Journal of Mathematical Research & Exposition
Aug., 2008, Vol. 28, No. 3, pp. 579–588
DOI:10.3770/j.issn:1000-341X.2008.03.017
Http://jmre.dlut.edu.cn

Strong Convergence Theorems of Viscosity Approximation for Accretive Operators

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Abstract Let E be a real Banach space and let A be an m-accretive operator with a zero. Define a sequence $\{x_n\}$ as follows: $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_{r_n}x_n$, where $\{\alpha_n\}$, $\{r_n\}$ are sequences satisfying certain conditions, and J_r denotes the resolvent $(I+rA)^{-1}$ for $r > 1$. Strong convergence of the algorithm $\{x_n\}$ is obtained provided that E either has a weakly continuous duality map or is uniformly smooth.

Keywords fixed point; nonexpansive mapping; m-accretive operator; viscosity approximation; weakly continuous duality map; uniformly smooth Banach space.

Document code A
MR(2000) Subject Classification 47H06; 47H10
Chinese Library Classification O177

1. Introduction

In the sequel, we assume that E is a real Banach space with norm $\|\cdot\|$, denote the fixed point set by $F(T) = \{x \in E; Tx = x\}$, the weak convergence by $\rightharpoonup$, the strong convergence by $\rightarrow$.

A mapping T with its domain $D(T)$ and range $R(T)$ in E is called nonexpansive (respectively contractive) if for all $x, y \in D(T)$ such that $\|Tx - Ty\| \le \|x - y\|$ (respectively $\|Tx - Ty\| \le \alpha\|x - y\|$ for some $0 < \alpha < 1$). Let Π_C denote the set of all contractions on C.

A classical way to study the nonexpansive mappings is to use the following[1,2]: for t$\in (0,1)$, define a mapping T_t^f: $T_t x = tu + (1-t)Tx, x \in C$, where $u \in C$ is a fixed point. Banach's contraction mapping Principle guarantees that T_t has a fixed point x_t in C. In the case that T has a fixed point, Browder[1] proved that if E is a Hilbert space, then x_t does converge strongly to a fixed point of T that is nearest to u. Reich[2] extended Browder's result to a uniformly Banach space and the limit defines the unique sunny nonexpansive retraction from C onto $F(T)$. Very recently Xu[3] extended Reich's result to a Banach space which has a weakly continuous duality map. And Xu[3] proved strong convergence theorems by the following iterative method assuming that either E is uniformly smooth or E has a weakly continuous duality map: $x_{n+1} = \alpha_n u + (1-\alpha_n)J_{r_n}x_n$, $n \ge 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{r_n\}$ is a sequence of positive numbers, and the initial guess $x_0 \in C$ is arbitrarily.

Received date: 2006-06-02; **Accepted date**: 2006-08-28
Foundation item: the National Natural Science Foundation of China (No. 10771050).

Viscosity approximation methods for nonexpansive mappings or nonexpansive nonself-mapping have been studied by several authors. It is our purpose in this paper to use the method to approximate the fixed point of accretive operators which improves the recent results.

2. Preliminaries

Let $\varphi : [0, \infty) =: R^+ \to R^+$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Such a φ is called a gauge. Associated with a gauge φ is the duality map $J_\varphi : E \to E^*$ defined by

$$J_\varphi(x) = \{f \in E^* : \langle x, f\rangle = \|x\| \cdot \varphi(\|x\|),\ \|f\| = \varphi(\|x\|)\}, \quad x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between E and E^*. In the case that $\varphi(t) = t$, we write J for J_φ and call J the normalized duality map.

Definition 1.1 *A Banach space E is said to have a weakly continuous duality mapping if there exists a gauge function φ such that J_φ is single-valued and weak to weak star sequentially continuous.*

It is known that l^p $(1 < p < \infty)$ has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$. Setting

$$\Phi(t) = \int_0^t \varphi(\tau)\mathrm{d}\tau, \quad t \geq 0,$$

then one sees that Φ is a convex function and

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in E,$$

where ∂ denotes the subdifferential in the sense of convex analysis. The subdifferential inequality

$$\Phi(\|y\|) \geq \Phi(\|x\|) + \langle y - x, j_x\rangle, \quad x, y \in E, \quad j_x \in J_\varphi(x),$$

implies that the inequality

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_{x+y}\rangle, \quad x, y \in E, \quad j_{x+y} \in J_\varphi(x + y).$$

If E is smooth, then J_φ is single-valued and hence the inequality above is reduced to

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle, \quad x, y \in E.$$

It is well known that if E is a uniformly smooth, then J is single-valued and is uniformly continuous on bounded subsets of E. we shall denote the single-valued duality map by j.

Definition 1.2 *Let $S(E) = \{x \in E : \|x\| = 1\}, E$ is said to be uniformly smooth provided that the limit*

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for each $x, y \in S(E)$.

Definition 1.3 *An operator A with domain $D(A)$ and range $R(A)$ in X is said to be accretive if,* *for each $x_i \in D(A)$ and $y_i \in Ax_i (i = 1, 2)$, there exists a $j \in J(x_2 - x_1)$ such that $\langle y_2 - y_1, j\rangle \geq 0$, where J is the duality map from E to the dual space E^*.*

An accretive operator A is m-accretive if $R(I + \lambda A) = E$ for all $\lambda > 0$.

Denote by F the zero set of A; i.e.,

$$F := A^{-1}(0) = \{x \in D(A) : 0 \in Ax\}.$$

Denote by J_r the resolvent of A for $r > 0$: $J_r = (I + rA)^{-1}$.

It is known that J_r is a nonexpansive mapping from E to $C := \overline{D(A)}$.

For the proof of our main results, we shall need the following lemmas.

Lemma 2.1[2] *Let E be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $x \mapsto tu + (1 - t)Tx$ converges strongly as $t \to 0$ to fixed point of T. Define $Q : C \to F(T)$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$; that is, Q satisfies the property:*

$$\langle u - Qu, J(z - Qu)\rangle \leq 0,\ u \in C,\ z \in F(T).$$

Lemma 2.2[4] *Let $\{\alpha_n\}$ be a nonnegative real sequence that satisfies the condition: $\alpha_{n+1} \leq (1-\beta_n)\alpha_n + \beta_n\gamma_n$ for all $n \geq n_0$, where the sequence $\beta_n \in [0, 1]$, and $\{\gamma_n\}$ satisfies the conditions:*

(i) $\lim_{n\to\infty} \beta_n = 0$; (ii) $\Sigma_{n=0}^{\infty}\beta_n = \infty$; (iii) $\lim_{n\to\infty} \gamma_n = 0$, then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.3 (The Sub-differential Inequality) *Let E be a Banach space, J the normalized duality mapping from E into 2^{E^*}, $\forall x, y \in E$, $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \forall j(x + y) \in J(x + y)$.*

Lemma 2.4[5] (The Resolvent Identity) *For $\lambda, \mu > 0$, there holds the identity: $J_\lambda x = J_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda x)$, $x \in E$.*

Lemma 2.5[6] *Assume that $c_2 \geq c_1 > 0$. Then $\|J_{c_1}x - x\| \leq \|J_{c_2}x - x\|$ for all $x \in E$.*

Lemma 2.6[7] *Assume that E has a weakly continuous duality map J_φ with gauge φ.*

(i) For all $x, y \in E$, there holds the inequality

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle.$$

(ii) Assume a sequence $\{x_n\}$ in E is weakly convergent to a point x. Then there holds the identity

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|).$$

3. Main results

Let E be a real Banach space, C a nonempty closed convex subset of E and T be a nonexpansive mapping from C into itself with $F(T) \neq \emptyset$. For $t \in (0, 1)$ and $f \in \Pi_C$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) + (1 - t)Tx$ on C; that is

$$x_t = tf(x_t) + (1 - t)Tx_t. \tag{1}$$

Theorem 3.1 *Let E be a reflexive Banach space and has a weakly continuous duality map J_φ with a gauge φ. Let C be a closed convex subset of E and T be a nonexpansive mapping from C into itself with $F(T) \neq \emptyset$, $f \in \Pi_C$. For $t \in (0,1)$, $x_t \in C$ is the unique solution in C to Eq.(3.1). Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges strongly to a fixed point of T. If we define $Q : \Pi_C \to F(T)$ by*

$$Q(f) := \lim_{t\to 0} x_t, f \in \Pi_C, \tag{2}$$

then $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad f \in \Pi_C, p \in F(T). \tag{3}$$

In particular, if $f = u \in C$ is a constant, then (2) reduces to the sunny nonexpansive retraction from C onto $F(T)$,

$$\langle Q(u)-u, J(Q(u)-p)\rangle \le 0, \quad u \in C, p \in F(T).$$

Proof Necessity. Assume that $F(T) \neq \emptyset$. Take $p \in F(T)$, for $t \in (0,1)$.

$$\begin{aligned}\|x_t - p\| &= \|t(f(x_t)-p) + (1-t)(Tx_t - p)\| \\ &\le t\|f(x_t)-p\| + (1-t)\|Tx_t - p\| \\ &\le t\|f(x_t)-f(p)\| + t\|f(p)-p\| + (1-t)\|Tx_t - p\| \\ &\le (1-t+\alpha t)\|x_t - p\| + t\|f(p)-p\|.\end{aligned}$$

We obtain $\|x_t - p\| \le \frac{1}{1-\alpha}\|f(p)-p\|$. Therefore $\{x_t\}$ is bounded.

Sufficiency. Assume that $\{x_t\}$ is bounded as $t \to 0^+$. Assume that $t_n \to 0^+$ and $\{x_{t_n}\}$ is bounded. Since E is reflexive, we may assume that $x_{t_n} \rightharpoonup z$ for some $z \in C$. Since J_φ is weakly continuous, by Lemma 2.6, we have $\limsup_{n\to\infty} \Phi(\|x_{t_n} - x\|) = \limsup_{n\to\infty} \Phi(\|x_{t_n} - z\|) + \Phi(\|x - z\|), \forall x \in E$. Put $g(x) = \limsup_{n\to\infty} \Phi(\|x_{t_n} - x\|), x \in E$. It follows that $g(x) = g(z) + \Phi(\|x - z\|), \forall x \in E$. $\{x_n\}$ is bounded, so are $\{f(x_{t_n})\}$ and $\{Tx_{t_n}\}$, we get

$$\|x_{t_n} - Tx_{t_n}\| = \frac{t_n}{1-t_n}\|f(x_{t_n}) - x_{t_n}\| \to 0.$$

$$\begin{aligned}g(Tz) &= \limsup_{n\to\infty} \Phi(\|x_{t_n} - Tz\|) = \limsup_{n\to\infty} \Phi(\|Tx_{t_n} - Tz\|) \\ &\le \limsup_{n\to\infty} \Phi(\|x_{t_n} - z\|) = g(z).\end{aligned} \tag{4}$$

On the other hand,

$$g(Tz) = g(z) + \Phi(\|Tz - z\|). \tag{5}$$

Combining (4) and (5) yields that $\Phi(\|Tz - z\|) \le 0$. Hence $Tz = z$, and $z \in F(T)$.

Next we claim that $\{x_t\}$ converges strongly to a fixed point of T provided that it remains bounded as $t \to 0^+$. Let $\{t_n\}$ be a sequence in $(0,1)$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup z$, as $n \to \infty$. Then $z \in F(T)$ by the above arguments. We show that $x_{t_n} \to z$. In fact, by Lemma 2.6,

$$\begin{aligned}\Phi(\|x_{t_n} - z\|) \le& \Phi((1-t_n)\|Tx_{t_n} - z\|) + t_n\langle f(x_{t_n}) - z, J_\varphi(x_{t_n} - z)\rangle \\ \le& (1-t_n)\Phi(\|x_{t_n} - z\|) + t_n\langle f(x_{t_n}) - f(z) + f(z) - z, J_\varphi(x_{t_n} - z)\rangle \\ \le& (1-t_n)\Phi(\|x_{t_n} - z\|) + t_n\|f(x_{t_n}) - f(z)\|\varphi(\|x_{t_n} - z\|) +\end{aligned}$$

$$\begin{aligned}&t_n\langle f(z)-z, J_\varphi(x_{t_n}-z)\rangle\\ \le&(1-t_n)\Phi(\|x_{t_n}-z\|)+\alpha t_n\|x_{t_n}-z\|\varphi(\|x_{t_n}-z\|)+\\ &t_n\langle f(z)-z, J_\varphi(x_{t_n}-z)\rangle.\end{aligned}$$

This implies that

$$\Phi(\|x_{t_n}-z\|)\le\alpha\|x_{t_n}-z\|\varphi(\|x_{t_n}-z\|)+\langle f(z)-z, J_\varphi(x_{t_n}-z)\rangle.$$

Since $x_{t_n}\rightharpoonup z$, as $n\to\infty$, we have $J_\varphi(x_{t_n}-z)\rightharpoonup 0$, and by the continuous strictly increasing property of the gauge function, we have $\varphi(\|x_{t_n}-z\|)\to 0$. Then $\Phi(\|x_{t_n}-z\|)\to 0$. Hence $x_{t_n}\to z$. Finally, we prove that the entire net $\{x_t\}$ converges strongly. To this end, we assume that two null sequences $\{t_n\}$ and $\{s_n\}$ in $(0,1)$ are such that $x_{t_n}\to z$ and $x_{s_n}\to z'$. We claim that $z=z'$. In fact, for $p\in F(T)$,

$$\begin{aligned}\langle x_t-Tx_t, J_\varphi(x_t-p)\rangle&=\Phi(\|x_t-p\|)+\langle p-Tx_t, J_\varphi(x_t-p)\rangle\\ &\ge\Phi(\|x_t-p\|)-\|p-Tx_t\|\|J_\varphi(x_t-p)\|\\ &\ge\Phi(\|x_t-p\|)-\Phi(\|x_t-p\|)=0.\end{aligned}$$

On the other hand, $x_t-Tx_t=\frac{t}{1-t}(f(x_t)-x_t)$. For $t\in(0,1)$ and $p\in F(T)$, $\langle x_t-f(x_t), J_\varphi(x_t-p)\rangle\le 0$. In particular, $\langle x_{t_n}-f(x_{t_n}), J_\varphi(x_{t_n}-p)\rangle\le 0$ and $\langle x_{s_n}-f(x_{s_n}), J_\varphi(x_{s_n}-p)\rangle\le 0$. Letting $n\to\infty$, we have $\langle z-f(z), J_\varphi(z-p)\rangle\le 0$ and $\langle z'-f(z'), J_\varphi(z'-p)\rangle\le 0$. Adding up, we get $\langle -z'-(f(z)-f(z')), J_\varphi(z-z')\rangle\le 0$, i.e., $\langle z-z'), J_\varphi(z-z')\rangle\le\langle f(z)-f(z'), J_\varphi(z-z')\rangle\le\alpha\langle z-z', J_\varphi(z-z')\rangle$, $\Phi(\|z-z'\|)\le\alpha\Phi(\|z-z'\|)$, we have $z=z'$ and $\{x_t\}$ converges strongly.

Define $Q:\Pi_C\to F(T)$ by $Q(f)=\lim_{t\to 0}x_t, f\in\Pi_C$. Since we have proved that for all $t\in(0,1)$ and $p\in F(T)$, $\langle x_t-f(x_t), J_\varphi(x_t-p)\rangle\le 0$. Letting $t\to 0$, we have $\langle (I-f)Q(f), J_\varphi(Q(f)-p)\rangle\le 0$. This implies that

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle\le 0,$$

since $J_\varphi(x)=\frac{\varphi(\|x\|)}{\|x\|}J(x)$ for $x\ne 0$.

If $f=u$ is a constant, then $\langle x_t-u, J_\varphi(x_t-p)\rangle\le 0$, letting $t\to 0$, we have $\langle Q(u)-u, J_\varphi(Q(u)-p)\rangle\le 0$, which implies

$$\langle Q(u)-u, J(Q(u)-p)\rangle\le 0,\quad u\in C, p\in F(T),$$

since $J_\varphi(x)=\frac{\varphi(\|x\|)}{\|x\|}J(x)$ for $x\ne 0$.

Hence Q reduces to the sunny nonexpansive retraction from C to $F(T)$. This completes the proof. □

Next we consider the following iteration process

$$\begin{cases}x_0\in C,\\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)J_{r_n}x_n,\quad n\ge 0,\end{cases}\tag{6}$$

where $\alpha_n\in(0,1), r_n\in(0,\infty)$. In the case $f=u$ is a constant, the Eq.(6) reduces to the following:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0, \end{cases} \tag{7}$$

which has been investigated in [3] and [8]. Let $\{x_n\}$ be defined by (7). In [8] the authors established strong convergence theorems under the conditions E is uniformly smooth and has a weakly continuous duality map J_φ with some gauge φ; in [3] the authors obtained the strong convergence theorems under the assumptions that E is uniformly smooth or E is reflexive and has a weakly continuous duality map.

One question arises naturally: Can the iterative algorithms used in [3] and [8] be extended to iterative algorithm (6)? We give a positive answer to this problem.

Theorem 3.2 *Suppose E is a reflexive Banach space and has a weakly continuous duality map J_φ with some gauge φ, A is an m-accretive operator in E such that $C = \overline{D(A)}$. Assume that*

(i) $\alpha_n \to 0$;

(ii) $\Sigma_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $r_n \to \infty$.

Let $\{x_n\}$ be defined by (6). Then $\{x_n\}$ converges strongly to a point in F.

Proof First notice that $\{x_n\}$ is bounded. In fact, take a fixed $p \in F$

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|J_{r_n} x_n - p\| \\ &\leq \alpha_n \alpha \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &= (1 - \alpha_n + \alpha\alpha_n)\|x_n - p\| + \alpha_n \|f(p) - p\|, \end{aligned}$$

by induction, we obtain $\|x_n - p\| \leq \max\{\frac{1}{1-\alpha}\|f(p) - p\|, \|x_0 - p\|\} = M$, for all $n \geq 0$. Therefore $\{x_n\}$ is bounded, so are $\{J_{r_n} x_n\}, \{f(x_n)\}$, hence $\|x_{n+1} - J_{r_n} x_n\| = \alpha_n \|f(x_n) - J_{r_n} x_n\| \to 0$. Next we prove that

$$\limsup_{n\to\infty} \langle f(p) - p, J_\varphi(x_n - p)\rangle \leq 0, \quad \text{where} \quad p = Q(f). \tag{8}$$

Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(p) - p, J_\varphi(x_n - p)\rangle = \lim_{k\to\infty} \langle f(p) - p, J_\varphi(x_{n_k} - p)\rangle. \tag{9}$$

Since E is reflexive, we may assume that $x_{n_k} \rightharpoonup \widetilde{x}$. Moreover, $\|x_{n+1} - J_{r_n} x_n\| \to 0$, we obtain $J_{r_{n_k-1}} x_{n_k-1} \rightharpoonup \widetilde{x}$.

Taking the limit as $k \to \infty$ in the relation $[J_{r_{n_k-1}} x_{n_k-1}, A_{r_{n_k-1}} x_{n_k-1}] \in A$, we get $[\widetilde{x}, 0] \in A$. That is $\widetilde{x} \in F$. Hence by (9) and Lemma 2.1, we have

$$\limsup_{n\to\infty} \langle f(p) - p, J_\varphi(x_n - p)\rangle = \langle f(p) - p, J_\varphi(\widetilde{x} - p)\rangle \leq 0.$$

Finally, to prove that $x_n \to p$. Using Lemma 2.6,

$$\begin{aligned} \Phi(\|x_{n+1} - p\|) &= \Phi(\|(1-\alpha_n)(J_{r_n} x_n - p) + \alpha_n(f(x_n) - p)\|) \\ &\leq (1-\alpha_n)\Phi(\|x_n - p\|) + \alpha_n \langle f(x_n) - f(p) + f(p) - p, J_\varphi(x_{n+1} - p)\rangle \end{aligned}$$

$$\leq (1-\alpha_n+\alpha\alpha_n)\Phi(\|x_n-p\|)+\alpha_n\langle f(p)-p, J_\varphi(x_{n+1}-p)\rangle$$
$$=(1-(1-\alpha)\alpha_n)\Phi(\|x_n-p\|)+(1-\alpha)\alpha_n\frac{1}{1-\alpha}\langle f(p)-p, J_\varphi(x_{n+1}-p)\rangle.$$

An application of Lemma 2.2 yields that $\Phi(\|x_{n+1}-p\|)\to 0$; i.e., $\|x_n-p\|\to 0$.

Theorem 3.3 *Suppose E is a reflexive Banach space and has a weakly continuous duality map J_φ with some gauge φ, A is an m-accretive operator in E such that $C=\overline{D(A)}$. Assume that*

(i) $\alpha_n\to 0, \Sigma_{n=0}^{\infty}\alpha_n=\infty$ and $\Sigma_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ (e.g. $\alpha_n=\frac{1}{n}$);

(ii) $r_n\geq\varepsilon$ for all n and $\Sigma_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$ (e.g. $r_n=1+\frac{1}{n}$).

Let $\{x_n\}$ be defined by (6). Then $\{x_n\}$ converges strongly to a point in F.

Proof From (6) we have

$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)J_{r_n}x_n,$$

$$x_n=\alpha_{n-1}f(x_{n-1})+(1-\alpha_{n-1})J_{r_{n-1}}x_{n-1},$$

and

$$\begin{aligned}x_{n+1}-x_n=&(\alpha_n-\alpha_{n-1})(f(x_n)-J_{r_{n-1}}x_{n-1})+\\&(1-\alpha_n)(J_{r_n}x_n-J_{r_{n-1}}x_{n-1})+\alpha_{n-1}(f(x_n)-f(x_{n-1})).\end{aligned}\tag{10}$$

If $r_{n-1}\leq r_n$, using the resolvent identity $J_{r_n}x_n=J_{r_{n-1}}(\frac{r_{n-1}}{r_n}x_n+(1-\frac{r_{n-1}}{r_n})J_{r_n}x_n)$, we obtain that

$$\begin{aligned}\|J_{r_n}x_n-J_{r_{n-1}}x_{n-1}\|&\leq\frac{r_{n-1}}{r_n}\|x_n-x_{n-1}\|+(1-\frac{r_{n-1}}{r_n})\|J_{r_n}x_n-x_{n-1}\|\\&\leq\|x_n-x_{n-1}\|+(\frac{r_n-r_{n-1}}{r_n})\|J_{r_n}x_n-x_{n-1}\|\\&\leq\|x_n-x_{n-1}\|+\frac{1}{\varepsilon}\|r_n-r_{n-1}\|\|J_{r_n}x_n-x_{n-1}\|.\end{aligned}\tag{11}$$

It follows from (10) that

$$\begin{aligned}\|x_{n+1}-x_n\|\leq&M(\|\alpha_n-\alpha_{n-1}\|)+\|r_{n+1}-r_n\|)+(1-\alpha_n)\|x_n-x_{n-1}\|+\\&\alpha_{n-1}\alpha\|x_n-x_{n-1}\|\\\leq&(1-(1-\alpha)\alpha_n)\|x_n-x_{n-1}\|+M_1(\|\alpha_n-\alpha_{n-1}\|)+\|r_{n+1}-r_n\|),\end{aligned}\tag{12}$$

where $M_1>0$ is some approximate constant. Similarly we can prove (12) if $r_{n-1}\geq r_n$. By the Conditions (i), (ii), (iii) and Lemma 2.2, we can conclude that $\|x_{n+1}-x_n\|\to 0$. This implies that

$$\|x_n-J_{r_n}x_n\|\leq\|x_{n+1}-x_n\|+\|x_{n+1}-J_{r_n}x_n\|.\tag{13}$$

It follows that $\|A_{r_n}x_n\|=\frac{1}{r_n}\|x_n-J_{r_n}x_n\|\leq\frac{1}{\varepsilon}\|x_n-J_{r_n}x_n\|$.

Now if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging weakly to a point $\widetilde{x}$, then taking the limit as $k\to\infty$ in the relation $[J_{r_{n_k}}x_{n_k}, A_{r_{n_k}}x_{n_k}]\in A$, we get $[\widetilde{x},0]\in A$. That is $\widetilde{x}\in F$. Then all weak limit points of $\{x_n\}$ are zeros of A. The rest of the proof follows from the corresponding parts of Theorem 3.2.

Now consider the framework of uniformly smooth Banach space. Since F is the fixed point set of nonexpansive mapping J_r for all $r > 0$, there exists a unique sunny nonexpansive retraction Q from C onto F. In particular, for each $n \geq 1$, we have

$$Q(f) = \lim_{t \to 0} z_{t,n}, \quad f \in \Pi_C, \tag{14}$$

where $z_{t,n} \in C$ is the unique point in C such that

$$z_{t,n} = tf(z_{t,n}) + (1-t)J_{r_n} z_{t,n}, \tag{15}$$

$\{z_{t,n}\}$ is uniformly bounded; in fact, $\|z_{t,n} - p\| \leq \frac{1}{1-\alpha}\|f(p) - p\|$ for all $t \in (0,1), n \geq 1$ and $p \in F$.

Lemma 3.4 *The limit in (14) is uniform for $n \geq 1$.*

Proof It is sufficient to show that for any positive integer n_t (may depend on $t \in (0,1)$), if $z_{t,n_t} \in C$ is the unique point in C such that

$$z_{t,n_t} = tf(z_{t,n_t}) + (1-t)J_{r_{n_t}} z_{t,n_t}. \tag{16}$$

Then $\{z_{t,n_t}\}$ converges as $t \to 0$ to a point in F. For short, put $w_t = z_{t,n_t}, v_t = J_{r_{n_t}}$, it follows that

$$w_t = tf(w_t) + (1-t)v_t w_t. \tag{17}$$

Note that $F(v_t) = F$ for all t, $\{w_t\}$ is bounded since $\|w_t - p\| \leq \frac{1}{1-\alpha}\|f(p) - p\|$ for all $t \in (0,1)$ and $p \in F$, so are $\{f(w_t)\}, \{v_t w_t\}$. Then $\|w_t - v_t w_t\| = t\|f(w_t) - v_t w_t\| \to 0$ (as $t \to 0$). Since $r_n \geq \varepsilon$, for all n, by Lemma 2.5, we have

$$\|w_t - J_\varepsilon w_t\| \leq 2\|w_t - J_{r_{n_t}} w_t\| = 2\|w_t - v_t w_t\| \to 0. \tag{18}$$

Let $\{t_k\}$ be a sequence in $(0,1)$ such that $t_k \to 0$ as $k \to \infty$. Define a function g on C by $g(w) = \mathrm{LIM}_k \frac{1}{2}\|w_{t_k} - w\|^2, w \in C$, where LIM denotes a Banach limit on l^∞. Let $K := \{w \in C : g(w) = min\{g(y) : y \in C\}\}$, then K is a nonempty closed convex bounded subset of C. K is also invariant under the nonexpansive mapping J_ε. Indeed, noting (18), for $w \in K$, $g(J_\varepsilon w) = \mathrm{LIM}_k \frac{1}{2}\|w_{t_k} - J_{\varepsilon w}\|^2 \leq \mathrm{LIM}_k \frac{1}{2}\|w_{t_k} - w\|^2 = g(w)$. Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings and J_ε is a nonexpansive mapping of C, J_ε has a fixed point in K, say w'. Now since w' is also a minimizer of g over C, for $w \in C$,

$$0 \leq \frac{g(w') + \lambda(w - w') - g(w')}{\lambda} = \mathrm{LIM}_k \frac{1/2\|w_{t_k} - w' + \lambda(w' - w)\|^2 - 1/2\|w_{t_k} - w'\|^2}{\lambda}.$$

E is uniformly smooth, the duality map J is uniformly continuous on bounded sets, letting $\lambda \to 0^+$ in the last equation:

$$0 \leq \mathrm{LIM}_k \langle w' - w, J(w_{t_k} - w')\rangle, \quad w \in C. \tag{19}$$

Since $w_{t_k} - w' = t_k(f(w_{t_k}) - w') + (1-t_k)(v_{t_k} w_{t_k} - w')$,

$$\begin{aligned}\|w_{t_k} - w'\|^2 =& t_k\langle f(w_{t_k}) - w', J(w_{t_k} - w')\rangle + (1-t_k)\langle v_{t_k} w_{t_k} - w', J(w_{t_k} - w')\rangle \\ \leq& \alpha t_k \|w_{t_k} - w'\|^2 - t_k\langle f(w') - w'), J(w_{t_k} - w')\rangle +\end{aligned}$$

$$\begin{aligned}&(1-t_k)\|v_{t_k}w_{t_k}-w'\|\|J(w_{t_k}-w')\|\\ \le&(\alpha t_k+1-t_k)\|w_{t_k}-w')\|^2+t_k\langle f(w')-w',J(w_{t_k}-w')\rangle.\end{aligned}$$

It follows that

$$\|w_{t_k}-w'\|^2\le\frac{1}{1-\alpha}\langle f(w')-w',J(w_{t_k}-w')\rangle. \tag{20}$$

Letting $w=f(w')$ in (19), we get $\mathrm{LIM}\|w_{t_k}-w'\|^2\le 0$. Therefore $\{w_{t_k}\}$ contains a subsequence, $\{w_{t_k}\}$ converging strongly to w (say). By (18), w is also a fixed point of J_ε; i.e., a point in F.

To prove that the entire net $\{w_t\}$ converges strongly, assume $\{s_k\}$ is another null subsequence in $(0,1)$ such that $w_{s_k}\to w_2$. Then $w_2\in F$. Repeating the argument of (20), we get

$$\|w_t-w'\|^2\le\frac{1}{1-\alpha}\langle f(w')-w',J(w_t)-w')\rangle,\quad\forall w'\in F.$$

In particular,

$$\|w_2-w_1\|^2\le\frac{1}{1-\alpha}\langle f(w_1)-w_1,J(w_2-w_1)\rangle;$$

$$\|w_1-w_2\|^2\le\frac{1}{1-\alpha}\langle f(w_2)-w_2,J(w_1-w_2)\rangle.$$

Adding up the last two inequalities, $\|w_1-w_2\|^2\le 0$, that is $w_1=w_2$.

Theorem 3.5 *Suppose E is a uniformly smooth Banach space, A is an m-accretive operator in E such that $C=\overline{D(A)}$. Assume that*

(i) $\alpha_n\to 0,\Sigma_{n=0}^{\infty}\alpha_n=\infty$ and $\Sigma_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ (e.g. $\alpha_n=\frac{1}{n}$);

(ii) $r_n\ge\varepsilon$ for all n and $\Sigma_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$ (e.g. $r_n=1+\frac{1}{n}$).

Then $\{x_n\}$ converges strongly to a point in F.

Proof It is easy to see that $\{x_n\}$ is bounded, so are $\{J_{r_n}x_n\},\{f(x_n)\}$. We show that $\limsup_{n\to\infty}\langle f(z)-z,J(x_n-z)\rangle\le 0$, where

$$z=Q(f)=\lim_{z\to 0}z_{t,n},\ \ z_{t,n}=tf(z_{t,n})+(1-t)J_{r_n}z_{t,n},$$

$$\|z_{t,n}-x_n\|^2=t(f(z_{t,n})-x_n)+(1-t)(J_{r_n}z_{t,n}-x_n).$$

Putting $a_n(t)=\|J_{r_n}x_n-x_n\|(2\|z_{t,n}-x_n\|+\|J_{r_n}x_n-x_n\|)\to 0(n\to\infty)$ and by Lemma 2.3

$$\begin{aligned}\|z_{t,n}-x_n\|^2\le&(1-t)^2\|J_{r_n}z_{t,n}-x_n\|^2+2t\langle f(z_{t,n})-x_n,J(z_{t,n}-x_n)\rangle\\ \le&(1-t)^2(\|J_{r_n}z_{t,n}-J_{r_n}x_n\|+\|J_{r_n}x_n-x_n\|)^2+\\ &2t\langle f(z_{t,n})-z_{t,n},J(z_{t,n}-x_n)\rangle+2t\|z_{t,n}-x_n\|^2\\ \le&(1-t)^2\|z_{t,n}-x_n\|^2+a_n(t)+\\ &2t\langle f(z_{t,n})-z_{t,n},J(z_{t,n}-x_n)\rangle+2t\|z_{t,n}-x_n\|^2\end{aligned}$$

which yields that $\langle f(z_{t,n})-z_{t,n},J(x_n-z_{t,n})\rangle\le\frac{t}{2}\|z_{t,n}-x_n\|^2+\frac{1}{2t}a_n(t)$.

It follows that $\limsup\langle f(z_{t,n})-z_{t,n},J(x_n-z_{t,n}\rangle\le\frac{t}{2}M$, where $M>0$ is a constant such that $M\ge\|z_{t,n}-x_n\|^2$ for all $n\ge 1$ and $t\in(0,1)$. Hence

$$\limsup\langle f(z_{t,n})-z_{t,n},J(x_n-z_{t,n}\rangle\le 0. \tag{21}$$

J is uniformly continuous on bounded sets and the uniform convergence of $\{z_{t,n}\}$ to $Q(f)$, so we can interchange the two limits and deduce that

$$\limsup\langle f(z)-z, J(x_n-z)\rangle \leq 0. \tag{22}$$

Finally to show $x_n \to z$. $x_{n+1}-z=\alpha_n(f(x_n)-z)+(1-\alpha_n)(J_{r_n}x_n-z)$. Applying the Lemma 2.3, we get

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq &(1-\alpha_n)^2\|J_{r_n}x_n-z\|^2+2\alpha_n\langle f(x_n)-z, J(x_{n+1}-z)\rangle\\ \leq &(1-\alpha_n)^2\|x_n-z\|^2+2\alpha_n\langle f(x_n)-f(z)+f(z)-z, J(x_{n+1}-z)\rangle\\ \leq &(1-\alpha_n)^2\|x_n-z\|^2+2\alpha_n\alpha\|x_n-z\|\|x_{n+1}-z\|+\\ &2\alpha_n\langle f(z)-z, J(x_{n+1}-z)\rangle\\ \leq &(1-\alpha_n)^2\|x_n-z\|^2+\alpha\alpha_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2)+\\ &2\alpha_n\langle f(z)-z, J(x_{n+1}-z)\rangle.\end{aligned}$$

It then follows that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq \frac{1-(2-\alpha)\alpha_n+\alpha_n^2}{1-\alpha\alpha_n}\|x_n-z\|^2+\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(z)-z, J(x_{n+1}-z)\rangle\\ &\leq \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}\|x_n-z\|^2+\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(z)-z, J(x_{n+1}-z)\rangle+M\alpha_n^2\end{aligned}$$

Put $\widetilde{\alpha_n}=\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}$, $\widetilde{\beta_n}=\frac{M(1-2\alpha\alpha_n)\alpha_n}{1-\alpha}+\frac{1}{1-\alpha}\langle f(z)-z, J(x_{n+1}-z)\rangle$, then we get

$$\|x_{n+1}-z\|^2 \leq (1-\widetilde{\alpha_n})\|x_n-z\|^2+\widetilde{\alpha_n}\widetilde{\beta_n}. \tag{23}$$

From the assumptions (i) and (22), $\widetilde{\alpha_n}\to 0$, $\Sigma\widetilde{\alpha_n}=\infty$, $\limsup\widetilde{\beta_n}\leq 0$. Using the Lemma 2.2 to (23) we conclude $x_n \to z$ as $n\to\infty$.

Remark The results in our paper improve the results in [3]. When $f=u$ is a constant, the results reduce to the ones in [3], which are special cases of ours.

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