Journal of Mathematical Research & Exposition
Aug., 2008, Vol. 28, No. 3, pp. 535–540
DOI:10.3770/j.issn:1000-341X.2008.03.011
Http://jmre.dlut.edu.cn

Weighted Composition Operators from A^p_α to $A^\infty(\varphi)$

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Abstract Suppose $u \in H(D)$ and ϕ is an analytic map of the unit disk D into itself. Define the weighted composition operator $uC_\phi : uC_\phi(f) = uf \circ \phi$, for all $f \in H(D)$. In this paper, we get necessary and sufficient conditions for the bounded and compact weighted composition operators from the weighted Bergman spaces A^p_α to $A^\infty(\varphi)$ $(A^\infty_0(\varphi))$ spaces.

Keywords A^p_α space; $A^\infty(\varphi)$ space; composition operator.

Document code A
MR(2000) Subject Classification 47B33; 47B38; 46E15; 30D15
Chinese Library Classification O177.2; O174.5

1. Introduction

Let $H(D)$ be the collection of all analytic functions on the unit disk D. For $0 < p < \infty$, $-1 < \alpha < \infty$, the weighted Bergman space A^p_α is defined by

$$A^p_\alpha = \{f \in H(D) : \|f\|^p_{\alpha,p} = (\alpha+1)\int_D |f(z)|^p(1-|z|^2)^\alpha \mathrm{d}A(z) < \infty\},$$

where $\mathrm{d}A(z)$ is the normalized Lebesgue measure on D and A^p_α is a Banach space with the above norm, for $1 \le p < \infty$.

Let φ denote the normal function on $[0,1)$. We define

$$A^\infty(\varphi) = \{f \in H(D) : \sup_{z\in D} \varphi(|z|)|f(z)| < \infty\};$$

$$A^\infty_0(\varphi) = \{f \in H(D) : \lim_{|z|\to 1} \varphi(|z|)|f(z)| = 0\}.$$

It is clear that $A^\infty(\varphi)$ $(A^\infty_0(\varphi))$ is the Bers-type space H^∞_α (little Bers-type space $H^\infty_{\alpha,0}$) for $\varphi(t) = (1-t^2)^\alpha$ $(0 < t < 1, 0 < \alpha < \infty)$. $A^\infty(\varphi)$ is a Banach space under the norm $\|f\|_\varphi = \sup_{z\in D} \varphi(|z|)|f(z)|$, and $A^\infty_0(\varphi)$ is the closed subspace of $A^\infty(\varphi)$.

Let ϕ be an analytic self-map of D and $u \in H(D)$. The weighted composition operator uC_ϕ is defined by

$$uC_\phi(f) = uf \circ \phi, \quad f \in H(D).$$

Received date: 2006-05-15; **Accepted date**: 2007-07-13
Foundation item: the National Natural Science Foundation of China (No. 10471039); the Natural Science Foundation of the Education Committee of Jiangsu Province (No. 06KJD110175, 07KJB110115).

It is obvious that the operator uC_ϕ is linear. We can regard this operator as a generalization of a multiplication operator M_u and a composition operator C_ϕ.

Jiang and Li[1] gave the characterizations of the bounded and compact composition operators from H_α^∞ and $H_{\alpha,0}^\infty$ to the other spaces of analytic functions. He and Jiang[2] discussed the composition operators on H_α^∞ and $H_{\alpha,0}^\infty$, and also generalized the corresponding results to $A^\infty(\varphi)$. Li[3] studied the bounded and compact weighted composition operator from Hardy space to Bers-type space. The purpose of this paper is to characterize boundedness and compactness of the weighted composition operators uC_ϕ from A_α^p to $A^\infty(\varphi)$ ($A_0^\infty(\varphi)$).

In this paper, we will always use the letter C to denote a positive constant, which may change from one equation to the next. The constants usually depend on a and other fixed parameters.

2. Main results

Lemma 2.1[4] *Suppose $0 < p < \infty$, $-1 < \alpha < \infty$. If $f \in A_\alpha^p$, then $|f(z)| \le \frac{C\|f\|_{\alpha,p}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}$ for any $z \in D$, and the equality holds if and only if f is constant multiple of $f_w(z) = (\frac{1-|w|^2}{(1-\overline{w}z)^2})^{\frac{2+\alpha}{p}}$, where $w \in D$.*

Lemma 2.2 *A closed set E in $A_0^\infty(\varphi)$ is compact if and only if E is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in E} \varphi(|z|)|f(z)| = 0.$$

The proof of Lemma 2.2 is similar to that of Lemma 2.1 in [2] and here is omitted.

Theorem 2.1 *Let ϕ be an analytic self-map of D and $u \in H(D)$, $0 < p < \infty$, $-1 < \alpha < \infty$. Then uC_ϕ is bounded from A_α^p to $A^\infty(\varphi)$ if and only if*

$$\sup_{z\in D} \frac{\varphi(|z|)|u(z)|}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} < \infty.$$

Proof Sufficiency. Assume $f \in A_\alpha^p$. By Lemma 2.1, we have

$$|f(z)| \le \frac{C\|f\|_{\alpha,p}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}, \quad \forall\ z \in D.$$

Therefore,

$$\begin{aligned}\|uC_\phi(f)\|_\varphi &= \sup_{z\in D} \varphi(|z|)|u(z)||f(\phi(z))| \\ &\le \sup_{z\in D} \varphi(|z|)|u(z)| \frac{C\|f\|_{\alpha,p}}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} \\ &\le C\|f\|_{\alpha,p}.\end{aligned}$$

It follows that uC_ϕ is a bounded operator.

Necessity. Assume that $uC_\phi : A_\alpha^p \to A^\infty(\varphi)$ is bounded. Let $f(z) \equiv 1$. Then $f \in A_\alpha^p$ and $uC_\phi(f) = u \in A^\infty(\varphi)$. Let $w = \phi(\lambda)$ ($\lambda \in D$), $f_w(z) = (\frac{1-|w|^2}{(1-\overline{w}z)^2})^{\frac{2+\alpha}{p}}$. It is easy to see that

$$\|uC_\phi(f_w)\|_\varphi \le C\|f_w\|_{\alpha,p} = C.$$

Thus

$$\varphi(|z|)|u(z)||f_w(\phi(z))| \le C, \quad \forall\ z \in D.$$

Choose $z = \lambda$, we have

$$\frac{\varphi(|\lambda|)|u(\lambda)|}{(1-|\phi(\lambda)|^2)^{\frac{2+\alpha}{p}}} \le C.$$

For any fixed $\delta \in (0,1)$, the above inequality indicates that

$$\sup\{\frac{\varphi(|\lambda|)|u(\lambda)|}{(1-|\phi(\lambda)|^2)^{\frac{2+\alpha}{p}}} : \lambda \in D, |\phi(\lambda)| > \delta\} < \infty.$$

On the other hand, since $u \in A^\infty(\varphi)$, we conclude

$$\frac{\varphi(|\lambda|)|u(\lambda)|}{(1-|\phi(\lambda)|^2)^{\frac{2+\alpha}{p}}} \le \frac{\varphi(|\lambda|)|u(\lambda)|}{(1-\delta^2)^{\frac{2+\alpha}{p}}} \le \frac{\|u\|_\varphi}{(1-\delta^2)^{\frac{2+\alpha}{p}}},$$

where $|\phi(\lambda)| \le \delta$. Therefore,

$$\sup\{\frac{\varphi(|\lambda|)|u(\lambda)|}{(1-|\phi(\lambda)|^2)^{\frac{2+\alpha}{p}}} : \lambda \in D, |\phi(\lambda)| \le \delta\} < \infty.$$

Consequently,

$$\sup_{z\in D} \frac{\varphi(|z|)|u(z)|}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} < \infty.$$

Corollary 2.1 *Let ϕ be an analytic self-map of D, $0 < p < \infty$, $-1 < \alpha < \infty$. Then C_ϕ is bounded from A^p_α to $A^\infty(\varphi)$ if and only if*

$$\sup_{z\in D} \frac{\varphi(|z|)}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} < \infty.$$

Theorem 2.2 *Let ϕ be an analytic self-map of D and $u \in H(D)$, $0 < p < \infty$, $-1 < \alpha < \infty$. Then uC_ϕ is compact from A^p_α to $A^\infty(\varphi)$ if and only if $u \in A^\infty(\varphi)$ and*

$$\lim_{|\phi(z)|\to 1} \frac{\varphi(|z|)|u(z)|}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} = 0.$$

Proof Sufficiency. Let $\|f_n\|_{\alpha,p} \le C$ and $\{f_n\}$ converges to 0 uniformly on compact subsets of D. It suffices to prove that $\lim_{n\to\infty} \|uC_\phi(f_n)\|_\varphi = 0$. By the assumption, for any $\varepsilon > 0$ there exists $r \in (0,1)$, such that for $|\phi(z)| > r$,

$$\frac{\varphi(|z|)|u(z)|}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} < \varepsilon.$$

Using Lemma 2.1 for any n and $|\phi(z)| > r$ gives

$$\varphi(|z|)|uC_\phi(f_n(z))| \le C\frac{\varphi(|z|)|u(z)|\|f_n\|_{\alpha,p}}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} \le C\varepsilon.$$

On the other hand, since $\{f_n\}$ converges to 0 uniformly on compact subsets $\{w : |w| \le r\}$ of D, there exists N, for above ε and $n \ge N$,

$$|f_n(\phi(z))| < \varepsilon, \quad |\phi(z)| \le r.$$

So, for $|\phi(z)| \le r$ and $n \ge N$, we have

$$\varphi(|z|)|uC_\phi(f_n(z))| \le \varphi(|z|)|u(z)|\varepsilon.$$

Since $u \in A^\infty(\varphi)$, we deduce that $\lim_{n\to\infty} \|uC_\phi(f_n)\|_\varphi = 0$.

Necessity. Assume that $uC_\phi : A^p_\alpha \to A^\infty(\varphi)$ is compact. By choosing $f(z) \equiv 1$ we obtain $u \in A^\infty(\varphi)$. If

$$\lim_{|\phi(z)|\to 1} \frac{\varphi(|z|)|u(z)|}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} \neq 0,$$

then there exist $\delta > 0$ and $\{z_n\} \subset D$, such that $|\phi(z_n)| \to 1$ $(n \to \infty)$, and for any n,

$$\frac{\varphi(|z_n|)|u(z_n)|}{(1-|\phi(z_n)|^2)^{\frac{2+\alpha}{p}}} \ge \delta.$$

Let

$$w_n = \phi(z_n), \quad f_n(z) = \left(\frac{1-|w_n|^2}{(1-\overline{w_n}z)^2}\right)^{\frac{2+\alpha}{p}}.$$

It is easy to see that , $\|f_n\|_{\alpha,p} = 1$ and $f_n \to 0$ on compact subsets of D. Thus, $\lim_{n\to\infty} \|uC_\phi(f_n)\|_\varphi = 0$. But, for above $\delta > 0$ and any n, we have

$$\begin{aligned}\|uC_\phi(f_n)\|_\varphi &\ge \varphi(|z_n|)|u(z_n)||f_n(\phi(z_n))| \\ &= \varphi(|z_n|)|u(z_n)|\frac{1}{(1-|\phi(z_n)|^2)^{\frac{2+\alpha}{p}}} \\ &\ge \delta,\end{aligned}$$

which contradicts the compactness of the weighted composition operator uC_ϕ. Therefore,

$$\lim_{|\phi(z)|\to 1} \frac{\varphi(|z|)|u(z)|}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} = 0.$$

Corollary 2.2 *Let ϕ be an analytic self-map of D, $0 < p < \infty$, $-1 < \alpha < \infty$. Then C_ϕ is compact from A^p_α to $A^\infty(\varphi)$ if and only if*

$$\lim_{|\phi(z)|\to 1} \frac{\varphi(|z|)}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} = 0.$$

Theorem 2.3 *Let ϕ be an analytic self-map of D and $u \in H(D)$, $0 < p < \infty$, $-1 < \alpha < \infty$. Then the following statements are equivalent:*

(1) $uC_\phi : A^p_\alpha \to A^\infty_0(\varphi)$ is bounded;

(2) $uC_\phi : A^p_\alpha \to A^\infty_0(\varphi)$ is compact;

(3) $\lim_{|z|\to 1} \frac{\varphi(|z|)|u(z)|}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} = 0$.

Proof (3) $\Rightarrow$ (2) Let $f \in A^p_\alpha$ and $\|f\|_{\alpha,p} \le 1$. By Lemma 2.1, we have

$$\varphi(|z|)|u(z)||f(\phi(z))| \le C\|f\|_{\alpha,p} \frac{\varphi(|z|)|u(z)|}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}}.$$

From the assumption, it follows

$$\lim_{|z|\to 1} \varphi(|z|)|u(z)||f(\phi(z))| = 0.$$

So, $uC_\phi(f) \in A^\infty_0(\varphi)$.

Suppose $E = \{uC_\phi(f) : \|f\|_{\alpha,p} \le 1\}$. For any $\{g_n\} \subset E$ converging to an analytic function g, there exists $\{f_n \in A^p_\alpha : \|f_n\|_{\alpha,p} \le 1\}$, such that $g_n = uC_\phi(f_n)$. By Lemma 2.1 and the Montel's theorem, there exists $\{f_{n_k}\}$ converging to an analytic function f uniformly on compact subsets of D. Thus we have

$$\begin{aligned}
&(\alpha+1)\int_D (1-|z|^2)^\alpha |f(z)|^p \mathrm{d}A(z)\\
&= (\alpha+1)\int_D \lim_{k\to\infty}(1-|z|^2)^\alpha |f_{n_k}(z)|^p \mathrm{d}A(z)\\
&\le \underline{\lim}_{k\to\infty}(\alpha+1)\int_D (1-|z|^2)^\alpha |f_{n_k}(z)|^p \mathrm{d}A(z)\\
&\le 1,
\end{aligned}$$

which implies that $f \in A^p_\alpha$ and $\|f\|_{\alpha,p} \le 1$. By Lemma 2.1 and the assumption, for any $\varepsilon > 0$, there exists $\delta > 0$, such that for $\delta < |z| < 1$, we have

$$|\varphi(|z|)u(z)(f_{n_k}(\phi(z)) - f(\phi(z)))| \le C\frac{\varphi(|z|)|u(z)|}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}} < \varepsilon.$$

For $|z| \le \delta$, there exists $K > 0$, such that for $k > K$,

$$|\varphi(|z|)u(z)(f_{n_k}(\phi(z)) - f(\phi(z)))| < \varepsilon.$$

It follows that

$$\|uC_\phi(f_{n_k}) - uC_\phi(f)\|_\varphi \to 0, \quad k\to\infty.$$

Therefore $g = uC_\phi(f)$ and $g \in E$. So E is the closed subset in $A^\infty_0(\varphi)$. Since

$$\lim_{|z|\to 1}\sup_{\|f\|_{\alpha,p}\le 1}\varphi(|z|)|u(z)||f(\phi(z))| = 0,$$

using Lemma 2.2, we have $uC_\phi : A^p_\alpha \to A^\infty_0(\varphi)$ is compact.

(2) $\Rightarrow$ (1) It is not difficult and omitted.

(1) $\Rightarrow$ (3) Suppose

$$f_n(z) = \Big(\frac{1-|\phi(\lambda_n)|^2}{(1-\overline{\phi(\lambda_n)}z)^2}\Big)^{\frac{2+\alpha}{p}},$$

where $\lambda_n \in D$ and $|\lambda_n| \to 1$ $(n\to\infty)$. Thus $f_n \in A^p_\alpha$ and $uC_\phi(f_n) \in A^\infty_0(\varphi)$. Consequently, for any $n \in N$, we have

$$\lim_{|z|\to 1}\varphi(|z|)|u(z)f_n(\phi(z))| = 0,$$

that is, for any $\varepsilon > 0$, there exists $\delta > 0$, such that for $\delta < |z| < 1$, we have

$$\varphi(|z|)|u(z)f_n(\phi(z))| < \varepsilon, \quad \forall\, n \in N.$$

Since $|\lambda_n| \to 1$ $(n\to\infty)$, there exists $N > 0$, for $n > N$, we get $\delta < |\lambda_n| < 1$. Thus

$$\frac{\varphi(|\lambda_n|)|u(\lambda_n)|}{(1-|\phi(\lambda_n)|^2)^{\frac{2+\alpha}{p}}} = \varphi(|\lambda_n|)|u(\lambda_n)f_n(\phi(\lambda_n))| < \varepsilon.$$

Therefore,

$$\lim_{n\to\infty}\frac{\varphi(|\lambda_n|)|u(\lambda_n)|}{(1-|\phi(\lambda_n)|^2)^{\frac{2+\alpha}{p}}}=0.$$

Consequently,

$$\lim_{|z|\to 1}\frac{\varphi(|z|)|u(z)|}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}}=0.$$

Corollary 2.3 *Let ϕ be an analytic self-map of D, $0 < p < \infty$, $-1 < \alpha < \infty$. Then the following statements are equivalent:*

(1) $C_\phi : A^p_\alpha \to A^\infty_0(\varphi)$ *is bounded;*

(2) $C_\phi : A^p_\alpha \to A^\infty_0(\varphi)$ *is compact;*

(3) $\lim_{|z|\to 1}\frac{\varphi(|z|)}{(1-|\phi(z)|^2)^{\frac{2+\alpha}{p}}}=0$.

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