsilon\tau} E\|\xi\|^p \geq 0, \\
C_3 :=& 2^{p-1}\tau^{\frac{p}{2}} e^{\varepsilon\tau}[\tau^{\frac{p}{2}}(K_2 + K_3\frac{e^{\varepsilon\tau}}{\eta}) + [\frac{p(p-1)}{2}]^{\frac{p}{2}}(\check{\beta} + \check{\delta}\frac{e^{\varepsilon\tau}}{\eta})] \geq 0.
\end{aligned}$$

Substituting (15) and (21) into (14) yields

$$EV(x(t),t,r(t)) \le C_4 + \lambda \int_0^t e^{\varepsilon s} E|x(s)|^p \mathrm{d}s, \tag{22}$$

for all $t \ge \tau$, where

$$C_4 := C_1 + (p-1)\check{q}\check{\delta}\sigma^{-\frac{p}{2}} \frac{\tau}{\eta} e^{\varepsilon\tau} E\|\xi\|^p + \check{q}\theta^{-p} K_1 C_2,$$

$$\begin{aligned}\lambda :=& \mu + pq_i\alpha_i + (p-1)\check{q}\theta^{\frac{p}{p-1}} + \frac{1}{2}(p-1)(p-2)\check{q}\sigma^{\frac{p}{p-2}} + (p-1)\check{q}\check{\beta}\sigma^{-\frac{p}{2}} + \\ & (p-1)\check{q}\check{\delta}\sigma^{-\frac{p}{2}} \frac{e^{\varepsilon\tau}}{\eta} + \check{q}\theta^{-p} K_1 C_3. \end{aligned} \tag{23}$$

Now we try to find the best θ and σ. Let

$$\frac{\partial\lambda}{\partial\theta} = 0 \quad \text{and} \quad \frac{\partial\lambda}{\partial\sigma} = 0.$$

We can get

$$\theta = (K_1C_3)^{\frac{p-1}{p^2}}, \quad \sigma = (\check{\beta} + \check{\delta}\frac{e^{\varepsilon\tau}}{\eta})^{\frac{2(p-2)}{p^2}}. \tag{24}$$

Substituting (24) into (23) gives

$$C_4 := C_1 + (p-1)\check{q}\check{\delta}(\check{\beta} + \check{\delta}\frac{e^{\varepsilon\tau}}{\eta})^{-\frac{p-2}{p}} \frac{\tau}{\eta} e^{\varepsilon\tau} E\|\xi\|^p + \check{q}(K_1C_3)^{-\frac{p-1}{p}} K_1 C_2 \ge 0,$$

$$\lambda = \mu + pq_i\alpha_i + p\check{q}(K_1C_3)^{\frac{1}{p}} + \frac{1}{2}(p-1)(p-2)\check{q}(\check{\beta} + \check{\delta}\frac{e^{\varepsilon\tau}}{\eta})^{\frac{2}{p}}. \tag{25}$$

Putting (9) into (25), we get

$$\lambda \le 0.$$

It follows from (22) that

$$EV(x(t),t,r(t)) \le C_4, \quad t \ge \tau. \tag{26}$$

Noting

$$EV(x(t),t,r(t)) \ge \hat{q} e^{\varepsilon t} E|x(t)|^p, \tag{27}$$

we obtain

$$E|x(t)|^p \le e^{-\varepsilon t} \frac{C_4}{\hat{q}}, \quad t \ge \tau. \tag{28}$$

Consequently

$$\limsup_{t\to\infty} \frac{1}{t} \ln(E|x(t)|^p) \le -\varepsilon < 0. \tag{29}$$

In other words, the trivial solution of equation (2) is pth moment exponentially stable. The proof is completed. □

4. The almost sure exponential stability

We now begin to discuss the almost sure exponential stability for equation (2).

Theorem 3 *Suppose hypotheses (H1) and (H2) hold, $p \ge 2$. Assume that the trivial solution of equation (2) is pth moment exponentially stable. Then the trivial solution of equation (2) is* *almost sure exponentially stable.*

Proof Fix the initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau,0],R^n)$ arbitrarily and write $x(t,\xi)=x(t)$. By Theorem 2, there is a positive constant C_5 such that

$$E|x(t)|^p \leq C_5 e^{-\varepsilon t}, \quad t \geq \tau. \tag{30}$$

Let $\bar{k}$ be an integer sufficiently large and $\mu = \frac{\tau}{\bar{k}}, k=\bar{k}+1, k=\bar{k}+2,\ldots$. We have

$$\begin{aligned}
&E[\sup_{(k-1)\mu\leq t\leq k\mu}|x(t)|^p]\\
&\leq 3^pE|x((k-1)\mu)|^p + 3^pE(\int_{(k-1)\mu}^{k\mu}|f(x(s),x(s-\tau_1(s)),s,r(s))|\mathrm{d}s)^p +\\
&\quad 3^pE(\sup_{(k-1)\mu\leq t\leq k\mu}\int_{(k-1)\mu}^{t}|g(x(s),x(s-\tau_2(s)),s,r(s))\mathrm{d}W(s)|^p).
\end{aligned} \tag{31}$$

By (30), we have

$$E|x((k-1)\mu)|^p \leq C_5 e^{-\varepsilon(k-1)\mu}. \tag{32}$$

Applying Hölder's inequality and hypothesis (H2), we can get

$$\begin{aligned}
&E(\int_{(k-1)\mu}^{k\mu}|f(x(s),x(s-\tau_1(s)),s,r(s))|\mathrm{d}s)^p\\
&\leq \mu^{p-1}\int_{(k-1)\mu}^{k\mu}E|f(x(s),x(s-\tau_1(s)),s,r(s))|^p\mathrm{d}s\\
&\leq \mu^{p-1}K_2\int_{(k-1)\mu}^{k\mu}E|x(s)|^p\mathrm{d}s + \mu^{p-1}K_3\int_{(k-1)\mu}^{k\mu}E|x(s-\tau_1(s))|^p\mathrm{d}s\\
&\leq \mu^pK_2\sup_{(k-1)\mu\leq s\leq k\mu}E|x(s)|^p + \mu^pK_3\sup_{(k-1-\bar{k})\mu\leq s\leq k\mu}E|x(s)|^p.
\end{aligned}$$

One can also obtain

$$\begin{aligned}
&E(\sup_{(k-1)\mu\leq t\leq k\mu}\int_{(k-1)\mu}^{t}|g(x(s),x(s-\tau_2(s)),s,r(s))\mathrm{d}W(s)|^p)\\
&\leq C_pE(\int_{(k-1)\mu}^{k\mu}|g(x(s),x(s-\tau_2(s)),s,r(s))|^2\mathrm{d}s)^{\frac{p}{2}}\\
&\leq C_p\mu^{\frac{p}{2}-1}\int_{(k-1)\mu}^{k\mu}E|g(x(s),x(s-\tau_2(s)),s,r(s))|^p\mathrm{d}s\\
&\leq C_p\mu^{\frac{p}{2}-1}\check{\beta}\int_{(k-1)\mu}^{k\mu}E|x(s)|^p\mathrm{d}s + C_p\mu^{\frac{p}{2}-1}\check{\delta}\int_{(k-1)\mu}^{k\mu}E|x(s-\tau_2(s))|^p\mathrm{d}s\\
&\leq C_p\mu^{\frac{p}{2}}\check{\beta}\sup_{(k-1)\mu\leq s\leq k\mu}E|x(s)|^p + C_p\mu^{\frac{p}{2}}\check{\delta}\sup_{(k-1-\bar{k})\mu\leq s\leq k\mu}E|x(s)|^p,
\end{aligned} \tag{34}$$

where C_p is the constant given by the Burkholder-Davis-Gundy inequality. We have used Burkholder-Davis-Gundy inequality, Hölder's inequality and hypothesis (H1) in (34). Substituting (32), (33) and (34) into (31) yields

$$E[\sup_{(k-1)\mu\leq t\leq k\mu}|x(t)|^p]$$

$$
\begin{aligned}
&\le 3^p C_5 e^{-\varepsilon(k-1)\mu} + 3^p(\mu^p K_2 + C_p \mu^{\frac{p}{2}} \check{\beta}) \sup_{(k-1)\mu \le s \le k\mu} E|x(s)|^p + \\
&\quad 3^p(\mu^p K_3 + C_p \mu^{\frac{p}{2}} \check{\delta}) \sup_{(k-1-\bar{k})\mu \le s \le k\mu} E|x(s)|^p \\
&\le 3^p C_5 e^{-\varepsilon(k-1)\mu} + 3^p(\mu^p K_2 + C_p \mu^{\frac{p}{2}} \check{\beta}) C_5 e^{-\varepsilon(k-1)\mu} + \\
&\quad 3^p(\mu^p K_3 + C_p \mu^{\frac{p}{2}} \check{\delta}) C_5 e^{-\varepsilon(k-1-\bar{k})\mu} \\
&\le C_6 e^{-\varepsilon k\mu},
\end{aligned}
\tag{35}
$$

where

$$C_6 := 3^p(1 + \mu^p K_2 + C_p \mu^{\frac{p}{2}} \check{\beta}) C_5 e^{\varepsilon\mu} + 3^p(\mu^p K_3 + C_p \mu^{\frac{p}{2}} \check{\delta}) C_5 e^{\varepsilon(\mu+\tau)}.$$

By Chebyshev's inequality and (35), we can get

$$P\{\omega : \sup_{(k-1)\mu \le t \le k\mu} |x(t)| > e^{\frac{-\varepsilon k\mu}{2p}}\} \le C_6 e^{\frac{-\varepsilon k\mu}{2}}.$$

In view of the well-known Borel-Cantelli lemma, we see that for almost all $\omega \in \Omega$,

$$\sup_{(k-1)\mu \le t \le k\mu} |x(t)| \le e^{\frac{-\varepsilon k\mu}{2p}}, \tag{36}$$

holds for all but finitely many k. Hence there exists a $k_0(\omega)$, for all $\omega \in \Omega$ excluding a P-null set, for which (36) holds whenever $k \ge k_0$. Consequently, for almost all $\omega \in \Omega$,

$$\frac{1}{t} \ln |x(t)| \le -\frac{\varepsilon k\mu}{2pt} \le -\frac{\varepsilon}{2p},$$

if $(k-1)\mu \le t \le k\mu$. Therefore

$$\limsup_{t \to \infty} \frac{1}{t} \ln |x(t)| \le -\frac{\varepsilon}{2p}. \quad \text{a.s.} \tag{37}$$

The proof is completed. $\square$

References

[1] ARNOLD L. *Stochastic Differential Equations: Theory and Applications* [M]. John Wiley & Sons, New York-London-Sydney, 1974.

[2] BASAK G K, BISI A, GHOSH M K. *Stability of a random diffusion with linear drift* [J]. J. Math. Anal. Appl., 1996, **202**(2): 604–622.

[3] GHOSH M K, ARAPOSTATHIS A, MARCUS S I. *Optimal control of switching diffusions with application to flexible manufacturing systems* [J]. SIAM J. Control Optim., 1993, **31**(5): 1183–1204.

[4] KHASMINSKII R Z. *Stochastic Stability of Differential Equations* [M]. Sijthoff and Noordhoff, Rockville, MD, 1981.

[5] KOLMANOVSKII V B, MYSHKIS A D. *Applied Theory of Functional-Differential Equations* [M]. Kluwer Academic Publishers Group, Dordrecht, 1992.

[6] KOLMANOVSKII V B, NOSOV V R. *Stability of Functional Differential Equations* [M]. Academic Press, London, 1986.

[7] LUO Jiaowan. *Comparison principle and stability of Ito stochastic differential delay equations with Poisson jump and Markovian switching* [J]. Nonlinear Anal., 2006, **64**(2): 253–262.

[8] MAO Xuerong. *Stability of Stochastic Differential Equations with Respect to Semimartingales* [M]. Pitman Research Notes in Mathematics Series, 251. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1991.

[9] MAO Xuerong. *Exponential Stability of Stochastic Differential Equations* [M]. Marcel Dekker, Inc., New York, 1994.

[10] MAO Xuerong. *Stochastic Differential Equations and Their Applications* [M]. Horwood Publishing Limited, Chichester, 1997.

[11] MAO Xuerong, MATASOV A, PIUNOVSKIY A B. *Stochastic differential delay equations with Markovian switching* [J]. Bernoulli, 2000, **6**(1): 73–90.

[12] MAO Xuerong, SHAIKHET L. *Delay-dependent stability criteria for stochastic differential delay equations with Markovian switching* [J]. Stab. Control Theory Appl., 2000, **3**(2): 88–102.

[13] MOHAMMED S E A. *Stochastic Functional Differential Equations* [M]. Research Notes in Mathematics, 99. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[14] SHAIKHET L. *Stability of stochastic hereditary systems with Markov switching* [J]. Theory of Stochastic Processes, 1996, **18**(2): 180–184.

[15] SKOROKHOD A V. *Asymptotic Methods in the Theory of Stochastic Differential Equations* [M]. American Mathematical Society, Providence, RI, 1989.