Journal of Mathematical Research & Exposition
Aug., 2008, Vol. 28, No. 3, pp. 529–534
DOI:10.3770/j.issn:1000-341X.2008.03.010
Http://jmre.dlut.edu.cn

Simultaneous Stabilization for a Family of Plants

LU Yu Feng, XU Xiao Ping
(Department of Applied Mathematics, Dalian University of Technology, Liaoning 116024, China)
(E-mail: lyfdlutl@yahoo.com.cn)

Abstract In this paper, we discuss the problem of simultaneous stabilization for plants more than three by using Youla parametrization[1] and give a necessary and sufficient condition for simultaneous stabilization.

Keywords complete nest; nest algebra; linear system; stabilizable.

Document code A
MR(2000) Subject Classification 93D99
Chinese Library Classification O231.9

1. Preliminaries

The problem of simultaneous stabilization which is using a single controller to stabilize many plants is very important in the field of robust control[2−4]. In fact, we sometimes use a series of linear systems to approach a nonlinear system for solving some stabilization problem about nonlinear system. Therefore, we convert the nonlinear problem into simultaneous stabilization problem. This problem was put forward by Saeks, Murroy and Vidyadagar, Viswanadham[5,6]. There are two kinds of methods in studying simultaneous stabilization problem: One is based on the frequency domain method and the other the state space method.

In this paper we consider the following problem of simultaneous stabilization: Given L_0, L_1, $\ldots$, $L_n \in \mathcal{L}$, when does there exist $C \in \mathcal{L}$ for which $\{L_0, C\}, \{L_1, C\}, \ldots, \{L_n, C\}$ are stable?

We recall some basic concepts[7] that will be useful in this paper. First, we introduce the definition about complete nest and nest algebra.

Definition 1.1 *A family $\mathcal{N}$ of closed subspaces of the Hilbert space $\mathcal{H}$ is a complete nest if*

(1) $\{0\}, \mathcal{H} \in \mathcal{N}$.

(2) For $N_1, N_2 \in \mathcal{N}$, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.

(3) If N_α is a subfamily in $\mathcal{N}$, then $\cap_\alpha N_\alpha$ and $\vee_\alpha N_\alpha$ are also in $\mathcal{N}$.

Every set $\mathcal{P}$ of projections in $\mathcal{L}(\mathcal{H})$ determines an algebra Alg$\mathcal{P}$ of operators,

$$\mathrm{Alg}\mathcal{P} = \{T \in \mathcal{L}(\mathcal{H}) : (I - P)TP = 0\}.$$

If $\mathcal{N}$ is a nest, and $\mathcal{P}$ is its associated family of orthogonal projections, Alg$\mathcal{P}$ is called a nest algebra.

Received date: 2006-10-11; **Accepted date**: 2007-03-27
Foundation item: the National Natural Science Foundation of China (No. 10671028).

Suppose $\mathcal{H}$ is a complex separable Hilbert space, and $\mathcal{P}$ is a complete nest on $\mathcal{H}$. We parametrize $\mathcal{P}$ and write $\mathcal{P} = \{P_t : t \in \Gamma\}$. Let $Q_t = I - P_t, \mathcal{R} = \{Q_t : t \in \Gamma\}$. We assume $P_{t_1} \leq P_{t_2}$ for $t_1 \leq t_2$, and for each $t \in \Gamma$ we define a seminorm on $\mathcal{H}$ by

$$\|x\|_t = \|P_t x\|, x \in \mathcal{H}, P_t \neq I.$$

The family $\{\| \|_t : t \in \Gamma\}$ of seminorms defines a topology on $\mathcal{H}$, called the resolution topology. Convergence in this topology is described as follows: a sequence $\{x_n\}$ converges to $x \in \mathcal{H}$ if, for all seminorms, $\|x_n - x\|_t \longrightarrow 0$. The resolution topology is a metric topology[6]. Let $\mathcal{H}_e$ denote the completion of the metric space $\mathcal{H}$.

Definition 1.2 *A linear transformation T on $\mathcal{H}_e$ is causal if for each $t \in \Gamma$, $P_t T = P_t T P_t$. A linear system on $\mathcal{H}_e$ is a causal linear transformation on $\mathcal{H}_e$, which is continuous with respect to the resolution topology.*

It is clear that the set of linear systems on $\mathcal{H}_e$ is an algebra. We denote this algebra by $\mathcal{L}$.

Definition 1.3 *A linear transformation $T : \mathcal{H}_e \longrightarrow \mathcal{H}_e$ is stable if there exists $M > 0$ such that for each $x \in \mathcal{H}_e$ and $t \in \Gamma, \|Tx\|_t \leq M\|x\|_t$.*

We denote the set of stable linear transformations on $\mathcal{H}_e$ by $\mathcal{S}$. Then $\mathcal{S}$ is a weakly closed algebra containing the identity.

The following proposition is the Theorem 5.4.2 of [7].

Proposition 1.1 *The following are equivalent:*

(1) T on $\mathcal{H}_e$ is stable.

(2) T is causal and $T|\mathcal{H}$ is a bounded operator.

(3) $T \in \mathcal{L}$ and is the extension to $\mathcal{H}_e$ of an operator in $\mathrm{Alg}\mathcal{R}$.

This proposition allows us to identify the algebra $\mathcal{S}$ of stable operators on $\mathcal{H}_e$ with the nest algebra $\mathrm{Alg}\mathcal{R}$. The restriction of $T \in \mathcal{S}$ to $\mathcal{H}$ is in $\mathrm{Alg}\mathcal{R}$ and the extension of $S \in \mathrm{Alg}\mathcal{R}$ to $\mathcal{H}_e$ is in $\mathcal{S}$.

2. Main results

From now, let $\mathcal{H}$ be the collection of sequences $\{x_n\}$ such that

$$\sum |x_i|^2 < \infty.$$

Here $|x|$ denotes the standard Euclidean norm on C. Then $\mathcal{H}$ is a Hilbert space with an inner product

$$(x, y) = \sum_{n=1}^{\infty} x_n \bar{y}_n.$$

It is easy to check that

$$\mathcal{H}_e = \{\langle x_0, x_1, x_2, \ldots\rangle : x_i \in C\}.$$

For each $n \geq 0$, let P_n denote the standard truncation projection defined on $\mathcal{H}$ by

$$P_n\langle x_0, x_1, \ldots, x_n, x_{n+1}, \ldots\rangle = \langle x_0, x_1, \ldots, x_n, 0, 0, \ldots\rangle.$$

Let L and C be given causal linear systems and consider the standard feedback configuration with plant L and compensator C, where the closed loop system equation for this configuration is

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} I & C \\ L & -I \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

The system is well posed if the internal input $e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ can be expressed as a causal function of the external input $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. This is equivalent to ([7], Chapter 6) requiring that $\begin{pmatrix} I & C \\ L & -I \end{pmatrix}$ be invertible. This inverse is easily computed formally and is given by the transfer matrix

$$H(L, C) = \begin{pmatrix} (I + CL)^{-1} & C(I + LC)^{-1} \\ L(I + CL)^{-1} & -(I + LC)^{-1} \end{pmatrix}.$$

L and C may not be stable. This means that there may be an input u in $\mathcal{H}$ such that Lu or Cu may not be in $\mathcal{H}$. Let $\mathcal{D}(L) = \{u \in \mathcal{H} : Lu \in \mathcal{H}\}$ and $\mathcal{D}(C) = \{u \in \mathcal{H} : Cu \in \mathcal{H}\}$. Then $\begin{pmatrix} I & C \\ L & -I \end{pmatrix}$ can be regarded as a linear transformation from $\mathcal{D}(L) \bigoplus \mathcal{D}(C)$ into $\mathcal{H} \bigoplus \mathcal{H}$.

Definition 2.1 *The closed loop system determined by the plant L and compensator C is stable if all the entries of $H(L, C)$ are stable systems on $\mathcal{H}$. The plant L is stabilizable if there exists a causal linear system C such that the closed loop system determined by L and C is stable.*

In order to characterize the stabilizable systems, we need the notions of right and left strong representations for a causal linear system. Recall that the graph of a linear transformation L with domain $\mathcal{D}(L)$ in $\mathcal{H}$ is $G(L) = \left\{ \begin{pmatrix} x \\ Lx \end{pmatrix} : x \in \mathcal{D}(L) \right\}$.

The following definitions are from [1] (also see [7], Chapter5).

Definition 2.2 *A plant L has a strong right representation $\begin{pmatrix} M \\ N \end{pmatrix}$ with M and N stable if*

(1) $G(L) = \operatorname{Ran} \begin{pmatrix} M \\ N \end{pmatrix}$;

(2) $\begin{pmatrix} M \\ N \end{pmatrix}$ *has a stable left inverse; there exist X, Y stable such that* $[Y, X] \begin{pmatrix} M \\ N \end{pmatrix} = I$.

L has a strong left representation $[-\hat{N}, \hat{M}]$ with $\hat{M}, \hat{N}$ stable if

(1) $G(L) = \operatorname{Ker}[-\hat{N}, \hat{M}]$.

(2) $[-\hat{N}, \hat{M}]$ *has a stable right inverse; there exist $\hat{X}, \hat{Y}$ stable such that*

$$[-\hat{N}, \hat{M}] \begin{pmatrix} -\hat{X} \\ \hat{Y} \end{pmatrix} = I.$$

The following proposition is Youla parametrization theorem, and was proved in [1] (also see [7], Chapter 6).

Proposition 2.1 *A causal linear system L is stabilizable if and only if L has a strong right and a strong left representation. If this is the case, the representations can be chosen so that we have the double Bezout identity*

$$\begin{pmatrix} Y & X \\ -\hat{N} & \hat{M} \end{pmatrix}\begin{pmatrix} M & -\hat{X} \\ N & \hat{Y} \end{pmatrix} = \begin{pmatrix} M & -\hat{X} \\ N & \hat{Y} \end{pmatrix}\begin{pmatrix} Y & X \\ -\hat{N} & \hat{M} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

A causal linear system C stabilizes L if and only if it has a strong right representation $\begin{pmatrix} \hat{Y} - NQ \\ \hat{X} + MQ \end{pmatrix}$ and a strong left representation $[-(X + Q\hat{M}), Y - Q\hat{N}]$ for some stable Q.

We now turn to the problem of simultaneous stabilization. Given $L_0, L_1, \ldots, L_n \in \mathcal{L}$, when does there exist $C \in \mathcal{L}$ for which $\{L_0, C\}, \{L_1, C\}, \ldots, \{L_n, C\}$ are stable?

Theorem 2.1 *Given $L_0, L_1, \ldots, L_n \in \mathcal{L}$ stabilizable. Define $A_j = Y_0 M_j + X_0 N_j, B_j = -\hat{N}_0 M_j + \hat{M}_0 N_j (j = 1, 2, \ldots, n)$, where $M_i, N_i, X_i, Y_i, \hat{M}_i, \hat{N}_i, \hat{X}_i, \hat{Y}_i (i = 0, 1, 2, \ldots, n) \in \mathcal{S}$, are associated with L_i, and $\begin{pmatrix} M_i \\ N_i \end{pmatrix}$ and $[-\hat{N}_i, \hat{M}_i]$ are, respectively, strong right and left representations for L_i that satisfy the double Bezout identity*

$$\begin{pmatrix} Y_i & X_i \\ -\hat{N}_i & \hat{M}_i \end{pmatrix}\begin{pmatrix} M_i & -\hat{X}_i \\ N_i & \hat{Y}_i \end{pmatrix} = \begin{pmatrix} M_i & -\hat{X}_i \\ N_i & \hat{Y}_i \end{pmatrix}\begin{pmatrix} Y_i & X_i \\ -\hat{N}_i & \hat{M}_i \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then there exists $C \in \mathcal{L}$, which simultaneously stabilizes $L_0, L_1, \ldots, L_n$ if and only if there exists $T \in \mathcal{S}$ such that $A_j + TB_j$ $(j = 1, 2, \ldots, n)$ are invertible in $\mathcal{S}$.

Proof Since $L_0, L_1, \ldots, L_n$ are stabilizable, the compensators that stabilize them are given, respectively, by the strong left representations

$$[-(X_0 + Q_0\hat{M}_0), Y_0 - Q_0\hat{N}_0], [-(X_1 + Q_1\hat{M}_1), Y_1 - Q_1\hat{N}_1], \ldots, [-(X_n + Q_n\hat{M}_n), Y_n - Q_n\hat{N}_n],$$

where $Q_i \in \mathcal{S}(i = 0, 1, 2, \ldots, n)$. Thus $L_0, L_1, \ldots, L_n$ can be simultaneously stabilized if and only if there exist $Q_{ii} \in \mathcal{S}(i = 0, 1, 2, \ldots, n)$ such that

$$\begin{aligned}[-(X_0 + Q_{00}\hat{M}_0), Y_0 - Q_{00}\hat{N}_0] &= Z_1[-(X_1 + Q_{11}\hat{M}_1), Y_1 - Q_{11}\hat{N}_1] = \cdots \\ &= Z_n[-(X_n + Q_{nn}\hat{M}_n), Y_n - Q_{nn}\hat{N}_n]\end{aligned}$$

for some invertible $Z_i (i = 1, 2, \ldots, n)$ in $\mathcal{S}$. This is equivalent to

$$X_0 + Q_{00}\hat{M}_0 = Z_1(X_1 + Q_{11}\hat{M}_1) = \cdots = Z_n(X_n + Q_{nn}\hat{M}_n),$$
$$Y_0 - Q_{00}\hat{N}_0 = Z_1(Y_1 - Q_{11}\hat{N}_1) = \cdots = Z_n(Y_n - Q_{nn}\hat{N}_n).$$

Rewrite them as

$$[I, Q_{00}]\begin{pmatrix} Y_0 & X_0 \\ -\hat{N}_0 & \hat{M}_0 \end{pmatrix} = Z_1[I, Q_{11}]\begin{pmatrix} Y_1 & X_1 \\ -\hat{N}_1 & \hat{M}_1 \end{pmatrix} = \cdots = Z_n[I, Q_{nn}]\begin{pmatrix} Y_n & X_n \\ -\hat{N}_n & \hat{M}_n \end{pmatrix}$$

or

$$[I, Q_{00}]\begin{pmatrix} Y_0 & X_0 \\ -\hat{N}_0 & \hat{M}_0 \end{pmatrix}\begin{pmatrix} M_1 & -\hat{X}_1 \\ N_1 & \hat{Y}_1 \end{pmatrix} = Z_1[I, Q_{11}]$$

$$[I, Q_{00}] \begin{pmatrix} Y_0 & X_0 \\ -\hat{N}_0 & \hat{M}_0 \end{pmatrix} \begin{pmatrix} M_2 & -\hat{X}_2 \\ N_2 & \hat{Y}_2 \end{pmatrix} = Z_2[I, Q_{22}]$$

$$\vdots$$

$$[I, Q_{00}] \begin{pmatrix} Y_0 & X_0 \\ -\hat{N}_0 & \hat{M}_0 \end{pmatrix} \begin{pmatrix} M_n & -\hat{X}_n \\ N_n & \hat{Y}_n \end{pmatrix} = Z_n[I, Q_{nn}]$$

or

$$[I, Q_{00}] \begin{pmatrix} A_1 & U_1 \\ B_1 & V_1 \end{pmatrix} = Z_1[I, Q_{11}]$$

$$[I, Q_{00}] \begin{pmatrix} A_2 & U_2 \\ B_2 & V_2 \end{pmatrix} = Z_2[I, Q_{22}] \qquad (*)$$

$$\vdots$$

$$[I, Q_{00}] \begin{pmatrix} A_n & U_n \\ B_n & V_n \end{pmatrix} = Z_2[I, Q_{nn}],$$

where $U_i = -Y_0\hat{X}_i + X_0\hat{Y}_i, V_i = -N_0\hat{X}_i + M_0\hat{Y}_i$ $(i = 0, 1, 2, \ldots, n)$.

$\Rightarrow$. Suppose there exist $Q_{00}, Q_{11}, \ldots, Q_{nn}, Z_1, Z_2, \ldots, Z_n \in \mathcal{S}$ with Z_i $(i = 0, 1, 2, \ldots, n)$ invertible, which satisfy the equation $(*)$. Then take $T = Q_{00}, A_j + Q_{00}B_j = Z_j (j = 1, 2, \ldots, n)$ are invertible in $\mathcal{S}$ as required.

$\Leftarrow$. Conversely, if there exists $T \in \mathcal{S}$ such that $A_j + Q_{00}B_j = Z_j$ $(j = 1, 2, \ldots, n)$ are invertible in $\mathcal{S}$, take $Z_j = A_j + Q_{00}B_j (j = 1, 2, \ldots, n)$, $Q_{00} = T$, and $Q_{ii} = Z_j^{-1}(U_j + Q_{00}V_j)$ $(j = 1, 2, \ldots, n)$. These satisfy the equation $(*)$ and the proof is completed. $\square$

Corollary 2.1 *Given L_0, L_1, $L_2 \in \mathcal{L}$ stabilizable. Define $A_1 = Y_0M_1 + X_0N_1, A_2 = Y_0M_2 + X_0N_2, B_1 = -\hat{N}_0M_1 + \hat{M}_0N_1, B_2 = -\hat{N}_0M_2 + \hat{M}_0N_2$, where M_i, N_i etc., are associated with L_i as in Theorem 2.1. Then there exists $C \in \mathcal{L}$, which simultaneously stabilizes L_0, L_1, L_2 if and only if there exists $T \in \mathcal{S}$ such that $A_i + TB_i$ $(i = 1, 2)$ are invertible in $\mathcal{S}$.*

Theorem 2.2 *Given $L_0, L_1, \ldots, L_n \in \mathcal{L}$ stabilizable. Define $\tilde{A}_j = \hat{N}_jM_0 - \hat{M}_jN_0, \tilde{B}_j = \hat{N}_i\hat{X}_0 + \hat{M}_jY_0 (j = 1, 2, \ldots, n)$, where $M_i, N_i, X_i, Y_i, \hat{M}_i, \hat{N}_i, \hat{X}_i, \hat{Y}_i (i = 0, 1, 2, \ldots, n) \in \mathcal{S}$, are associated with L_i, and $\begin{pmatrix} M_i \\ N_i \end{pmatrix}$ and $[-\hat{N}_i, \hat{M}_i]$ are, respectively, strong right and left representations for L_i that satisfy the double Bezout identity*

$$\begin{pmatrix} Y_i & X_i \\ -\hat{N}_i & \hat{M}_i \end{pmatrix} \begin{pmatrix} M_i & -\hat{X}_i \\ N_i & \hat{Y}_i \end{pmatrix} = \begin{pmatrix} M_i & -\hat{X}_i \\ N_i & \hat{Y}_i \end{pmatrix} \begin{pmatrix} Y_i & X_i \\ -\hat{N}_i & \hat{M}_i \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then there exists $C \in \mathcal{L}$, which simultaneously stabilizes $L_0, L_1, \ldots, L_n$ if and only if there exists $T \in \mathcal{S}$ such that $\tilde{B}_j + \tilde{A}_jT$ $(j = 1, 2, \ldots, n)$ are invertible in $\mathcal{S}$.

Proof Since $L_0, L_1, \ldots, L_n$ are stabilizable, by Youla parametrization theorem, the compensators that stabilize them are given, respectively, by the strong right representations, and the proof is similar to Theorem 2.1. □

Remark 2.1 If $\mathcal{H}$ is the direct sum of finite-dimensional Euclidean spaces, then all results in this section are true.

References

[1] DALE W N, SMITH M C. *Stabilizability and existence of system representations for discrete-time time-varying systems* [J]. SIAM J. Control Optim., 1993, **31**(6): 1538–1557.

[2] GHOSH B K, BYRNES C I. *Simultaneous stabilization and simultaneous pole-placement by nonswitching dynamic compensation* [J]. IEEE Trans. Automat. Control, 1983, **28**(6): 735–741.

[3] EMRE E. *Simultaneous stabilization with fixed closed-loop characteristic polynomial* [J]. IEEE Trans. Automat. Control, 1983, **28**(1): 103–104.

[4] VIDYASAGAR M. *Control System Synthesis* [M]. MIT Press, Cambridge, MA, 1985.

[5] VIDYASAGAR M, VISWANADHAM N. *Algebraic design techniques for reliable stabilization* [J]. IEEE Trans. Automat. Control, 1982, **27**(5): 1085–1095.

[6] SAEKS R, MURRAY J. *Fractional representation, algebraic geometry, and the simultaneous stabilization problem* [J]. IEEE Trans. Automat. Control, 1982, **27**(4): 895–903.

[7] FEINTUCH A. *Robust Control Theory in Hilbert Space* [M]. Springer-Verlag, New York, 1998.