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Normality Family and Shared Functions by Meromorphic Functions and Its Differential Polynomials

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Abstract In this paper, we obtain the following normal criterion: Let $\mathcal{F}$ be a family of meromorphic functions in domain $D \subset \mathbf{C}$, all of whose zeros have multiplicity $k+1$ at least. If there exist holomorphic functions $a(z)$ not vanishing on D, such that for every function $f(z) \in \mathcal{F}$, $f(z)$ shares $a(z)$ IM with $L(f)$ on D, then $\mathcal{F}$ is normal on D, where $L(f)$ is linear differential polynomials of $f(z)$ with holomorphic coefficients, and k is some positive numbers. We also proved coressponding results on normal functions.

Keywords meromorphic functions; differential polynomials; normal criterion; shared functions.

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1. Introduction and main results

Suppose that $f(z)$ is meromorphic functions in plane domain $D \subset \mathbf{C}$, and a is a complex value, $a \in \mathbf{C}$. Let

$$\overline{E}_f(a) = \{f^{-1}(a)\} \cap D = \{z \in D | f(z) = a\}.$$

We say that f shares a IM with g in D if $\overline{E}_f(a) = \overline{E}_g(a)$.

Fang[1] obtained the following result.

Theorem A *Suppose that $\mathcal{F}$ is a family of meromorphic functions in plane domain $D \subset \mathbf{C}$ and a is a nonzero complex value. For every function $f \in \mathcal{F}$, if $f(z) \neq 0$, and $\overline{E}_f(a) = \overline{E}_{f^{(k)}}(a)$, then $\mathcal{F}$ is normal in D, where k is a positive integer.*

Fang and Zalcman[2] extended Theorem A into the case that $f(z)$ has mulitple zeros and obtained the following result.

Theorem B *Let $\mathcal{F}$ be a family of meromorphic functions in domain $D \subset \mathbf{C}$, all of whose zeros have multiplicity $k+1$ at least. If for every function $f \in \mathcal{F}$, $\overline{E}_f(a) = \overline{E}_{f^{(k)}}(b)$, $a \neq 0$ and $b \neq 0$, then $\mathcal{F}$ is normal in D.*

Suppose that $a_i(z)(i = 1, 2, \ldots, k)$ are analytic functions in D. Let

$$L(f) = a_k(z)f^{(k)}(z) + a_{k-1}(z)f^{(k-1)}(z) + \cdots + a_1(z)f'(z).$$

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Then we call $L(f)$ a linear differential polynomial about f with holomorphic coefficients.

In this paper, with the method similar to the one used by Fang and Zalcman in [2], allowing functions $f \in \mathcal{F}$ in Theorem A to have multiple zeros, we shall consider whether Theorem A still holds when "$f^{(k)}$" is replaced by $L(f)$ and the complex value a is also replaced by holomorphic function $a(z)$ which does not vanish in D, and obtain our first result as follows.

Theorem 1.1 *Suppose that $\mathcal{F}$ is a family of functions meromorphic in plane domain $D \subset \mathbf{C}$, all of whose zeros have multiplicity $k+1$ at least. If there exists a holomorphic function $a(z)$ in D, which does not vanish in D, such that $\overline{E}_f(a(z)) = \overline{E}_{L(f)}(a(z))$, then $\mathcal{F}$ is normal in D, where k is a positive integer and $L(f)$ is a linear differential polynomial about f defined above.*

Theorem 1.1 gives some abundant conditions on which $\mathcal{F}$ is normal when the linear differential polynomial $L(f)$ about f with holomorphic coefficients shares the holomorphic function $a(z)$ IM with $f(z)$. For these conditions, we have the following notes:

Remark 1 The restrictions of multiple zeros of $f(z)$ in Theorem 1.1 is essential. For example, taking a family of functions[3]

$$\mathcal{F} = \{f_n(z) | f_n(z) = \frac{e^{(n+1)z} - a}{n+1} + a\},$$

we can see that zeros of $f_n(z)$ are simple and $\overline{E}_{f_n}(a(z)) = \overline{E}_{f'_n}(a(z))$, but $\mathcal{F}$ is not normal in the unit disc Δ, here a is a nonzero finite complex number.

Remark 2 For a positive integer k, $k \geq 2$, and a family of $\mathcal{F}$, $\mathcal{F} = \{f_n(z) | f_n(z) = n^{k-1} z^{k-1} e^z, n, k \in \mathbf{N}\}$, it is clear that zeros of $f_n(z)$ have multiplicity $k-1$. Writing $L(f_n(z))$ as a linear differential polynomial,

$$L(f_n(z)) = \sum_{i=1}^{k} (-1)^{k-i} C_k^i f_n^{(i)}(z),$$

we have $L(f_n(z)) \equiv f_n(z)$, so $f_n(z)$ shares any $a(z)$ IM with $L(f_n(z))$, but $\mathcal{F}$ is not normal in Δ. This also implies that the restrictions of zeros of $f(z)$ in Theorem 1.1 is necessary for the case $k \geq 2$.

Remark 3 The requirement of which holomorphic functions $a(z)$ does not assume zero is also necessary. For example, for a family $\mathcal{F}$, $\mathcal{F} = \{f_n(z) | f_n(z) = n^{k+1} z^{k+1},\ n, k \in \mathbf{N}\}$, and holomorphic function $a(z) = z$, it is clear that all zeros of $f_n(z)$ have multiplicity $k+1$, and $\overline{E}_{f_n}(a(z)) = \overline{E}_{f_n^{(k)}}(a(z))$, but $\mathcal{F}$ is not normal in Δ.

In fact, by the same method as used in the proof of Theorem 1.1, we may obtain a corresponding result as follows.

Theorem 1.2 *Let $\mathcal{F}$ be a family of functions meromorphic in plane domain $D \subset \mathbf{C}$, all of whose zeros have multiplicity $k+1$ at least, and let $a(z) \neq 0$ and $b(z) \neq 0$ be two holomorphic functions which do not take zero value in D. If for every function $f \in \mathcal{F}$, $\overline{E}_f(a(z)) = \overline{E}_{L(f)}(b(z))$, then $\mathcal{F}$ is normal in Δ, where k is a positive integer and $L(f)$ is a linear differential polynomial about $f(z)$ defined above.*

For meromorphic functions $f(z)$ in Δ, we call it a normal functions if there exists positive numbers $M > 0$, such that $(1-|z|)f^{\#}(z) \leq M$ for any $z \in \mathbf{C}$, where $f^{\#}(z) = |f'(z)|/(1+|f(z)|^2)$ is a spherical derivative of $f(z)$.

Suppose that there exists a property P of function families $\mathcal{F}$ in the plane domain D, with which a family $\mathcal{F}$ is normal in D. Then for every function $f \in \mathcal{F}$, Pang and Bergweiler bring forward whether f is normal function. In this paper, for the case that $a(z)$ and coefficients of a linear differential polynomial $L(f)$ about $f(z)$ are constants, attaching a condition that $\overline{E}_{L(f)}(0) \subset \overline{E}_f(0)$, we obtain that function $f(z)$ which satisfies the conditions in Theorem 1.1 or Theorem 1.2 must be a normal function. But it is valuable to consider whether the extra condition $\overline{E}_{L(f)}(0) \subset \overline{E}_f(0)$ is essential.

Theorem 1.3 *Let $f(z)$ be meromorphic function in Δ, all of whose zeros have mulitiplicity $k+1$ at least, and $\overline{E}_{L(f)}(0) \subset \overline{E}_f(0)$. Suppose that for nonzero complex numbers $a \neq 0$ and $b \neq 0$, $\overline{E}_f(a) = \overline{E}_{L(f)}(b)$. Then $f(z)$ is normal function in Δ. Where*

$$L(f) = a_k f^{(k)}(z) + a_{k-1} f^{(k-1)}(z) + \cdots + a_1 f'(z),$$

here $a_1, a_2, \ldots, a_k$ are complex constants, $a_k \neq 0$ and k is a positive integer with $k \geq 2$.

2. Some lemmas

To complete the proof of Theorem 1.1, we need some lemmas as follows.

Lemma 2.1[4] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ, all of whose zeroes have multiplicity at least k, and suppose there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$:*

(a) A number r, $0 < r < 1$;

(b) Points z_n, $|z_n| < r$;

(c) Functions $f_n \in \mathcal{F}$, and

(d) Positive numbers $\rho_n \to 0$ such that

$$\frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\alpha}} = g_n(\xi) \to g(\xi)$$

locally uniform with respect to the spherical metric, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbf{C}$ such that

$$g^{\#}(\xi) \leq g^{\#}(0) = kA + 1.$$

Lemma 2.2[3] *Suppose that $R(z)$ is a nonconstant rational function, all of whose zeros have multiplicity $k+1$ at least. If for any nonzero complex constant $b, b \neq 0$, $R^{(k)}(z) \neq b$, then*

$$R(z) = \frac{(\gamma z + \delta)^{k+1}}{\alpha z + \beta}.$$

Where $\alpha, \beta, \delta, \gamma$ are some complex constants, $\gamma^{k+1} k! = \alpha b$, $\alpha\gamma \neq 0$, $|\alpha| + |\delta| \neq 0$, and k is a positive integer.

Lemma 2.3[5] *Suppose that $f(z)$ is a meromorphic function with finite orders, all of whose zeros have multiplicity $k+1$ at least. Then $f^{(k)}(z)$ assumes any nonzero finite value b finite times, where k is a positive integer.*

Lemma 2.4[6,7] *Suppose that $f(z)$ is a meromorphic function in the plane, and k is a positive integer. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = e^{\alpha z+\beta}$ or $f(z) = (\alpha z+\beta)^{-n}$, where $\alpha \neq 0$, β are two complex numbers, and n is a positive integer.*

3. Proofs of Theorems

3.1 Proof of Theorem 1.1

Suppose $\mathcal{F}$ is not normal at $z_0 \in \Delta$. Without loss of generality, we write $z_0 = 0$, then from Lemma 2.1, there exist $f_n \in \mathcal{F}$, point sequences $z_n \to 0$, and positive numbers $\rho_n \to 0^+$, such that $\rho_n^{-\alpha} f_n(z_n + \rho_n(\xi))$ locally uniformly converges on functions $g(\xi)$ with respect to the spherical metric, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbf{C}$, whose spherical derivative satisfies $g^{\#}(\xi) \leq g^{\#}(0) = kA+1$, that is, $g(\xi)$ is of finite order. We may assert that the following conclusion is true.

(i) All zeros of $g(\xi)$ have multiplicity $k+1$ at least;

(ii) $g^{(k)}(\xi) \neq 1$;

(iii) All poles of $g(\xi)$ are multiple.

In fact, suppose that point $\xi_0 \in \mathbf{C}$ is a zero of $g(\xi)$, $g(\xi_0) = 0$. Since $g(\xi)$ is not identical constant, there exists ξ_n such that $g_n(\xi_n) = \rho_n^{-k} f_n(z_n + \rho_n \xi)$ for n sufficiently large, that is, $f_n(z_n + \rho_n \xi) = 0$. Because all zeros of f have multiplicity $k+1$ at least, $f_n^{(m)}(z_n + \rho_n \xi) = 0$, $m = 1, 2, \ldots, k$. Thus,

$$g_n^{(m)}(\xi) = \rho_n^{m-k} f_n^{(m)}(z_n + \rho_n \xi) = 0, \quad m = 1, 2, \ldots, k.$$

Then $g^{(m)}(\xi_0) = 0, m = 1, 2, \ldots, k$, that is, all zeros of $g(\xi)$ have multiplicity $k+1$ at least. The assertion (i) holds.

Suppose that there exists $\xi_0 \in \mathbf{C}$ such that $g^{(k)}(\xi_0) = 1$. Then $g^{(k)}(\xi_0) \neq \infty$. We can see that $g^{(k)}(\xi)$ is not identical constant 1. Otherwise, $g^{(k)}(\xi) \equiv 1$ for any $\xi \in \mathbf{C}$. Therefore, $g(\xi)$ is a polynoimal about ξ with degree k, all of whose zeros have multiplicity k at most. This contradicts the assertion (i). Again because of

$$\begin{aligned} L(f_n)(z_n + \rho_n \xi) =& a_k(z_n + \rho_n \xi) f_n^{(k)}(z_n + \rho_n \xi) + \cdots + a_1(z_n + \rho_n \xi) f_n'(z_n + \rho_n \xi) \\ =& a_k(z_n + \rho_n \xi) g_n^{(k)}(z_n + \rho_n \xi) + \cdots + a_1(z_n + \rho_n \xi) \rho_n^{k-1} g_n'(z_n + \rho_n \xi), \end{aligned} \quad (3.1)$$

we have that $L(f)(z_n + \rho_n \xi) - a(z_n + \rho_n \xi) \to g^{(k)}(\xi) - 1$ locally uniformly with respect to the spherical metric. From Hurwitz Theorem, there exists $\xi_n \to \xi_0$ such that $L(f)(z_n + \rho_n \xi_n) = a(z_n + \rho_n \xi_n)$. Again from $\overline{E}_f(a(z)) = \overline{E}_{L(f)}(a(z))$, we deduce that $f_n(z_n + \rho_n \xi_n) = a(z_n + \rho_n \xi_n)$, that is,

$$g_n(\xi_n) = \rho_n^{-k} f_n(z_n + \rho_n \xi_n) = \rho_n^{-k} a(z_n + \rho_n \xi_n).$$

Thereby, $g(\xi_0) = \infty$, this contradicts that $g^{(k)}(\xi_0) \neq \infty$, so the assertion (ii) holds.

In the sequel, we shall prove that all poles of $g(\xi)$ are multiple.

If there exists ξ_0 ($\xi_0 \in \mathbf{C}$), such that $g(\xi_0) = \infty$, then we shall deduce that $(g(\xi)^{-1})' \mid_{\xi=\xi_0} = 0$.

Since $g(\xi)$ is not identical ∞, there exists a set $K = \{\xi : |\xi - \xi_0| < \delta\}$ in which $\frac{1}{g(\xi)}$ and $\frac{1}{g_n(\xi)}$ are holomorphic for n sufficiently large, and $\frac{1}{g_n(\xi)}$ uniformly converges to $\frac{1}{g(\xi)}$. Therefore,

$$\frac{1}{g_n(\xi)} - \frac{\rho_n^k}{a(z_n + \rho_n \xi)^{-1}} \longrightarrow \frac{1}{g(\xi)}$$

with convergence being uniform in K. From $\frac{1}{g(\xi)} \not\equiv \infty$, there exists ξ_n in K, $\xi_n \to \xi_0$, such that $g_n(\xi_n)^{-1} - \rho_n^k a(z_n + \rho_n \xi_n)^{-1} = 0$ for n sufficiently large, thus $f(z_n + \rho_n \xi_n) = a(z_n + \rho_n \xi_n)$. Then

$$L(f_n)(z_n + \rho_n \xi_n) = a(z_n + \rho_n \xi_n). \tag{3.2}$$

Writing

$$l(g_n(\xi)) = a_k(z_n + \rho_n \xi) g_n^{(k)}(\xi) + a_k(z_n + \rho_n \xi)\rho_n g_n^{(k)}(\xi) + \cdots + a_1(z_n + \rho_n \xi)\rho_n^{k-1} g_n'(\xi), \tag{3.3}$$

$$\frac{1}{g(\xi)} \equiv F(\xi), \quad \frac{1}{g_n(\xi)} \equiv F_n(\xi). \tag{3.4}$$

For the case $k = 1$, Theorem 1.1 is just Theorem A, so we omit the details. For the case $k = 2$, by (3.4), we have

$$g'(\xi) = -F' g^2, \tag{3.5}$$

$$g''(\xi) = -F'' g^2 + 2(F')^2 g^3, \quad F'' = -g'' g^{-2} + 2(F')^2 g, \tag{3.6}$$

thus $l(F) = a_2 F'' + a_1 \rho_n F' = -l(g) g^{-2} + 2a_2 (F')^2 g$. Then we have

$$\begin{aligned} \lim_{n\to+\infty} l(F_n(\xi_n)) &= \lim_{n\to+\infty} [-l(g_n) g_n^{-2} + 2a_2(F_n')^2 g_n(\xi_n)] \\ &= \lim_{n\to+\infty} [a(z_n + \rho_n \xi_n) g_n^{-2}(\xi_n) + 2a_2(F_n')^2 g_n(\xi_n)] \\ &= \lim_{n\to+\infty} [2a_2(F_n')^2 g_n(\xi_n)]. \end{aligned}$$

From $\lim_{n\to+\infty} g_n(\xi_n) = \infty$, we deduce that $g'(\xi_0) g^{-2}(\xi_0) = \lim_{n\to+\infty} (g_n' g_n^{-2})^2 = 0$, that is, ξ_0 is of multiple pole of $g(\xi)$ with order 2 at least.

For the case $k \geq 3$, from Equs. (3.5) and (3.6), and by mathematical inductive method, it follows that

$$g^{(k)} = -F^{(k)} g^2 + A_{k3} g^3 + A_{k4} g^4 + \cdots + A_{kk} g^k + (-1)^k k!(F')^k g^{k+1},$$

where $A_{kj} (j = 1, 2, \ldots, k)$ are some polynomials about $F', F'', \ldots, F^{(k)}$. Thus,

$$F^{(k)} = -g^{(k)} g^{-2} + (-1)^k k!(F')^k g^{k-1} A_{k3} g^1 + A_{k4} g^2 + \cdots + A_{kk} g^{k-2}. \tag{3.7}$$

Setting $B_0 = -a_1 \rho_n^{k-1} F'$, $B_t = (-1)^{t+1}(t+1)! a_{t+1} \rho_n^{k-t-1} (F')^{t+1} + \sum_{i=3}^k a_i \rho_n^{k-i} A_{i,t+2}$, $t = 1, 2, \ldots, k-2$, by Equs. (3.3)–(3.7) and the above signs, we have

$$l(F_n) = l(g_n^{-1}) = -l(g_n) g_n^{-2} + (-1)^k k! a_k (F')^k g_n^{k-1} + \sum_{t=0}^{k-2} B_t g_n^t. \tag{3.8}$$

Again by Equs. (3.1) and (3.2), Equ. (3.8) implies

$$l(F_n(\xi_n)) = -a(z_n+\rho_n\xi_n)g_n^{-2} + (-1)^k k! a_k (F')^k g_n^{k-1} + \sum_{t=0}^{k-2} B_t g_n^t.$$

From $\lim_{n\to+\infty} g_n(\xi_n) = \infty$ and $\lim_{n\to+\infty} B_0(\xi_n) = B_0(\xi_0) = 0$, we have

$$\lim_{n\to+\infty}[l(F_n(\xi_n))] = \lim_{n\to+\infty}\big\{[(-1)^k k! a_k (F')^k g_n^{k-2} + \sum_{t=1}^{k-2} B_t g_n^{t-1}]g_n\big\}.$$

Therefore, we have

$$\lim_{n\to+\infty}[(-1)^k k! a_k (F')^k g_n^{k-2} + \sum_{t=1}^{k-2} B_t g_n^{t-1}] = 0.$$

Similarly, from $\lim_{n\to+\infty} B_1(\xi_n) = B_1(\xi_0)$, we have

$$\lim_{n\to+\infty}[l(F_n(\xi_n))] = \lim_{n\to+\infty}\big\{[(-1)^k k! a_k (F')^k g_n^{k-3} + \sum_{t=2}^{k-2} B_t g_n^{t-2}]g_n\big\} = -B_1(\xi_0).$$

Again by $\lim_{n\to+\infty} g_n(\xi_n) = \infty$, we also have

$$\lim_{n\to+\infty}[(-1)^k k! a_k (F')^k g_n^{k-3} + \sum_{t=2}^{k-2} B_t g_n^{t-2}] = 0.$$

Going on with the similar duduction step by step, we have $\lim_{n\to+\infty}[(-1)^k k! a_k (F')^k] = 0$. Thereby, $(g^{-1})'|_{\xi=\xi_0} = \lim_{n\to+\infty} F_n'(\xi_n) = 0$, that is, ξ_0 is of multiple poles of $g(\xi)$, thus the assertion (iii) also holds.

Since $g(\xi)$ is of finite order, by assertions (i) and (ii), and Lemma 2.3, we have that $g(\xi)$ must be nonconstant rational function. Again from Lemma 2.2, we deduce that $g(\xi)$ only has simple poles, which contradicts that all poles of $g(\xi)$ are multiple. This completes proof of Theorem 1.1, thus $\mathcal{F}$ is normal in D.

3.2 Proof of Theorem 1.3

If $f(z)$ is not normal function in Δ, then there exists z_n, $|z_n|<1$, such that $\lim_{n\to+\infty}(1-|z_n|)f^{\#}(z_n) = \infty$. Let $f_n(z) = f(z_n + (1-|z_n|)z)$ and $\mathcal{F} = \{f_n\}$. By Marty's criterion, it is not difficult to see that $\mathcal{F}$ is not normal at $z=0$. From Lemma 2.1, there exists point sequences $\xi_n \to 0$, $\rho_n \to 0^+$, and one subsequence of $\mathcal{F}$, still denoted $\{f_n\}$ for this subsequence, such that

$$g_n(\xi) = f_n(\xi_n + \rho_n\xi) = f(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi) \to g(\xi)$$

locally uniformly with respect to the spherical metric, where $g^{(k)}(\xi)$ is not identical zeros. Then all zeros of $g(\xi)$ have multiplicity $k+1$ at least. We may assert that $\overline{E}_g(a) = \overline{E}_{g^{(k)}}(0) \subset \overline{E}_g(0)$.

(i) $\overline{E}_g(a) = \overline{E}_{g^{(k)}}(0)$.

Suppose that there exists $\xi_0 \in \mathbf{C}$ such that $g^{(k)}(\xi_0) = 0$. Since

$$\begin{aligned} &a_k g_n^{(k)}(\xi) + a_k(1-|z_n|)\rho_n g_n^{(k-1)}(\xi) + \cdots + a_1\left[(1-|z_n|)\rho_n\right]^{k-1} g_n'(\xi) - \left[(1-|z_n|)\rho_n\right]^k b \\ &\quad = \left[(1-|z_n|)\rho_n\right]^k L(f)(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi) - \left[(1-|z_n|)\rho_n\right]^k b \longrightarrow g^{(k)}(\xi) \end{aligned}$$

locally uniformly with respect to the spherical metric, and $g^{(k)}(\xi)$ is not identically zero, by Rouché Theorem, there exists point sequence $\xi_n^* \to \xi_0$, such that

$$[(1-|z_n|)\rho_n]^k L(f)(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi_n^*) - [(1-|z_n|)\rho_n]^k b = 0,$$

for n sufficiently large, that is, $L(f)(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi_n^*) = b$. Again from $\overline{E}_f(a) = \overline{E}_{L(f)}(b)$, we have

$$g_n(\xi_n^*) = f(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi_n^*) = a.$$

Thus, $g(\xi_0) = a$, that is, $\overline{E}_{g^{(k)}}(0) \subset \overline{E}_g(a)$.

Similarly, suppose there exists $\xi_0 \in \mathbf{C}$ such that $g(\xi_0) = a$. Since

$$g_n(\xi) - a = f(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi) - a \to g(\xi) - a$$

locally uniformly with respect to the spherical metric, and $g^{(k)}(\xi) \not\equiv 0$, $g(\xi) - a \not\equiv 0$. Then, from Rouché Theorem we have that there exists point sequence $\xi_n^* \to \xi_0$, such that

$$g_n(\xi_n^*) - a = f(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi_n^*) - a = 0$$

for n sufficiently large. Again from $\overline{E}_f(a) = \overline{E}_{L(f)}(b)$, we have

$$L(f)(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi_n^*) = b,$$

that is,

$$[(1-|z_n|)\rho_n]^k L(f)(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi_n^*) = [(1-|z_n|)\rho_n]^k b.$$

Then $g^{(k)}(\xi_0) = 0$. This shows that $\overline{E}_g(a) \subset \overline{E}_{g^{(k)}}(0)$, so the assertion (i) holds.

(ii) $\overline{E}_{g^{(k)}}(0) \subset \overline{E}_g(0)$.

By the same argument as the first part in proof of the assertion (i), suppose that there exists $\xi_0 \in \mathbf{C}$ such that $g^{(k)}(\xi_0) = 0$. Since

$$\begin{aligned} &a_k g_n^{(k)}(\xi) + a_k(1-|z_n|)\rho_n g_n^{(k-1)}(\xi) + \cdots + a_1 [(1-|z_n|)\rho_n]^{k-1} g_n'(\xi) \\ &= [(1-|z_n|)\rho_n]^k L(f)(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi) \longrightarrow g^{(k)}(\xi) \end{aligned}$$

locally uniformly with respect to the spherical metric, where $g^{(k)}(\xi) \not\equiv 0$, and by Rouché Theorem, we have that there exists point sequence $\xi_n^* \to \xi_0$, such that

$$[(1-|z_n|)\rho_n]^k L(f)(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi_n^*) = 0$$

for n sufficiently large. That is, $L(f)(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi_n^*) = 0$.

From $\overline{E}_{L(f)}(0) \subset \overline{E}_f(0)$, we may deduce that

$$g_n(\xi_n^*) = f(z_n + (1-|z_n|)\xi_n + (1-|z_n|)\rho_n\xi_n^*) = 0.$$

Then $g(\xi_0) = 0$. Thereby, $\overline{E}_{g^{(k)}}(0) \subset \overline{E}_g(0)$, which shows that the assertion (ii) also holds.

From the above assertions that $\overline{E}_g(a) = \overline{E}_{g^{(k)}}(0) \subset \overline{E}_g(0)$ and all zeros of $g(\xi)$ have multiplicity $k+1$ at least, we obtain $g^{(k)}(\xi) \neq 0$ and $g(\xi) \neq 0$.

Again from $\overline{E}_g(a) = \overline{E}_{g^{(k)}}(0)$, we have $g(\xi) \neq a$. On the other hand, from Lemma 2.4, we immediately obtain that the expression of $g(\xi)$ is either $g(\xi) = e^{\alpha z+\beta}$ or $g(\xi) = (\alpha z + \beta)^{-n}$.

Clearly, this contradicts $g(\xi) \neq a$. Therefore, $f(z)$ must be a normal function in Δ. So far, we give the complete proof of Theorem 1.3.

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